Solutions to Problems and Projects for Chapter 10

10.1. Consider the three forces acting at the point A. The slanting push of magnitude P, the horizontal pull H from left to right by the tie beam, and the upward push that balances $\frac{1}{2}$ of the load L. (The other half of the load L is balanced by the corresponding upward push at B.) These three forces are in equilibrium. By the force diagram of Figure 10.37,

$$\sin\alpha = \frac{\frac{L}{2}}{P} \text{ and } \tan\alpha = \frac{\frac{L}{2}}{H}.$$

The expressions for P and H follow directly from this.

10.2. Figure 10.38 tells us that $\sin 70^\circ = \frac{b}{r}$, so that $b = r\sin 70^\circ \approx (15.2)(0.94) \approx 14.3$ m. By the Pythagorean theorem $r^2 = b^2 + (r-a)^2$. Therefore $a^2 - 2ra + b^2 = 0$ and by the quadratic formula, $a = \frac{2r\pm\sqrt{4r^2-4b^2}}{2} = r \pm \sqrt{r^2-b^2}$. Since $a < r$, $a = r - \sqrt{r^2-b^2} \approx 15.2 - \sqrt{15.2^2 - 14.3^2} \approx 10$ m.

10.3. **i.** Since $b = 14.3$ m, $a = 10 - 3.1 = 6.9$ m, and $r^2 = b^2 + (r-a)^2$ by the Pythagorean theorem, we get $2ar = a^2 + b^2$ and hence $r = \frac{a^2+b^2}{2a} = \frac{6.9^2+14.3^2}{13.8} \approx 18.3$ m. Since $\sin\frac{\theta}{2} = \frac{b}{r} \approx \frac{14.3}{18.3} \approx 0.78$, it follows by the push of the inverse sine button of a calculator that $\frac{\theta}{2} \approx 51.4^\circ$ and hence that $\theta \approx 103^\circ$.

ii. We first turn to estimate the volume of the shell of the original dome. With the estimates $r \approx 18.3$ m, $R \approx 19.1$ m, and $\theta \approx 103^\circ$ in hand, return to Section 10.1.1 and Figure 10.3. Check, using Figure 10.3, that $a = r\cos\frac{\theta}{2} \approx 18.3\cos 51.4^\circ \approx 11.4$ m. (A word of caution is in order. The a of Figure 10.3 needed now and the a of Figure 10.38 mean different things.) Inserting the above values for r, R, and a into the volume formula

$$V = \pi\big[\tfrac{2}{3}R^3 - R^2a + \tfrac{1}{3}a^3\big] - \pi\big[\tfrac{2}{3}r^3 - r^2a + \tfrac{1}{3}a^3\big] = \pi\big[\tfrac{2}{3}(R^3 - r^3) - (R^2 - r^2)a\big].$$

derived in Section 10.1.1, we get

$$V \approx \pi\big[\tfrac{2}{3}(19.1^3 - 18.3^3) - (19.1^2 - 18.3^2)11.4\big] \approx 686.44\,\mathrm{m}^3$$

for the volume of the shell of the original dome. Given that the density of the original shell is assumed to be 1760 kg/m^3, we get that the mass m of the original shell is

$$m \approx 686.44 \times 1760 \approx 1{,}208{,}134 \approx 1{,}200{,}000\,\mathrm{kg}.$$

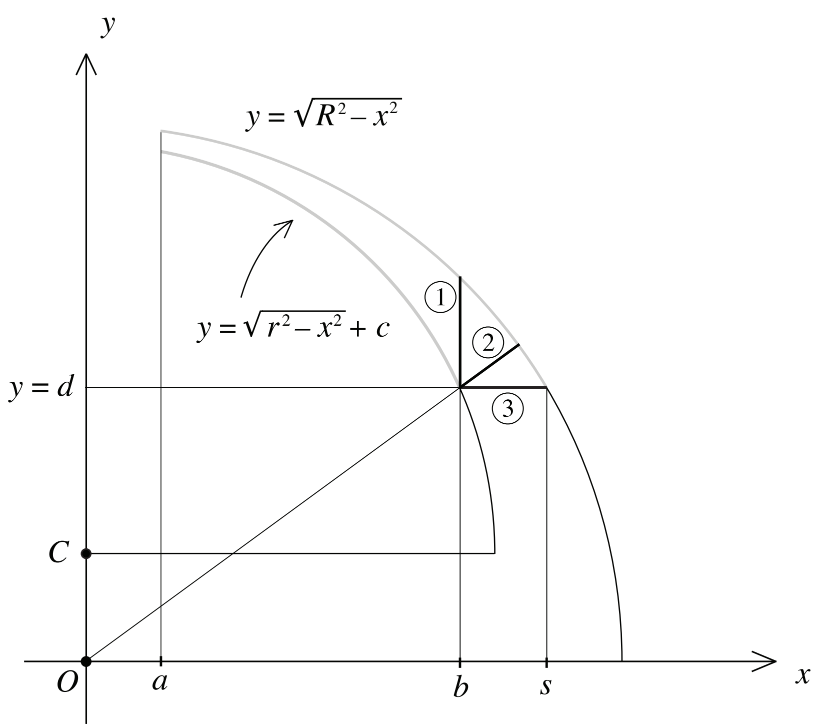
iii. The weight of the original shell is $W = mg \approx (1{,}208{,}134)(9.81) \approx 11{,}851{,}795\,\mathrm{N}$. So the weight per rib is $\frac{11{,}851{,}795}{40} \approx 296{,}295 \approx 296{,}000\,\mathrm{N}$. Consider Figure 10.4 but replace the angle 70° by the angle 51.4° from part (i). This provides the estimates

$$P \approx \frac{296{,}295}{\sin 51.4^\circ} \approx 379{,}000\,\mathrm{N} \text{ and } H \approx \frac{296{,}295}{\tan 51.4^\circ} \approx 237{,}000\,\mathrm{N}$$

for the magnitudes of these forces for the original dome.

What we can conclude by way of comparisons is this. The original dome was about 15% lighter than the current dome. But the fact that it was flatter meant that the slanting push of each rib was slightly greater than the slanting push of each rib of the current dome (379,000 N to 375,000 N). This difference is much more pronounced in the context of the lateral forces against the base of the dome. The lateral force of 237,000 N per rib against the base of the original dome was much greater than the 128,000 N per rib against the base of the current dome. There is little doubt that this significant reduction of the outward thrust of the shell against the base of the dome did much to increase the stability of the structure of the Hagia Sophia overall.

10.4. Figure 10.8 tells us that the shell is thinnest at the oculus. The shell's thickness there is equal to $\sqrt{R^2 - a^2} - (\sqrt{r^2 - a^2} + c)$. Inserting the data $R = 28.1, r = 21.6, a = 4.1$ and $c = 5.2$ all in meters, we get that this thickness is approximately 1.39 m. For the maximal thickness of the shell we'll consider several possibilities. They are labeled (1) (2) and (3) in the figure below. For (1) we get $\sqrt{R^2 - b^2} - (\sqrt{r^2 - b^2} + c)$. Using the earlier data as well as $b = 19.4$ m we get 5.63 m for this vertical thickness. For the slanting thickness (2) we get $R - \sqrt{b^2 + d^2}$. Since $d = 14.9$ m we get 3.64 m for the



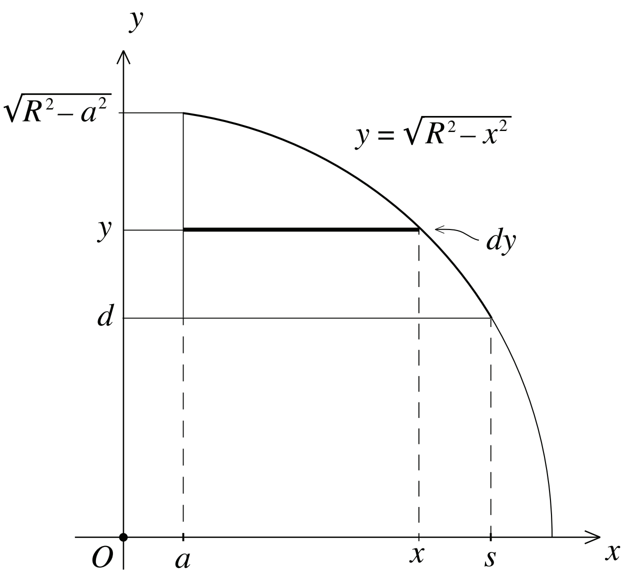
slanting thickness (2). As to the horizontal thickness (3) of the shell, note that it is simply $s - b$. Since $s = 23.8$ m, is $23.8 - 19.4 = 4.4$ m.

10.5. Figures (a) and (b) below refine Figures 10.39a and 10.39b, respectively. We will consider the volumes $\mathrm{V_a}$ and $\mathrm{V_b}$ given by the integrals

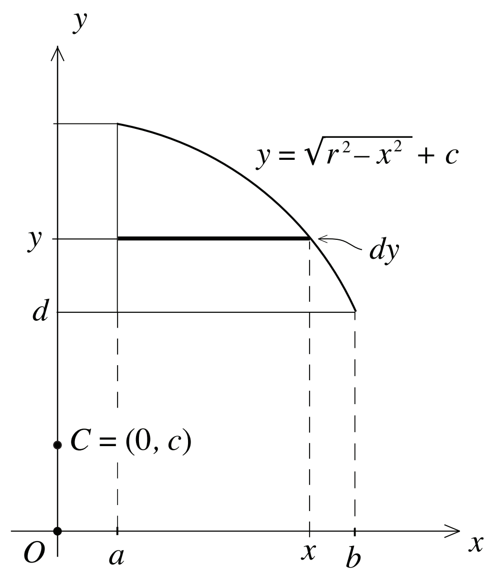
$$\mathrm{V_a} = \pi \int_d^{\sqrt{R^2-a^2}} \big((R^2 - y^2) - a^2\big)\, dy \quad \text{and} \quad \mathrm{V_b} = \pi \int_d^{\sqrt{r^2-a^2}+c} \big(r^2 - (y-c)^2 - a^2\big)\, dy$$

that theses figures give rise to and show that the difference $\mathrm{V_a} - \mathrm{V_b}$ represents the volume of the shell of the Pantheon.

Start with Figure (a). Focus on the horizontal strip of thickness dy. It starts at the point (a, y) and ends at the point (x, y). Its length is $x - a$ and x. Solving $x^2 + y^2 = R^2$ for x we get $x = \sqrt{R^2 - y^2}$. Revolving this horizontal strip one complete turn around



(a)



(b)

the y-axis, we get a volume element that is the difference between two discs of thickness dy one of radius $x = \sqrt{R^2 - y^2}$ and the other of radius a. It follows that this element has volume

$$\pi(\sqrt{R^2 - y^2})^2\, dy - \pi a^2\, dy = \pi\big((R^2 - y^2) - a^2\big)\, dy\,.$$

Summing all these volumes up between $y = d$ and $\sqrt{R^2 - a^2}$ gives us integral $(\mathrm{V_a})$ on the one hand, and on the other the volume obtained by revolving one complete turn around the y-axis the region under the graph of $y = \sqrt{R^2 - x^2}$, over the line $y = d$, and to the right of the line $x = a$. Doing the same thing with Figure (b) tells us that the integral $(\mathrm{V_b})$ above is equal to the volume obtained by revolving the region under the graph of $y = \sqrt{r^2 - x^2} + c$, over the line $y = d$, and to the right of the line $x = a$ one complete turn around the y-axis. The difference $(\mathrm{V_a}) - (\mathrm{V_b})$ between the integrals and the volumes is the volume of the shell of the dome of the Pantheon.

Now to the evaluation of the integrals. The antiderivatives are easy so that

$$
\begin{aligned}
\mathrm{V_a} = \pi\big((R^2-a^2)y - \tfrac{y^3}{3}\big)\Big|_d^{\sqrt{R^2-a^2}} &= \pi\big[\big((R^2-a^2)^{\frac{3}{2}} - \tfrac{1}{3}(R^2-a^2)^{\frac{3}{2}}\big) - \big((R^2-a^2)d - \tfrac{d^3}{3}\big)\big] \\
&= \pi\big(\tfrac{2}{3}(R^2-a^2)^{\frac{3}{2}} - (R^2-a^2)d + \tfrac{d^3}{3}\big)
\end{aligned}
$$

for the first integral, and

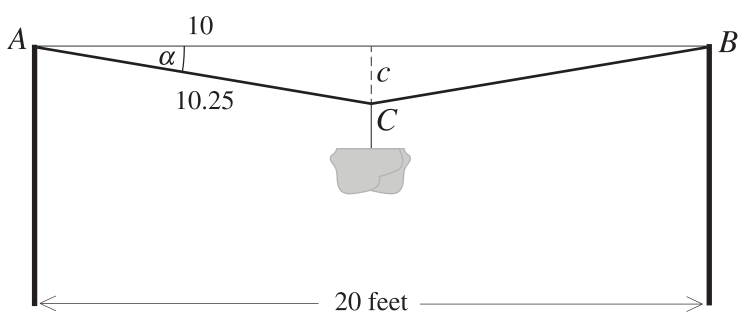
$$
\begin{aligned}
\mathrm{V_b} &= \pi\big((r^2-a^2)y - \tfrac{1}{3}(y-c)^3\big)\Big|_d^{\sqrt{r^2-a^2}+c} \\
&= \pi\big[\big((r^2-a^2)^{\frac{3}{2}} + (r^2-a^2)c - \tfrac{1}{3}(r^2-a^2)^{\frac{3}{2}}\big) - \big((r^2-a^2)d - \tfrac{1}{3}(d-c)^3\big)\big] \\
&= \pi\big(\tfrac{2}{3}(r^2-a^2)^{\frac{3}{2}} + (r^2-a^2)c\big) - (r^2-a^2)d + \tfrac{1}{3}(d-c)^3\big)
\end{aligned}
$$

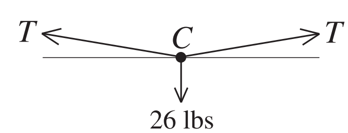
for the second.

The solution above gives the same value for the volume of the shell of the Pantheon as the solution in Section 10.1.2. When finding the numerical values, however, roundup procedures will provide somewhat different results. In particular, after substituting the data from 10.1.2 into the expressions for $\mathrm{V_a}$ and $\mathrm{V_b}$ we get $\mathrm{V_a} \approx 12{,}284\,\mathrm{m}^3$ and $\mathrm{V_b} \approx 7227\,\mathrm{m}^3$ so that $\mathrm{V_a} - \mathrm{V_b} \approx 12{,}284 - 7227 \approx 5057\,\mathrm{m}^3$. This is a little different than the approximation achieved in Section 10.1.2.

This "horizontal strategy" for determining the weight of the dome of the Pantheon together with more precise data about the location of the boundaries between the strata of concrete of different weights as illustrated by Figure 10.6, provides an approach for a much more accurate estimate of the weight of the shell of the dome than that of the inequality toward the end of Section 10.1.2.

10.6. The figure below on the right depicts the situation that the problem describes.



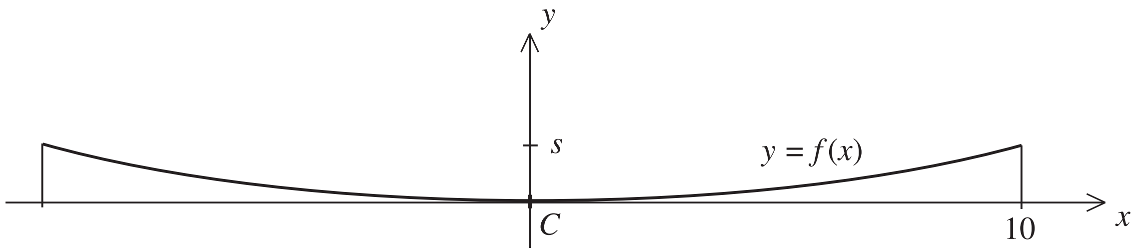


i. Let the distance from the point C to the horizontal AB be c. Refer to the figure and notice that $c^2 = 10.25^2 - 10^2 = 105.0625 - 100 = 5.0625$, so that $c = \sqrt{5.0625} = 2.25$ feet.

ii. The diagram on the left tells us that $\sin\alpha = \frac{c}{10.25} = \frac{2.25}{10.25}$. Let T be the tension in the line. The symmetry of the situation and the force diagram on the right tells us that $2T\sin\alpha = 26$ pounds. It follows that

$$T = \tfrac{(13)(10.25)}{2.25} \approx 59.22 \text{ pounds.}$$

10.7. Place an xy-coordinate system into the plane of Figure 10.40 so that the x-axis is parallel the line AB and C is at the origin. Let $y = f(x)$ be a function that has the clothesline as its graph. Since the weight of the socks is distributed evenly along the x-axis, the discussion of Section 8.3 applies with $d = 10, w = \frac{26}{20} = 1.3$ pounds/foot, and with an as yet undetermined sag s. It follows that $f(x) = \frac{s}{d^2}x^2$ with $0 \le x \le d$.



i. Since $f'(x) = \frac{2s}{d^2}x$, this is the length formula of Section 10.2 applied to the situation of the clothesline.

ii. We will use what we know to determine the sag s. It is a consequence of the conclusion of Problem 10.6i that $s < 2.25$ feet. Therefore $\frac{s}{2d} < \frac{2.25}{20} = 1.125 < \frac{1}{8}$. It follows that the approximation

$$L \approx d + \frac{2}{3}\frac{s}{d}s - \frac{2}{5}\left(\frac{s}{d}\right)^3 s = d + \frac{2}{3}\left(\frac{s}{d}\right)^2 d - \frac{2}{5}\left(\frac{s}{d}\right)^4 d$$

developed in Section 10.2 holds. As demonstrated there, this approximation is tight, so that we will take it to be an equality. Since the clothesline is 20.5 feet long, $L = 10.25$ feet, so that

$$10.25 = d + \frac{2}{3}\left(\frac{s}{d}\right)^2 d - \frac{2}{5}(\tfrac{s}{d})^4 d.$$

Since $d = 10$, we get $\frac{10.25}{10} = 1 + \frac{2}{3}(\frac{s}{10})^2 - \frac{2}{5}(\frac{s}{10})^4$ and therefore that

$$\tfrac{2}{5}(\tfrac{s}{10})^4 - \tfrac{2}{3}(\tfrac{s}{10})^2 + \tfrac{1}{40} = \tfrac{2}{5}(\tfrac{s}{10})^4 - \tfrac{2}{3}(\tfrac{s}{10})^2 + 1.025 - 1 = \tfrac{2}{5}(\tfrac{s}{10})^4 - \tfrac{2}{3}(\tfrac{s}{10})^2 + \tfrac{10.25}{10} - 1 = 0.$$

With $z = (\frac{s}{10})^2$ we see that $\frac{2}{5}z^2 - \frac{2}{3}z + \frac{1}{40} = 0$, and after multiplying through by 120, that

$$48z^2 - 80z + 3 = 0.$$

By the quadratic formula

$$z = \frac{80 \pm \sqrt{80^2 - 4(48)(3)}}{96} = \frac{80 \pm \sqrt{8^2 \cdot 10^2 - 8^2 \cdot 3^2}}{8 \cdot 12} = \frac{80 \pm 8\sqrt{10^2 - 3^2}}{8 \cdot 12} = \frac{10 \pm \sqrt{91}}{12}.$$

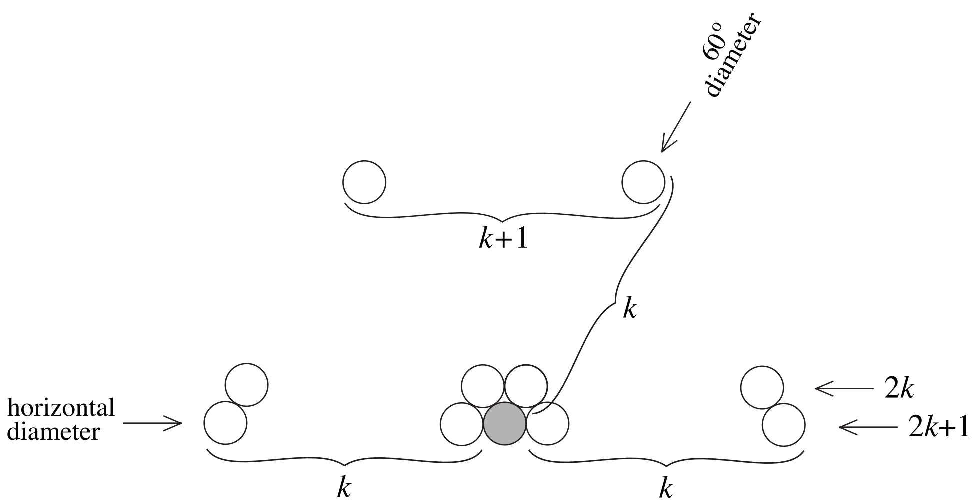
Since $z = (\frac{s}{10})^2$, we get $\frac{s}{10} = \sqrt{\frac{10 \pm \sqrt{91}}{12}}$ and $s = 10\sqrt{\frac{10 \pm \sqrt{91}}{12}}$. With the $+$ option, this is approximately equal to $s \approx 12.764$. Since $s < 2.25$, this is too large. With the $-$ option we get our result $s \approx 1.959$ feet.

iii. We now have all we need to apply the formulas

$$T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2 + 1} \quad \text{and} \quad T_0 = \frac{1}{2}\frac{wd^2}{s}$$

for the maximal tension T_d and the minimal T_0 tension in the clothesline. (Refer to Section 8.3.) Feeding $w = 1.3, d = 10$, and $s = 1.96$ into the formulas, we get $T_d \approx 35.62$ pounds and $T_0 \approx 33.16$ pounds.

10.8. The diagram below depicts some of the more important circles of the upper part of the hexagonal array that the problem descibes. This includes the circle at the center of the array (in grey) and an indication of the k circles on the horizontal diameter to the left and right of the circle at the center. The k circles on the upper part of one of the $60°$ diameters are indicated as well. These k circles tell us that the process described in the problem stops after k steps. Going up, each horizontal row of circles has one



fewer circle than the one below it. It follows that after k steps the highest horizontal row of the array has $(2k + 1) - k = k + 1$ circles in it (as the diagram indicates). The horizontal rows above the horizontal diameter form a trapezoid. It that has $2k$ circles in its bottom row of circles. In each successive row above this, there is one fewer circle. So there are $2k - 1$ circles in the row above the bottom row, $2k - 2$ circles in the row above that, and so on. Finally in the top row there are $k + 1$ circles. The circles below the horizontal diagonal form a trapezoid as well. There are $k + 1$ circles in its bottom row, one additional circle in each successive row going up, until the top row of the bottom trapezoid with its $2k$ circles is reached. The sum of the $2k$ circles of the bottom row of the upper trapezoid plus the $k + 1$ circles of the bottom row of the lower trapezoid is $2k + (k + 1) = 3k + 1$ circles. Going up and adding the circles of the next two rows of the two trapezoids we get $(2k - 1) + (k + 2) = 3k + 1$. Continuing the pattern and adding the circles of the two corresponding rows, we get $3k + 1$ each time. The last and kth step is the addition $(k + 1) + 2k = 3k + 1$ of the number of circles in the highest rows of the two trapezoids. It follows that the number of circles above and below the horizontal diagonal of the hexagonal array is $k(3k + 1) = 3k^2 + k$. Therefore the total number of circles in the hexagonal array is $(3k^2 + k) + (2k + 1) = 3k^2 + 3k + 1$.

The next three problems make use of the formula

$$L = \tfrac{d}{2}\sqrt{1+\left(\tfrac{2s}{d}\right)^2} \;+\; \tfrac{d^2}{4s}\ln\left(\tfrac{2s}{d}+\sqrt{1+\left(\tfrac{2s}{d}\right)^2}\right)$$

for one-half of the length L of the cable of a suspension bridge as well as the approximation

$$L \approx d + \tfrac{2}{3}\tfrac{s}{d}s - \tfrac{2}{5}\left(\tfrac{s}{d}\right)^3 s$$

both developed in Section 10.2.

10.9. The relevant data for the new Tacoma Narrows Bridge include its center span of 2800 feet and the sag in the main cable over the center span of about 280 feet. So we get $d = 1400$, $s = 280$, the value

$$L = \tfrac{1400}{2}\sqrt{1+\left(\tfrac{2\cdot 280}{1400}\right)^2} \;+\; \tfrac{1400^2}{4\cdot 280}\ln\left(\tfrac{2\cdot 280}{1400}+\sqrt{1+\left(\tfrac{2\cdot 280}{1400}\right)^2}\right) \approx 1436.48 \text{ feet}$$

and the approximation $L \approx 1400 + \frac{2}{3}\frac{280}{1400}280 - \frac{2}{5}\left(\frac{280}{1400}\right)^3 280 \approx 1436.44$ feet. So the approximation is tight. The length of the cable over the center span is 2872.96 feet.

10.10. What we need to know about the Verrazano Narrows Bridge is that its center span is 4260 feet and the sag in each of the main cables is 385 feet. So $d = 2130$ and $s = 385$. The value for L is

$$L = \tfrac{2130}{2}\sqrt{1+\left(\tfrac{2\cdot 385}{2130}\right)^2} \;+\; \tfrac{2130^2}{4\cdot 385}\ln\left(\tfrac{2\cdot 385}{2130}+\sqrt{1+\left(\tfrac{2\cdot 385}{2130}\right)^2}\right) \approx 2175.52 \text{ feet.}$$

and the approximation is $L \approx 2130 + \frac{2}{3}\frac{385}{2130}385 - \frac{2}{5}\left(\frac{385}{2130}\right)^3 385 \approx 2175.48$ feet.

10.11. A rough approximation for the ultimate strength can be gotten by noting that the cross-sectional area of a main cable is $\pi r^2 \approx \pi(18^2) \approx 1018$ in^2. Since the ultimate strength of a wire is 220,000 pounds/in^2, this provides the estimate $1018 \times 220{,}000 \approx 224{,}000{,}000$ pounds. Since the cable is pressed together from circular strands, provided with a coating of lead paste, and wrapped with additional wires, not all of the cross section of the cable is weight (or tension) bearing. A more accurate approach is to note that the cross-sectional area of a single wire is $\pi(\frac{0.196}{2})^2 \approx 0.030172$ in^2 so that the ultimate strength of a single wire is $(0.030172)(220{,}000) \approx 6638$ pounds/in^2. Multiplying this by the 27,572 wires of the cable this gives the approximation for the ultimate strength of the cable as $(6638)(27{,}572) \approx 183{,}000{,}000$ pounds.

The Akashi Straits Bridge was already studied in Problem 8.28. A comparison of the data provided there with that of the introduction to Problem 10.12 shows that there are differences. The center span is listed as 1990 m and again as 1991 m. The difference of 1 meter was the result of the powerful Kobe Earthquake of 1995. The towers—already built at the time—moved apart by about 1 meter (actually 0.8 m) in response to the seismic forces. While the towers suffered no damage of any consequence, their shift increased the center span accordingly. The second difference concerns the height of the towers. This is listed both as 282.8 m and 297 m. This discrepancy is explained by the fact that the first figure does not include the height of the saddle-like structures that brace the cables at the very top of each tower. With their height included, the towers rise to 297.3 m. We now turn to a study of the main cables of the Akashi Straits Bridge.

10.12. **i.** The hexagonal array of 127 steel wires in each strand conforms to the arrangement described in Section 10.2. It follows that $127 = 3k^2 + 3k + 1$ for some k. It is easy to see that $k = 6$. In general, the horizontal diameter has $2k + 1$ wires along it, a strand has 13 wires along the diameter.

ii. Given that a wire has a diameter of 5.23 millimeters, it has a cross-sectional area of $\pi(\frac{5.23}{2})^2 \approx 21.4829$ mm^2. Since the wires have an ultimate strength of 1800 N per mm^2, it follows that a single wire has an ultimate strength of about $(21.4829)(1800) \approx 38{,}669$ N.

iii. By summing up the ultimate strengths of the 36,830 steel wires in a cable, we get that a cable of the Akashi Straits Bridge has an ultimate strength of $(38{,}669)(36{,}830) \approx 1{,}424{,}000{,}000$ N.

iv. A look at Section 6.10 tells us that 1 newton corresponds to about 0.2248 pounds. It follows that the ultimate strength of each cable is $1{,}424{,}000{,}000(0.2248) \approx 319{,}000{,}000$ pounds. We saw in Section 10.2 that the ultimate strength of a cable of the George Washington Bridge is 175,000,000 pounds. So a main cable of the Akashi Straits Bridge is almost twice as strong as a main cable of the George Washington Bridge. This is in part explained by the fact that the diameter of the cable of the Akashi Straights Bridge is greater than that of the George Washington (about 1.12 meters compared to the 0.915 meters). Most of the difference, however, is due to the greater tensile strength of the steel wires used in the cable of the Akashi Straits Bridge.

10.13. The tractrix $y = T(x)$ of Figure 10.13 satisfies $\frac{dy}{dx} = T'(x) = -\frac{\sqrt{a^2-x^2}}{x}$. So by the length formula of Section 9.3, the length of the curve from $(a, 0)$ to $(c, T(c))$ is equal to

$$\int_c^a \sqrt{1 + T'(x)^2}\, dx = \int_c^a \sqrt{1 + \tfrac{a^2-x^2}{x^2}}\, dx = \int_c^a \sqrt{\tfrac{a^2}{x^2}}\, dx = \int_c^a \tfrac{a}{x}\, dx = a \ln x \,\Big|_c^a = a \ln \tfrac{a}{c}.$$

10.14. That $A(c) = \displaystyle\int_c^a \left(a \operatorname{sech}^{-1}\tfrac{x}{a} - \sqrt{a^2 - x^2}\right) dx$ follows from the formula for $T(x)$ derived in Section 10.3.

i. By Formula (23) of Section 9.11, with $u = \frac{x}{a}$ and hence $du = \frac{1}{a}dx$, we get

$$\tfrac{1}{a}\int \operatorname{sech}^{-1}\tfrac{x}{a}\, dx = \tfrac{x}{a} \operatorname{sech}^{-1}\tfrac{x}{a} + \sin^{-1}\tfrac{x}{a} + C.$$

Factoring a^2 from $(a^2 - x^2)$, we get $\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \frac{x^2}{a^2})} = a\sqrt{1 - (\frac{x}{a})^2}$ and hence that $\displaystyle\int \sqrt{a^2 - x^2}\, dx = a \int \sqrt{1 - (\tfrac{x}{a})^2}\, dx$. By applying Formula (16) of Section 9.11 with $u = \frac{x}{a}$ and $du = \frac{1}{a}dx$, we see that

$$\frac{1}{a}\int \sqrt{1 - (\tfrac{x}{a})^2}\, dx = \tfrac{x}{2a}\sqrt{1 - (\tfrac{x}{a})^2} + \tfrac{1}{2}\sin^{-1}\tfrac{x}{a} + C.$$

After multiplying both integrals above by a^2, we get

$$\begin{aligned}\int &\left(a\,\mathrm{sech}^{-1}\tfrac{x}{a} - \sqrt{a^2 - x^2}\right) dx \\ &= \left(ax\,\mathrm{sech}^{-1}\tfrac{x}{a} + a^2\sin^{-1}\tfrac{x}{a}\right) - \left(\tfrac{ax}{2}\sqrt{1 - (\tfrac{x}{a})^2} + \tfrac{a^2}{2}\sin^{-1}\tfrac{x}{a}\right) + C \\ &= ax\,\mathrm{sech}^{-1}\tfrac{x}{a} + \tfrac{a^2}{2}\sin^{-1}\tfrac{x}{a} - \tfrac{ax}{2}\sqrt{1 - (\tfrac{x}{a})^2} + C.\end{aligned}$$

From Figures 9.39 and 9.31 we know that $\mathrm{sech}^{-1}(1) = 0$ and $\sin^{-1}(1) = \frac{\pi}{2}$. So it follows that

$$\begin{aligned}A(c) &= \int_c^a \left(a\,\mathrm{sech}^{-1}\tfrac{x}{a} - \sqrt{a^2 - x^2}\right) dx = \left(ax\,\mathrm{sech}^{-1}\tfrac{x}{a} + \tfrac{a^2}{2}\sin^{-1}\tfrac{x}{a} - \tfrac{ax}{2}\sqrt{1 - (\tfrac{x}{a})^2}\right)\Big|_c^a \\ &= \tfrac{\pi a^2}{4} - ac\,\mathrm{sech}^{-1}\tfrac{c}{a} - \tfrac{a^2}{2}\sin^{-1}\tfrac{c}{a} + \tfrac{ac}{2}\sqrt{1 - (\tfrac{c}{a})^2}.\end{aligned}$$

ii. The limit formula in question is $\lim\limits_{x\to 0}\left(x\,\mathrm{sech}^{-1}x\right) = 0$. The property of $y = \sin^{-1}x$ can be read off from Figure 9.31. It is the fact that $\lim\limits_{x\to 0}\sin^{-1}x = 0$. Pushing c to zero in the expression for $A(c)$, tells us that $\lim\limits_{c\to 0}A(c) = \frac{1}{4}\pi a^2$.

10.15. **i.** The derivative of $f(x) = \frac{1}{x}$ is $f'(x) = -x^{-2}$. Therefore, $f'(x)^2 = (-x^{-2})^2 = x^{-4} = \frac{1}{x^4}$. Therefore

$$S(c) = \int_1^c 2\pi\tfrac{1}{x}\sqrt{1 + \tfrac{1}{x^4}}\, dx = \int_1^c 2\pi\tfrac{1}{x}\sqrt{\tfrac{x^4+1}{x^4}}\, dx = \int_1^c 2\pi\tfrac{1}{x^3}\sqrt{x^4 + 1}\, dx.$$

ii. The area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = c$ (for $c \geq 1$) is equal to $\displaystyle\int_1^c \tfrac{1}{x} = \ln x\,\Big|_1^c = \ln c - \ln 1 = \ln c$. Even though the natural log function ln grows very slowly, see Figure 7.42, it is a fact that $\lim\limits_{c\to+\infty}\ln c = +\infty$. To see this pick any number $d > 0$ (no matter how large) and notice that $\ln(e^{d+1}) = d + 1$. So the values of $y = \ln x$ eventually becomes larger than any positive number. Hence the consequence that $\lim\limits_{c\to+\infty}\ln c = +\infty$. It follows that the area under the full graph of $f(x) = \frac{1}{x}$ with $x \geq 1$ is infinite. Because $\frac{1}{x}\sqrt{1 + f'(x)^2} > \frac{1}{x}$, the area under the graph of $y = \frac{1}{x}\sqrt{1 + f'(x)^2}$ with $x \geq 1$ is infinite also. It follows that

$$\lim_{c\to+\infty}\int_1^c \tfrac{1}{x}\sqrt{1 + f'(x)^2}\, dx = \lim_{c\to+\infty}\int_1^c \tfrac{1}{x}\sqrt{1 + \tfrac{1}{x^4}}\, dx = +\infty.$$

Therefore by part i, the area $\lim\limits_{c\to+\infty}S(c)$ of the surface obtained by rotating the graph of $f(x) = \frac{1}{x}$ with $x \geq 1$ one revolution around the x-axis is infinite as well.

10.16. Turn to http://www.integral-calculator.com/# and type `1/x^3sqrt(x^4+1)` into the box containing `e^(x/2)* sin(ax)` and click Go!. In addition to the antiderivative already mentioned, there is also the antiderivative

$$\tfrac{1}{2}\ln\left(\sqrt{x^4+1}+x^2\right)-\tfrac{\sqrt{x^4+1}}{2x^2}+C.$$

Checking this is a slog also.

10.17. The derivative $x'(t) = 15t^4 - 195t^2 + 540 = 15(t^4 - 13t^2 + 36)$ provides the velocity $v(t)$ of the point. By applying the quadratic formula to $(t^2)^2 - 13t^2 + 36 = 0$ we see that the velocity of the point is zero when $t^2 = \frac{13\pm\sqrt{13^2-4\cdot 36}}{2} = \frac{13\pm\sqrt{25}}{2} = 4$ and 9. Therefore the point stops at times t equal to $-3, -2, 2,$ and 3. Checking at the instants $t = -4, -2\frac{1}{2} = -\frac{5}{2}, 0, \frac{5}{2}$ and 4, we see that

$$v(\pm 4) = 15\big((\pm 4)^4 - 13(\pm 4)^2 + 36\big) = 15(16^2 - 13\cdot 16 + 36) = 15\big(16(16-13)+36\big) > 0,$$

$$v(\pm\tfrac{5}{2}) = 15(\tfrac{25^2}{4^2} - 13\cdot\tfrac{25}{4} + 36) = 15(\tfrac{25}{4}(\tfrac{25}{4} - 13) + 36) = 15(\tfrac{25}{4}(-\tfrac{27}{4}) + 36) < 0, \text{ and}$$

$$v(0) = 15(0 - 0 + 36) > 0.$$

It follows that over the time intervals $[-5,-3), (-3,-2), (-2,2), (2,3)$ and $(3,+\infty)$, the point moves to the right, then left, then right, then left, and finally right again, respectively.

The acceleration is equal to $a(t) = v'(t) = 15(4t^3 - 26t) = 60t(t^2 - \frac{13}{2})$ so that the force at time t is $F(t) = m\cdot 60t(t^2 - \frac{13}{2})$. Since $t(t^2 - \frac{13}{2}) = t\big(t + \sqrt{\frac{13}{2}}\big)\big(t - \sqrt{\frac{13}{2}}\big)$, it follows that the force is zero at times

$$t = -\sqrt{\tfrac{13}{2}} \approx -2.55,\ t = 0, \text{ and } t = \sqrt{\tfrac{13}{2}} \approx 2.55.$$

Notice that the force is negative for $t < -\sqrt{\frac{13}{2}}$, positive over $-\sqrt{\frac{13}{2}} < t < 0$, negative over $0 < t < \sqrt{\frac{13}{2}}$, and positive again for $\sqrt{\frac{13}{2}} < t$.

Let's turn to the motion of the point on the x-axis. Since

$$x(-5) = 3(-5)^5 - 65(-5)^3 + 540(-5) + 3950 = 25(-375 + 325 - 108 + 158) = 0 \text{ and}$$
$$v(-5) = 15[(-5)^4 - 13(-5)^2 + 36] = 5040$$

the point starts at time $t = -5$ at the origin, moving to the right with an initial speed of 5040. With the force pushing it to the left, the point stops for the first time at $t = -3$ at position

$$x(-3) = 3(-3)^5 - 65(-3)^3 + 540(-3) + 3950 = 3356.$$

The force, still acting to the left, continues to push the point to the left. But at time $t = -\sqrt{\frac{13}{2}} \approx -2.55$ the force begins acting to the right, stopping the point at time $t = -2$ at position

$$x(-2) = 3(-2)^5 - 65(-2)^3 + 540(-2) + 3950 = 3294.$$

The force acts to the right until $t = 0$ and pushes the point to the right until then. At $t = 0$ the force begins to act to the left, slowing the point until it stops at time $t = 2$ at position

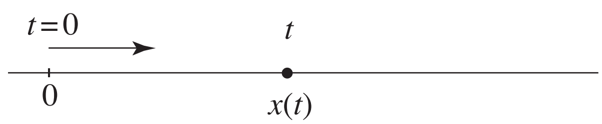
$$x(2) = 3(2)^5 - 65(2)^3 + 540(2) + 3950 = 4606.$$

The force still acting to the left, now moves the point to the left. At time $t = \sqrt{\frac{13}{2}} \approx -2.55$, the force changes direction for the last time and acts to the right. It slows the point to a stop at time $t = 3$ at position

$$x(3) = 3(3)^5 - 65(3)^3 + 540(3) + 3950 = 4544.$$

From that time on the force is positive. It drives the point to the right at ever increasing speeds.

10.18. The body is depicted below as moving on an x-axis. Its position at time $t = 0$ is the origin 0. Since $a(t) = a$ is constant and the velocity $v(t)$ is an antiderivative of $a(t)$, it follows that $v(t) = at + C$ with C a constant. Since $v(t) = 0$, $v(t) = at$. The distance that the body moves during time t is the coordinate $x(t)$ of its position. Since $x(t)$ is an



an antiderivative of $v(t) = at$, $x(t) = \frac{1}{2}at^2 + C$, again C a constant. Since $x(0) = 0$, $x(t) = \frac{1}{2}at^2$. The body's average speed from time $t = 0$ to t is $\frac{x(t)}{t} = \frac{1}{2}\frac{at^2}{t} = \frac{1}{2}at = \frac{1}{2}v'(t)$. Therefore its velocity $v'(t)$ at time t is equal to twice its average velocity over the time interval $[0, t]$.

10.19. At any time t, $y(t) = 1 - x(t)$. Therefore the point moves along the line $y = -x + 1$ through $(0, 1)$ with slope -1. Since $x(t) = t$ and $y(t) = 1 - t$, the point starts at time $t = 0$ from the position $(0, 1)$. Since its velocity in the x direction $x'(t) = 1$ is always positive, the point moves down the line and to the right. Its speed at any time $t \geq 0$ is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and hence constant.

10.20. At any time t, $y(t) = \frac{1}{5}x(t)^2$ so that the point moves along the parabola $y = \frac{1}{5}x^2$. It starts its trip at $(-10, \frac{1}{5}(-10)^2) = (-10, 20)$ and finishes at $(10, 20)$. Its speed at any time t is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1^2 + (\frac{2}{5}t)^2}$. It starts with a speed of $\sqrt{1^2 + (-\frac{2}{5}(10))^2} = \sqrt{1^2 + 4^2} = \sqrt{17}$, moves down the parabola attaining its minimum speed of 1 at time $t = 0$ at the bottom $(0, 0)$ of the parabola, and then moves up the parabola and reaches $(10, 20)$ with a speed of $\sqrt{17}$. If the mass m of the point is 1, then (using $F = ma$) the horizontal component of the force acting on it is $x''(t) = 0$ and the vertical component is $y''(t) = \frac{2}{5}$. So the force acting on the point is directed upward with magnitude $\frac{2}{5}$. It acts initially by slowing the downward motion of the point, stops it at $(0, 0)$, and then pushes the point upward.

10.21. Since $y(t) = \sqrt{t-1}$ needs to make sense, the motion ends at time $t = 1$. Since $x(t)^2 = t$ and $y(t)^2 = 1 - t$, it follows that $x(t)^2 + y(t)^2 = 1$. Therefore the point moves on the circle $x^2 + y^2 = 1$ starting at $(x(0), y(0)) = (0, 1)$. As t flows from $t = 0$ to $t = 1$, $x(t)$ increases to $x(1) = 1$ and $y(t)$ decreasing to $y(1) = 0$. So the point moves clockwise down to $(1, 0)$. Since $x'(t) = \frac{1}{2}t^{-\frac{1}{2}}$ and $y'(t) = -\frac{1}{2}(1-t)^{-\frac{1}{2}}$ its speed is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\frac{1}{4t} + \frac{1}{4(1-t)}} = \sqrt{\frac{(1-t)+t}{4t(1-t)}} = \frac{1}{2}\sqrt{\frac{1}{t(1-t)}}$. Notice that its initial and terminal speeds are both infinite.

10.22. Since $x(t)^3 = t$ and $y(t) = t = x(t)^3$, the point moves along the curve $y = x^3$. It starts at $(-10, -1000)$ and stops at $(10, 1000)$. Since $x'(t) = \frac{1}{3}t^{-\frac{2}{3}}$ and $y'(t) = 1$ the speed of the point is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\frac{1}{9t^{\frac{4}{3}}} + 1}$. Its acceleration in the x-direction is $x''(t) = -\frac{2}{9}t^{-\frac{5}{3}} = -\frac{2}{9t^{\frac{5}{3}}}$ and in the y-direction $y''(t) = 0$. Since its mass is equal to 1, the resultant of the forces acting on the point in the x- and y-directions is equal to $F(t) = -\frac{2}{9t^{\frac{5}{3}}}$ in the x-direction. The point starts at $(-10, -1000)$ with a speed of $\sqrt{\frac{1}{9(-10)^{\frac{4}{3}}} + 1} \approx 1.003$. For $t < 0$, the force $F(t)$ is positive and drives the point upward along the curve $y = x^3$. At $(0, 0)$ the point reaches infinite speed. Thereafter $t > 0$ and hence $F(t)$ is negative. So the point slows down and reaches $(10, 1000)$ with a speed of $\sqrt{\frac{1}{9(10)^{\frac{4}{3}}} + 1} \approx 1.003$ (the same as its initial speed).

10.23. We are given that $x'(t) = 2t$ and $y'(t) = t^3 + 4t$.

i. It follows that $x(t) = t^2 + C_1$ and $y(t) = \frac{1}{4}t^4 + 2t^2 + C_2$. Since the point is at $(-4, 3)$ at time $t = 0$, $C_1 = -4$ and $C_2 = 3$. Therefore $x(t) = t^2 - 4$ and $y(t) = \frac{1}{4}t^4 + 2t^2 + 3$. At time $t = 2$ the point is at $(0, 4 + 8 + 3) = (0, 15)$.

ii. Since $t^2 = x(t) + 4$, we see that

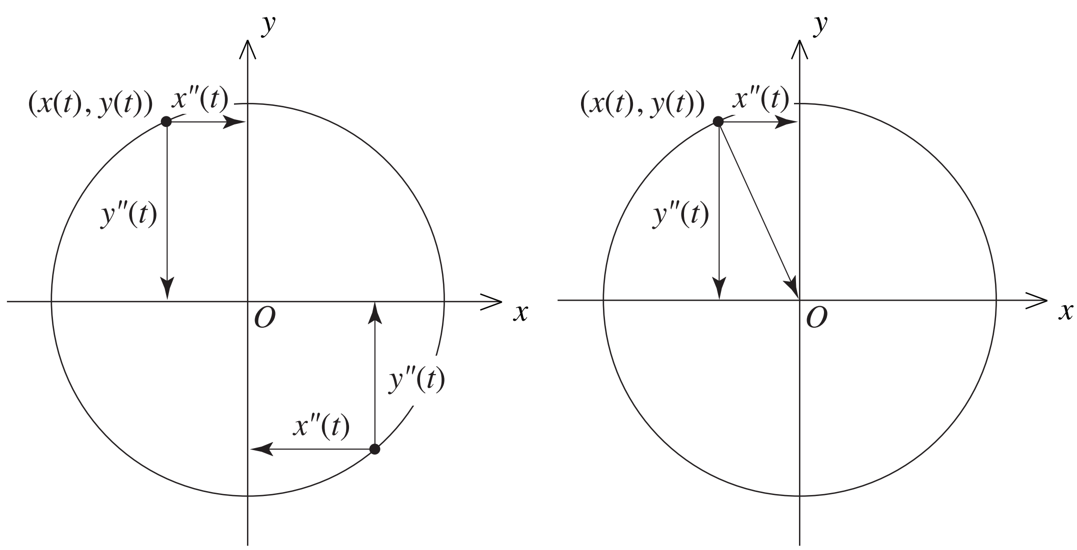
$$y(t) = \tfrac{1}{4}(x(t) + 4)^2 + 2(x(t) + 4) + 3 = \tfrac{1}{4}x(t)^2 + 4x(t) + 15\,.$$

So the point travels along the parabola $y = \frac{1}{4}x^2 + 4x + 15$.

10.24. The point starts its motion at $(x(0), y(0)) = (0, 0)$. Since $x'(t) = 1$ and $y'(t) = \cos t$, the velocity of the point is constant in the x-direction and varies between -1 and 1 in the y-direction. Since $x''(t) = 0$, the horizontal component of the force on the point is zero. It follows that the force on the point acts in the y-direction and is equal to $F(t) = y''(t) = -\sin t$. If the point is above the x-axis, then $y(t) = \sin t$ is positive. Since $F(t) = -\sin t$ is negative, the point is pushed downward. If the point is below the x-axis, then $y(t) = \sin t$ is negative. So $F(t) = -\sin t$ is positive and the point is pushed upward. In summary, the horizontal component of the point's speed is constant and the vertically acting force causes the up and down oscillation of the point as it moves along the curve $y = \sin x$.

10.25. Problem 7.37 already considered some of these concerns. Since $x(t)^2 + y(t)^2 = 1$, the point moves on the circle $x^2 + y^2 = 1$. When $t = 0$ the point is at $(1, 0)$. Since $y(t) = \sin t$ increases from 0 to 1 as t flows from $t = 0$ to $t = \frac{\pi}{2}$, the point moves from $(1, 0)$ to $(0, 1)$. As t varies from $t = \frac{\pi}{2}$ to $t = \frac{3\pi}{2}$, $y(t) = \sin t$ decreases from 1 to -1. It follows that the point continues its counterclockwise motion around the circle.

- **i.** Since $x'(t) = -\sin t$ and $y'(t) = \cos t$, $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$. So the speed of the point around the circle is constant and equal to 1.
- **ii.** We get $x''(t) = -\cos t = -x(t)$ and $y''(t) = -\sin t = -y(t)$. With the point P in any position (as in figure (a) below for instance) its coordinates $(x(t), y(t))$ determine the lengths of the components $x''(t)$ and $y''(t)$ of the force acting on the point. Since $x''(t)$ and $x(t)$ as well as $y''(t)$ and $y(t)$ have opposite signs, these components will always act in the direction of the x- and y-axes.
- **iii.** By the parallelogram law and figure (b), the force on P (the resultant of the two components $x''(t)$ and $y''(t)$) acts toward the origin O with magnitude 1.

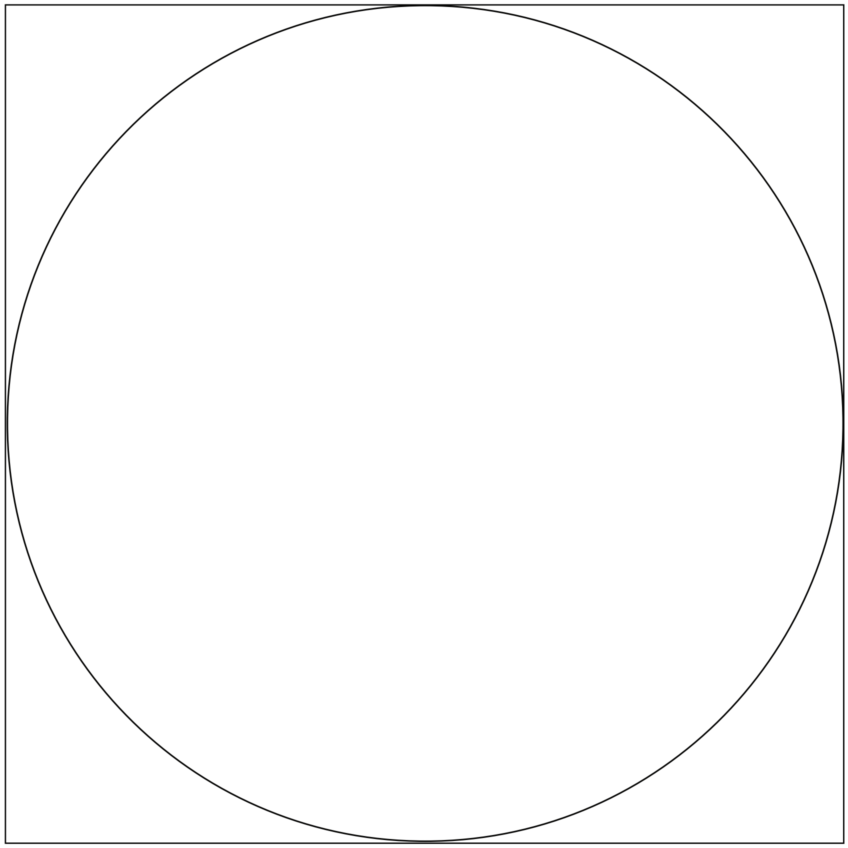


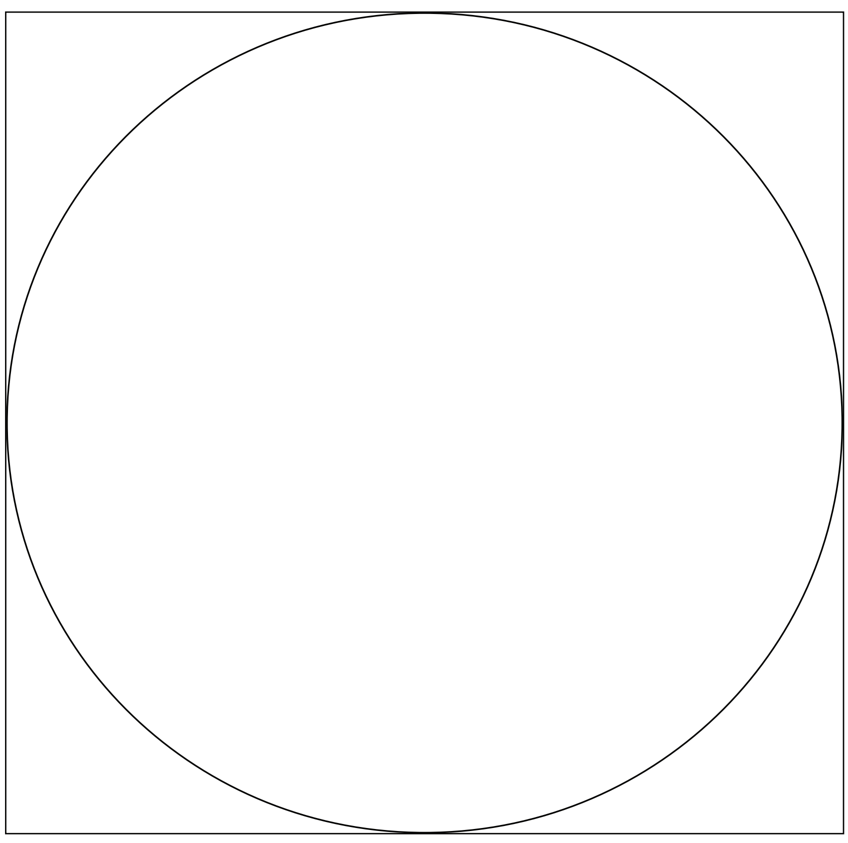
(a) (b)

10.26. The four equations are $at_1 + bt_1 = w, at_2 + bt_2 = 3w, bt_1 = 700$ and $bt_2 = w + 300$. There are several different ways to proceed. For example, since $t_1(a + b) = w$ and $t_2(a + b) = 3w$, it follows that $\frac{t_2}{t_1} = 3$ and $t_2 = 3t_1$. In addition to $bt_1 = 700$, we now know that $3bt_1 = w + 300$. So $w = 3(700) - 300 = 1800$ yards. Another way (as suggested in the parenthetical comment) is to divide $at_1 + bt_1 = w$ by $bt_1 = 700$ to get $\frac{a}{b} + 1 = \frac{w}{700}$ and $at_2 + bt_2 = 3w$ by $bt_2 = w + 300$ to get $\frac{a}{b} + 1 = \frac{3w}{w+300}$. So $\frac{w}{700} = \frac{3w}{w+300}$ and hence $w + 300 = 3(700)$. Again $w = 1800$ yards. So the prize goes to Marilyn.

10.27. Nothing to do here but chuckle.

10.28. A version of this problem was considered in Problem 3.24. Since 1 au about 150,000,000 km, the scale there was 6,000,000 km = 1 cm. The current 10,000,000 km = 3 cm is





larger. Table 10.1 informs us that Mercury's semimajor axis is approximately 57,909,000 km and that its eccentricity is close to $\varepsilon = 0.2056$. It follows that its semiminor axis is

$$b = \sqrt{a^2 - c^2} = \sqrt{a^2 - a^2\varepsilon^2} = a\sqrt{1-\varepsilon^2} = (57{,}909{,}000)\sqrt{1 - 0.2056^2} \approx 56{,}672{,}000 \text{ km}.$$

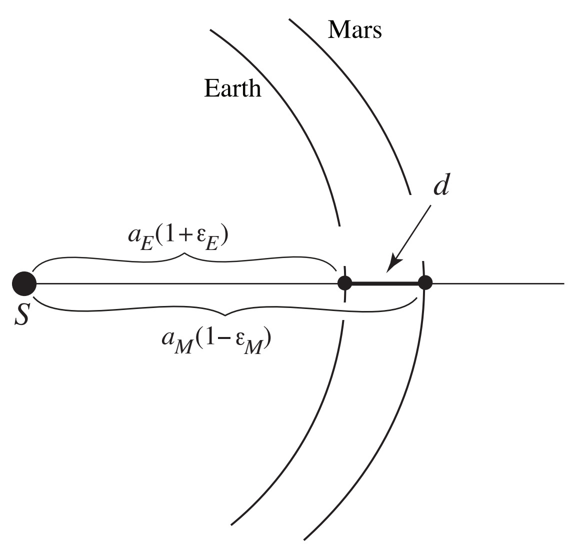
In terms of the given scale,

$$2a \approx 2(57{,}909{,}000) = 115{,}818{,}000 = 11.5818(10{,}000{,}000) = 11.5818(3) \approx 34.75 \text{ cm}$$

$$2b \approx 2(56{,}762{,}000) = 113{,}524{,}000 = 11.3524(10{,}000{,}000) = 11.3524(3) \approx 34.06 \text{ cm}.$$

Since $\frac{34.06}{34.75} \approx 0.98$, it follows that $2b$ is 98% of $2a$. The square of side $2a$ and the inscribed circle of radius a as well as the $2a \times 2b$ rectangle with the inscribed ellipse with semimajor axis a and semiminor axis b are both shown above. Both are drawn at a smaller scale than computed above so as to provide a side by side comparison. Which diagram corresponds to which situation?

10.29. Since Mars orbits outside Earth, the two planets are at the shortest possible distance from each other, let's call it d, at a moment when the following three things occur simultaneously: Mars is at perihelion so closest to the Sun, Earth is at aphelion so farthest from the Sun, and the Sun, Earth, and Mars are aligned. The figure below illustrates such an occurrence. It follows that $d = a_M(1 - \varepsilon_M) - a_E(1 + \varepsilon_E)$ where



a_M, a_E, ε_M, and ε_E are the semimajor axes of the orbits of Mars and Earth, respectively, and ε_M and ε_E are the eccentricities of their orbits. Putting in the data from Table 10.1, we get

$$d \approx 227{,}944{,}000(1 - 0.09339) - 149{,}598{,}000(1 + 0.01671) \approx 54{,}559{,}000 \text{ km}.$$

10.30. The relevant formula is $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$, where a is the semimajor axis of the orbit of either of the satellites, T is its period, and M is the mass of Eugenia. We'll use MKS, so that $G = 6.67384 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$. The fact that the two orbits are nearly circular means that we can take a to be the given radius in each case. So for the outer satellite, $a = 1{,}165{,}000\,\text{m}$ and, since its period is 4.7 days, $T \approx 4.7(24)(60)(60) \approx 406{,}000$ seconds. So we get that the mass of Eugenia is

$$M \approx \tfrac{4\pi^2 a^3}{GT^2} \approx \tfrac{4\pi^2(1{,}165{,}000^3)}{6.674\times10^{-11}\cdot 406{,}000^2} \approx 5.674 \times 10^{18}\,\text{kg}.$$

We can now use the formula $T^2 = \frac{4\pi^2 a^3}{GM}$ or $T = \sqrt{\frac{4\pi^2 a^3}{GM}}$ to get that

$$T = \sqrt{\tfrac{4\pi^2(611{,}000^3)}{(6.674\times10^{-11})(5.674\times10^{18})}} \approx 154{,}200 \text{ sec}$$

is the period of the inner satellite. This is equivalent to about $\frac{154{,}200}{(24)(60)(60)} \approx 1.8$ days.

10.31. From Table 10.1 we know that the period of Venus's orbit is approximately 0.6152 years or—since one year is about 365.2596 days (see Section 10.4.3)—approximately $0.6152(365.2596) \approx 224.71$ days. Each of the three computations proceeds from information about the angle α to an estimate of the elapsed time t. Gauss's formula connects α and β and Kepler's equation leads from β to t. The time t is the elapsed time from perihelion at $t = 0$. We'll use the rewritten versions $\tan\frac{\beta(t)}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\alpha(t)}{2}$ and $t = \frac{T}{2\pi}(\beta(t) - \varepsilon\sin\beta(t))$ of Gauss's and Kepler's equations with $T \approx 224.71$ days the period of Venus and $\varepsilon \approx 0.0068$ its eccentricity.

i. We'll compute the time it takes for α to sweep from 0° to 60° or $\frac{\pi}{3}$. Taking $\alpha(t) = \frac{\pi}{3}$, we get

$$\tan\tfrac{\beta(t)}{2} = \sqrt{\tfrac{1-\varepsilon}{1+\varepsilon}}\tan\tfrac{\pi}{6} \approx \sqrt{\tfrac{1-0.0068}{1+0.0068}}\cdot\tfrac{1}{\sqrt{3}} \approx 0.5734,$$

so that $\frac{\beta(t)}{2} \approx \tan^{-1}(0.5734) \approx 0.5206$ and $\beta(t) \approx 1.0413$ radians. So

$$t = \tfrac{T}{2\pi}(\beta(t) - \varepsilon\sin\beta(t)) \approx \tfrac{224.71}{2\pi}(1.0413 - 0.0068\sin(1.0413)) \approx 37.03 \text{ days}.$$

ii. Next we'll compute the time it takes for α to sweep from 60° to 120°. Since t is elapsed time from the perihelion position $\alpha = 0$, we'll compute the time for α to move from 0° to 120° and subtract the time calculated in part i. With $\alpha(t) = 120°$ or $\frac{2\pi}{3}$, we get

$$\tan\tfrac{\beta(t)}{2} = \sqrt{\tfrac{1-\varepsilon}{1+\varepsilon}}\tan\tfrac{\pi}{3} \approx \sqrt{\tfrac{1-0.0068}{1+0.0068}}\cdot\sqrt{3} \approx 1.7203,$$

and hence $\frac{\beta(t)}{2} \approx \tan^{-1}(1.7203) \approx 1.0442$ and $\beta(t) \approx 2.0885$ radians. So

$$t = \tfrac{T}{2\pi}(\beta(t) - \varepsilon\sin\beta(t)) \approx \tfrac{224.71}{2\pi}(2.0885 - 0.0068\sin(2.0885)) \approx 74.48 \text{ days}.$$

It follows that α takes approximately $74.48 - 37.03 = 37.45$ days to rotate from 60° to 120°.

iii. Since the segment from the Sun S to Venus sweeps out half the area of the ellipse in time $\frac{T}{2} \approx 112.35$ days, it follows that during this time α flows from $0°$ to $180°$ Therefore it takes $112.35 - 74.48 = 37.82$ days for α to rotate from $120°$ to $180°$.

10.32. In Section 10.4.2 we saw that the speed of a planet at any time t after it passes perihelion is given by $v(t) = \frac{2\pi a}{T}\sqrt{\frac{2a}{r(t)} - 1}$, where $r(t)$ is the distance from the planet to the Sun at that time, a is the semimajor axis of the orbit, and T is the period. The formula tells us that $v(t)$ is a maximum when $r(t)$ is at its minimum and that $v(t)$ is a minimum when $r(t)$ is at its maximum.

Since the minimal distance between S and the planet is its perihelion distance $a(1-\varepsilon)$, we see that

$$v_{\max} = \frac{2\pi a}{T}\sqrt{\frac{2a}{a(1-\varepsilon)} - 1} = \frac{2\pi a}{T}\sqrt{\frac{2-(1-\varepsilon)}{1-\varepsilon}} = \frac{2\pi a}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}.$$

Since the maximal distance between S and the planet is its aphelion distance $a(1+\varepsilon)$,

$$v_{\min} = \frac{2\pi a}{T}\sqrt{\frac{2a}{a(1+\varepsilon)} - 1} = \frac{2\pi a}{T}\sqrt{\frac{2-(1+\varepsilon)}{1+\varepsilon}} = \frac{2\pi a}{T}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}.$$

Inserting the orbital parameters for the Moon, $a = 384{,}400\,\text{km}$, $\varepsilon = 0.0549$, and $T = 27.3217\,\text{days}$ or $27.3217(24)(60)(60) \approx 2{,}360{,}600\,\text{seconds}$ into the formulas just derived, we get

$$v_{\max} = \frac{2\pi a}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \approx \frac{2\pi(3.844\times 10^5)}{2.3606\times 10^6}\sqrt{\frac{1+0.0549}{1-0.0549}} \approx 1.0810\,\text{km/s}$$

$$v_{\min} = \frac{2\pi a}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \approx \frac{2\pi(3.844\times 10^5)}{2.3606\times 10^6}\sqrt{\frac{1-0.0549}{1+0.0549}} \approx 0.9684\,\text{km/s}.$$

10.33. We'll compute with greater accuracy than the problem calls for. The fact that 1 au $= 149{,}597{,}892\,\text{km}$, the information in Table 10.1 tells us that the semimajor axis for Halley's orbit is $a = \frac{2{,}667{,}950{,}000}{149{,}597{,}892} \approx 17.8341\,\text{au}$. Taking the eccentricity of its orbit $\varepsilon = 0.9671$, we get $b = \sqrt{a^2 - c^2} = a\sqrt{1 - \frac{c^2}{a^2}} = a\sqrt{1-\varepsilon^2} = 4.5369\,\text{au}$ for the semiminor axis, $c = a\varepsilon = 17.2474\,\text{au}$, and $q = a - c = 17.8341 - 17.2474 = 0.5867\,\text{au}$ for the perihelion distance of Halley's orbit.

i. Since the center of Earth's orbit is $S = (c, 0)$ and its radius is 1, its orbit has equation is $(x-c)^2 + y^2 = 1$. The coordinates of the points of intersection H_1 and H_2 satisfy both $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $(x-c)^2 + y^2 = 1$, so that their common x-coordinate satisfies $\frac{x^2}{a^2} + \frac{1-(x-c)^2}{b^2} = 1$. So $b^2x^2 + a^2(1-(x-c)^2) = a^2b^2$ and hence $b^2x^2 + a^2 - a^2(x-c)^2 - a^2b^2 = 0$. After multiplying through by -1 and recalling that $a^2 = b^2 + c^2$, we get

$$0 = a^2(x-c)^2 - b^2x^2 - a^2 + a^2b^2 = a^2(x^2 - 2cx + c^2) - b^2x^2 - a^2 + a^2b^2$$

$$= (a^2 - b^2)x^2 - 2a^2cx + a^2c^2 + a^2b^2 - a^2 = c^2x^2 - 2a^2cx + (a^4 - a^2).$$

By the quadratic formula,

$$x = \frac{2a^2c \pm \sqrt{4a^4c^2 - 4c^2(a^4 - a^2)}}{2c^2} = \frac{2a^2c \pm \sqrt{4a^2c^2}}{2c^2} = \frac{2a^2c \pm 2ac}{2c^2} = \frac{a^2 \pm a}{c}.$$

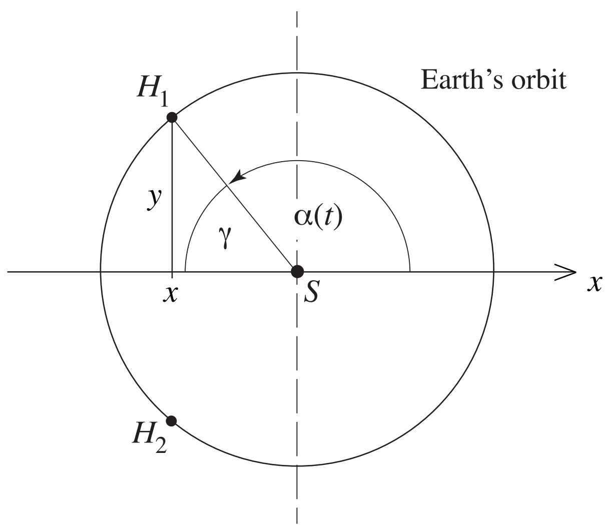
As already noted in the hint, the + option $x = \frac{a^2+a}{c}$ implies that $x = \frac{a^2+a}{c} = a(\frac{a}{c} + \frac{1}{c}) > a$ (since $\varepsilon = \frac{c}{a} < 1$). Figure 10.44 tells us that this cannot be since Halley's orbit intersects that x-axis at $x = a$. So the x-coordinate of both H_1 and H_2 is $\frac{a^2-a}{c}$.

To compute the y-coordinates of these two points, we can use the equation $(x-c)^2 + y^2 = 1$ of Earth's orbit. Since $y = \pm\sqrt{1-(x-c)^2}$, the y-coordinates of H_1 and H_2 are $y = \pm\sqrt{1-(\frac{a^2-a}{c} - c)^2}$. By simplifying, we get

$$1 - (\tfrac{a^2-a}{c} - c)^2 = 1 - \left(\tfrac{a^2-a-c^2}{c}\right)^2 = 1 - \tfrac{(b^2-a)^2}{c^2} = \tfrac{c^2 - b^4 + 2b^2a - a^2}{c^2}$$
$$= \tfrac{2b^2a - b^4 - b^2}{c^2} = \tfrac{b^2}{c^2}(2a - b^2 - 1).$$

So the y-coordinates of H_1 and H_2 are $\frac{b}{c}\sqrt{2a-b^2-1}$ and $-\frac{b}{c}\sqrt{2a-b^2-1}$, respectively. Putting in the numerical values (and rounding to 4 decimal place accuracy), we find that the x-coordinate of H_1 and H_2 is 17.4067 and the y-coordinates ± 0.9872, respectively.

ii. Since $r(t)$ is the distance from S to H_1 and H_1 lies on Earth's orbit $r(t) = 1$ (so that there is nothing to compute). Turning to the angle $\alpha(t)$ and the figure below, we see that $\sin\gamma = y = 0.9872$. So $\gamma \approx \sin^{-1}(0.9872) \approx 1.4106$. Therefore $\alpha(t) = \pi - \gamma \approx 3.1416 - 1.4106 \approx 1.7310$. Let's insert this information into the



equations

$$\tan\frac{\beta(t)}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\alpha(t)}{2} \quad \text{and} \quad t = \frac{T}{2\pi}(\beta(t) - \varepsilon\sin\beta(t))$$

of Gauss and Kepler for Halley's orbit, hence with $\varepsilon = 0.9671$ and $T = 75.32$ years. Doing so, we get $\tan\frac{\beta(t)}{2} = \sqrt{\frac{1-0.9671}{1+0.9671}}\tan\frac{1.7310}{2} \approx 0.1519$ so $\frac{\beta(t)}{2} \approx \tan^{-1}(0.1519) \approx 0.1507$ and $\beta(t) \approx 0.3015$. Putting this $\beta(t)$ into Kepler's equation, we get

$$t \approx \tfrac{75.32}{2\pi}(0.3015 - 0.9671\sin(0.3015)) \approx 0.1716 \text{ years.}$$

Since 1 year has approximately 365.2596 days (see Section 10.43), it takes Halley $t \approx 0.1716(365.2596) \approx 62.68$ days to travel from its perihelion position to the point H_1. By the symmetry of the situation, Halley takes about 125.36 days to travel from H_2 to H_1. This is the time that it will remain inside Earth's orbit.

iii. The inequality $|\beta(t) - \beta_i| < \varepsilon^i$ measures the difference between the ith approximation β_i and the value $\beta(t)$. What is called for is the smallest i for which $\varepsilon^i < 0.0002$. Making use of the fact that $\varepsilon < 0.9672$, we get by squaring 0.9672 again and again, that $\varepsilon^{256} < 0.000196$. So with $i = 256$ iterations, we can be sure that $|\beta(t) - \beta_i| < 0.0002$.

10.34. Let's do this for April 27th, 2017 at 9:32 am. The website tells us that in 2017 Earth's perihelion occurred on January 4th at 2:18 pm.

i. Starting with January 4th at 2:18 pm and counting 27 additional days for January, 28 for February, 31 for March, and taking 26 days from April, we get exactly 112 days between this perihelion and April 26th at 2:18 pm. Adding 12 hours brings us to 2:18 am on April 27th, and with 7 hours and 14 minutes more to the moment April 27th, 2017 at 9:32 am of interest. So the relevant t is 112 days, 19 hours, and 14 minutes, or $t = 112.8014$ days.

ii. The first thing to do is to is to compute the corresponding $\beta(t)$ of Kepler's equation using the successive approximation strategy of Step 3 of Section 10.4.1. We'll take the eccentricity of Earth's orbit to be $\varepsilon = 0.0167$ and the period of its orbit as $T = 365.2596$ days (from Section 10.4.3).

The first approximation is $\beta_1 = \frac{2\pi t}{T} = \frac{2\pi(112.8014)}{365.2596} \approx 1.9404$. Using

$$b_{i+1} = \tfrac{2\pi t}{T} + \varepsilon \sin\beta_i = \beta_1 + \varepsilon\sin\beta_i$$

again and again, we get

$$\begin{aligned}\beta_2 &\approx 1.9404 + 0.0167\sin(1.9404) \approx 1.9560,\\ \beta_3 &\approx 1.9404 + 0.0167\sin(1.9560) \approx 1.9559,\\ \beta_4 &\approx 1.9404 + 0.0167\sin(1.9559) \approx 1.9559.\end{aligned}$$

Since we have reached stable state, $\beta(t) \approx 1.9559$ radians. Taking the semimajor axis of Earth's orbit to be $a = 149{,}598{,}000$ km, we find (by Step 1 of Section 10.4.1) that the distance of the Earth from the Sun at time t is

$$r(t) = a(1 - \varepsilon\cos\beta(t)) \approx 149{,}598{,}000(1 - 0.0167\cos(1.9559)) \approx 150{,}536{,}000\,\text{km}.$$

Using Gauss's equation $\tan\frac{\alpha(t)}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\beta(t)}{2}$, we get

$$\tan\tfrac{\alpha(t)}{2} \approx \sqrt{\tfrac{1-0.0167}{1+0.0167}}\tan\tfrac{1.9559}{2} \approx 1.4598$$

and hence $\alpha(t) \approx 2\tan^{-1}(1.4598) \approx 1.9404$ or about $111.18°$.

iii. It remains to insert the above estimates into the speed formula

$$v(t) = \tfrac{2\pi a}{T}\sqrt{\tfrac{2a}{r(t)} - 1}.$$

Since Earth's speed is best given in km/sec, we'll convert $T = 365.2596$ days to $T = 365.2596(24)(60)(60) = 31{,}558{,}000$ seconds. So we get

$$v(t) \approx \tfrac{2\pi(149{,}598{,}000)}{31{,}558{,}000}\sqrt{\tfrac{2(149{,}598{,}000)}{150{,}536{,}000} - 1} \approx 29.5987 \text{ km/sec}.$$

10.35. Let's say a little more about these constants. The constant κ is specific to a given elliptical orbit. It is closely related to Kepler's second law. Let t be any time and suppose that the segment PS traces out the area A_t during time t. Then the ratio $\frac{A_t}{t}$ is a constant. It has the same value no matter what t is and no matter where in its orbit P moves. This is the constant κ. The fact that the area of an ellipse is $ab\pi$ where a and b are the semimajor and semiminor axis of the ellipse, tells us that $\kappa = \frac{ab\pi}{T}$ where T the period of the orbit. The latus rectum is also specific to a given orbit. Consider the line perpendicular to the focal axis of the ellipse through a focal point. It intersects the ellipse at two points. The latus rectum is the distance L between these two points. The standard equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ implies that $L = \frac{2b^2}{a}$. The constant K is given by Kepler's third law. For any P in elliptical orbit around the same S, the ratio $\frac{a^3}{T^2}$ is the same. This is the constant K. Finally, there is the constant G of universal gravitation. Newton's equality $K = \frac{a^3}{T^2} = \frac{GM}{4\pi^2}$ gives added precision to Kepler's third law. Since $\kappa^2 = \frac{a^2b^2\pi^2}{T^2}$ and $\frac{\pi^2}{2}LK = \frac{\pi^2}{2}\cdot\frac{2b^2}{a}\cdot\frac{a^3}{T^2}$, we get $\kappa^2 = \frac{\pi^2}{2}LK$. From $K = \frac{GM}{4\pi^2}$, we get that $\frac{\pi^2}{2}LK = \frac{\pi^2}{2}L\cdot\frac{GM}{4\pi^2} = \frac{GLM}{8}$. So $\kappa^2 = \frac{\pi^2}{2}LK = \frac{GLM}{8}$. Take square roots to get the rest.

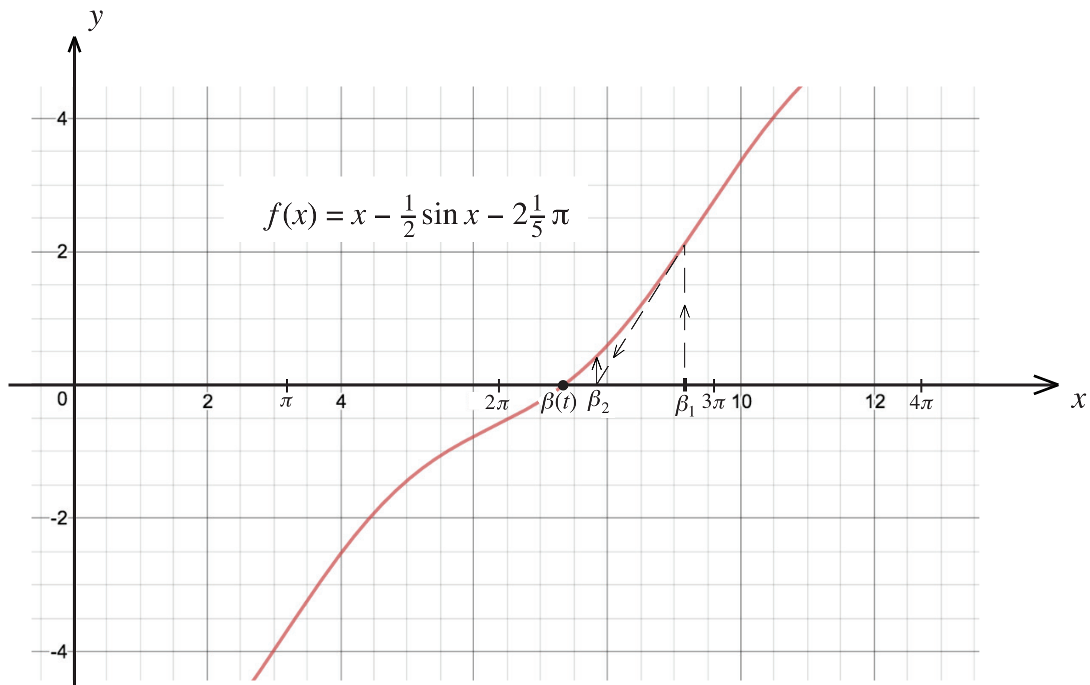
10.36. Suppose that P is at perihelion at time $t = 0$ and that after an elapsed time $t > 0$ the angle $\beta(t) = \pi$. Kepler's equation $\beta(t) - \varepsilon \sin\beta(t) = \frac{2\pi t}{T}$ tells us that $\pi - \varepsilon\cdot 0 = \frac{2\pi t}{T}$ and hence that $t = \frac{T}{2}$. By Figure 10.20, $\beta(t) = \pi$ corresponds to an apoapsis position. So periapsis to apoapsis takes time $\frac{T}{2}$. Since the time from one periapsis position to the next is the period T of the orbit, the time from apoapsis to periapsis is also $\frac{T}{2}$.

10.37. Figure 10.20 tells us that when $\beta(t) = \frac{\pi}{2}$, P has completed the first $\frac{1}{4}$th of its orbit. Inserting $\beta(t) = \frac{\pi}{2}$ into Kepler's equation $\beta(t) - \varepsilon\sin\beta(t) = \frac{2\pi t}{T}$ informs us that $\frac{\pi}{2} - \varepsilon = \frac{2\pi t}{T}$. Dividing through by 2π we get that $\frac{t}{T} = \frac{1}{4} - \frac{\varepsilon}{2\pi}$. So P completes the first $\frac{1}{4}$th of its orbit when $t = \frac{T}{4} - \frac{T\varepsilon}{2\pi}$. If the orbit is a flat ellipse with $\varepsilon \approx 1$, then $t \approx T(\frac{1}{4} - \frac{1}{2\pi}) \approx (0.0908)T \approx \frac{1}{11}T$.

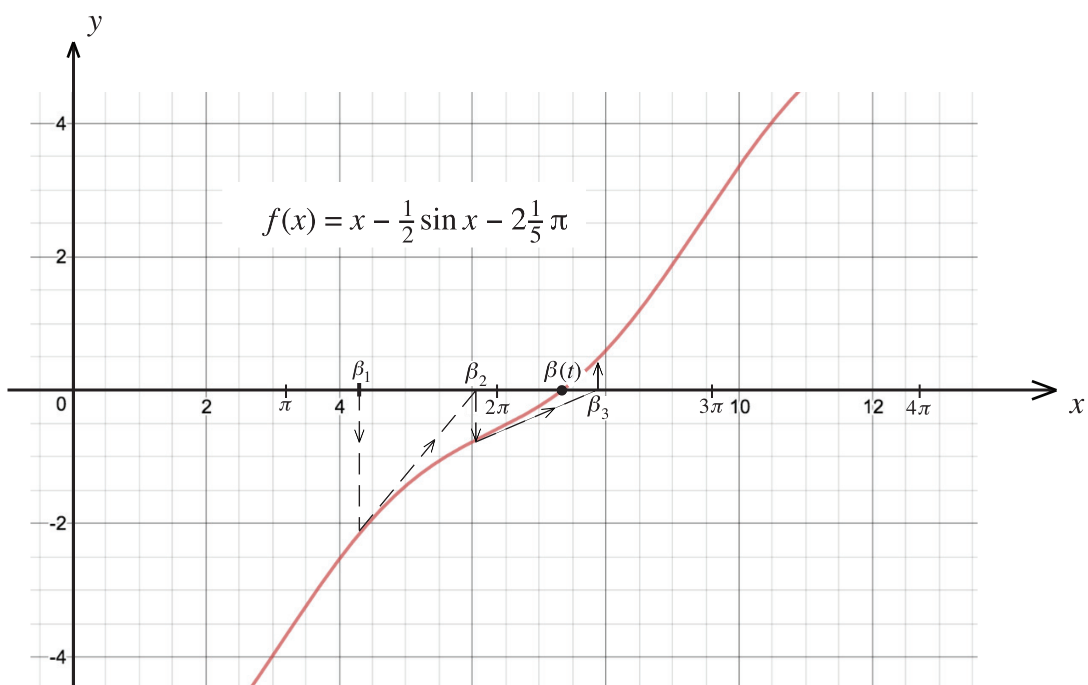
10.38. The problem has a mistake in its formulation. It is not the case in general that the approximation $\beta_{i+1} = \beta_i + \frac{\frac{2\pi t}{T} - (\beta_i - \varepsilon\sin\beta_i)}{1-\varepsilon\cos\beta_i}$ for the zero $\beta(t)$ of the function $f(x) = x - \varepsilon\sin x - \frac{2\pi t}{T}$ is better than β_i. Whether this is so or not depends on the graph of $f(x)$ between $\beta(t)$ and β_1. Since $f'(x) = 1 - \varepsilon\cos x$ and $1 \geq \cos x > \varepsilon\cos x$, $f'(x) > 0$ so that $y = f(x)$ is an increasing function. Because $f''(x) = \varepsilon\sin x$, the graph of $y = f(x)$

is concave up over the intervals $(0, \pi), (2\pi, 3\pi), (4\pi, 5\pi), \ldots$ and concave down over the intervals $(\pi, 2\pi), (3\pi, 4\pi), (5\pi, 6\pi), \ldots$. The assertions about the increasing nature of the function $f(x)$ and the concavity pattern do not depend on the particular ε.

We turn to examples with $\varepsilon = \frac{1}{2}$. Use https://www.desmos.com/calculator to obtain the graph of $y = x - \frac{1}{2}\sin x$. For any $t \geq 0$, the graph of $f(x) = x - \frac{1}{2}\sin x - \frac{2\pi t}{T}$ is gotten by shifting this graph downward by the distance $\frac{2\pi t}{T}$. Figures (a) and (b) consider the



(a)



(b)

specific case $t = 1.1T$. Since $\frac{2\pi t}{T} = 2\frac{1}{5}\pi$, this is the example $f(x) = x - \frac{1}{2}\sin x - 2\frac{1}{5}\pi$. The number $\beta(t)$ satisfies $\beta(t) - \frac{1}{2}\sin\beta(t) = 2\frac{1}{5}\pi$. So $x = \beta(t)$ is the solution of the equation $f(x) = x - \frac{1}{2}\sin x - 2\frac{1}{5}\pi = 0$. Newton's method generates the numbers

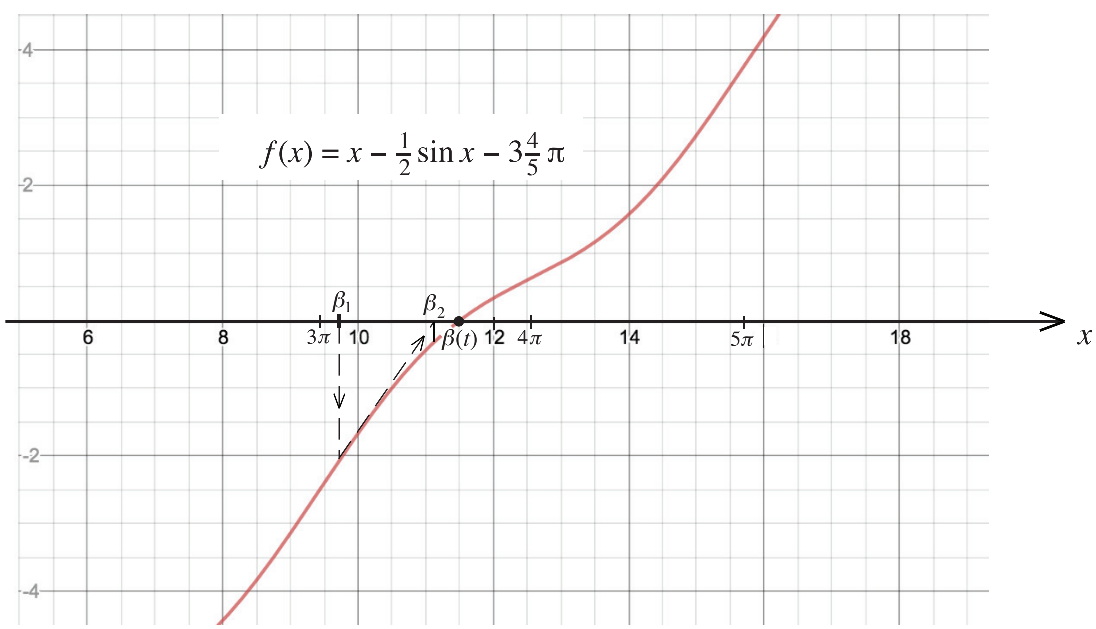
$$\beta_{i+1} = \beta_i - \tfrac{f(\beta_i)}{f'(\beta_i)} = \beta_i + \tfrac{2\frac{1}{5}\pi - (\beta_i - \frac{1}{2}\sin\beta_i)}{1 - \frac{1}{2}\cos\beta_i}.$$

If the method works as intended, then the sequence $\beta_1, \beta_2, \ldots$ consists of approximations β_i of $\beta(t)$ that become successively tighter and converge to $\beta(t)$.

Figure (a) depicts the graph of $f(x)$ and the solution $x = \beta(t)$ of $f(x) = 0$. It also depicts the initial guess β_1 at the solution $\beta(t)$ as well as the number β_2. Since $\beta(t) < \beta_1 < 3\pi$ and $y = f(x)$ is increasing and concave up over the interval $(2\pi, 3\pi)$, the sequence $\beta_1, \beta_2, \ldots$ converges to $\beta(t)$ as intended. Any subsequent β_{i+1} is a better approximation of $\beta(t)$ than β_i. This is illustrated in the figure and confirmed by the conclusion of Problem 7.91(i).

Figure (b) studies the same example. The only difference is the initial guess β_1. Since the third approximation β_3 satisfies $\beta(t) < \beta_3 < 3\pi$, this case reduces to the one depicted in Figure (a). So here too Newton's method converges to the solution $\beta(t)$.

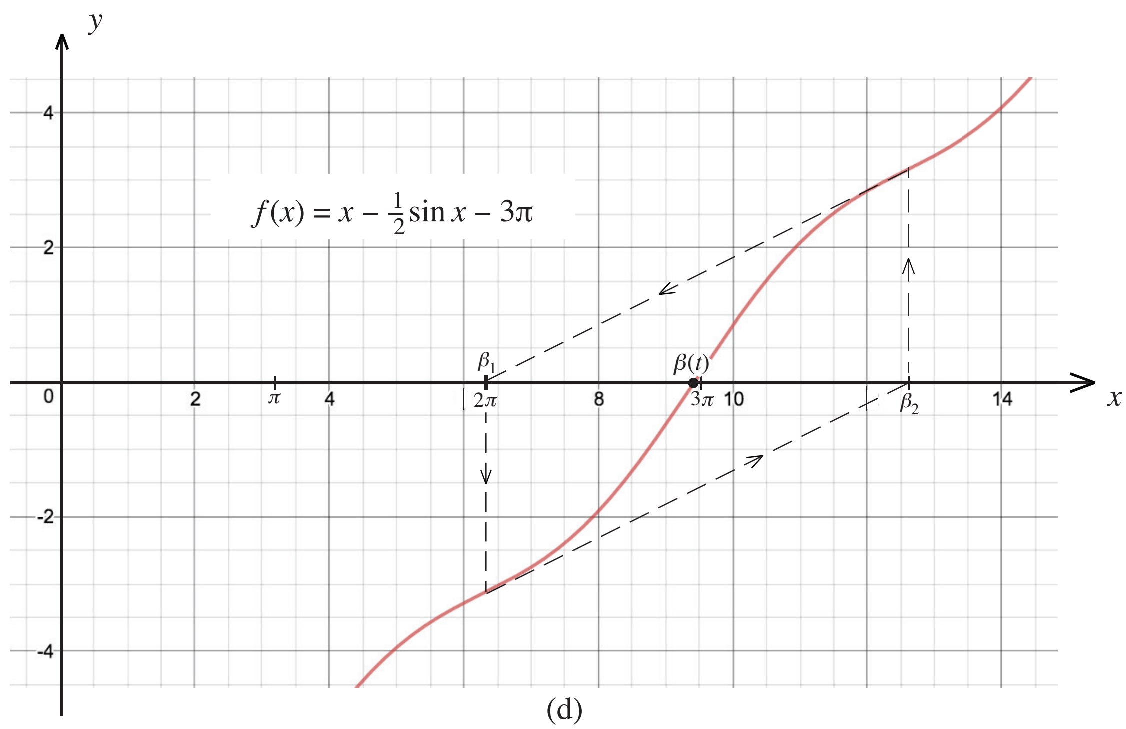
Figure (c) considers the example $f(x) = x - \frac{1}{2}\sin x - \frac{2\pi t}{T}$ with $t = 1.9T$, therefore with $\frac{2\pi t}{T} = 3.8\pi = 3\frac{4}{5}\pi$. The value $\beta(t)$ satisfies $\beta(t) - \frac{1}{2}\sin\beta(t) = 3\frac{4}{5}\pi$. It is therefore the solution of $f(x) = x - \frac{1}{2}\sin x - 3\frac{4}{5}\pi = 0$. Since $y = f(x)$ is increasing and concave down over the interval $(3\pi, 4\pi)$ and β_1 satisfies $3\pi < \beta_1 < \beta(t)$, Figure (c) tells us that here too the sequence $\beta_1, \beta_2, \ldots$ of Newton's method converges to $\beta(t)$. This is verified



(c)

in a more general context by the conclusion of Problem 7.91(ii).

We will see next that $f(x) = x - \frac{1}{2}\sin x - \frac{2\pi t}{T}$ with $t = 1.5T$ and hence with $\frac{2\pi t}{T} = 3\pi$ provides an example for which Newton's method does not converge. In this case, Newton's iteration formula is $\beta_{i+1} = \beta_i + \frac{3\pi - (\beta_i - \frac{1}{2}\sin\beta_i)}{1-\frac{1}{2}\cos\beta_i}$. Start with the guess $\beta_1 = 2\pi$. Then $\beta_2 = 2\pi + \frac{3\pi - 2\pi}{1-\frac{1}{2}} = 4\pi$. The next step $\beta_3 = 4\pi + \frac{3\pi-4\pi}{1-\frac{1}{2}} = 2\pi$. It follows that the β_i alternate between 2π and 4π. So the sequence that Newton's method gives



(d)

rise to is $2\pi, 4\pi, 2\pi, 4\pi \ldots$. So there is no convergence and no increasing accuracy. Figure (d) illustrates the infinite loop that is involved.

The site http://keisan.casio.com/exec/system/1244946907 carries out Newton's method for any differentiable function $f(x)$. The calculator

http://orbitsimulator.com/sheela/kepler.htm

computes the solutions of Kepler's equation for any ε but with restrictions on $\frac{2\pi t}{T}$.

10.39. The $\beta(t)$ we need is the solution of Kepler's equation $\beta(t) = \beta_1 + \varepsilon \sin\beta(t)$ with $\beta_1 = \frac{2\pi t}{T} = 2.924336$ and ε the eccentricity 0.016711 for Earth's orbit. So $x = \beta(t)$ is the solution of $f(x) = x - \varepsilon\sin x - \beta_1 = 0$.

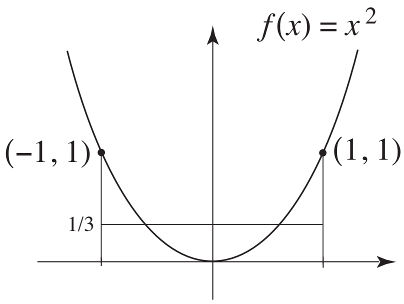
$$\beta_2 = \beta_1 - \frac{f(\beta_1)}{f'(\beta_1)} = \beta_1 + \frac{\beta_1 - (\beta_1 - \varepsilon\sin\beta_i)}{1-\varepsilon\cos\beta_1} = 2.924336 + \frac{0.016711\sin 2.924336}{1-0.016711\cos 2.924336} = 2.9278802.$$

After rounding to six decimal place accuracy, we get $\beta_2 = \beta(t) = 2.927880$.

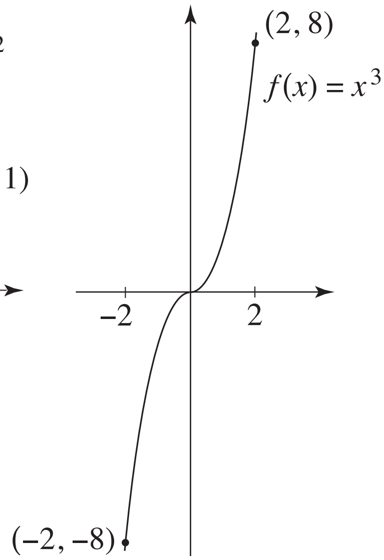
10.40. **i.** $\frac{1}{1-(-1)}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\cdot\frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{6}(1-(-1)) = \frac{1}{3}$.

ii. $\frac{1}{2-(-2)}\int_{-2}^{2} x^3\,dx = \frac{1}{4}\cdot\frac{1}{4}x^4\Big|_{-2}^{2} = \frac{1}{16}(16-(16)) = 0$.

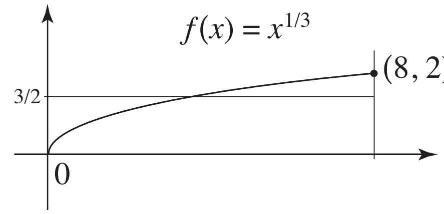
iii. $\frac{1}{8-0}\int_0^8 x^{\frac{1}{3}}\,dx = \frac{1}{8}\cdot\frac{3}{4}x^{\frac{4}{3}}\Big|_0^8 = \frac{3}{32}(8^{\frac{1}{3}})^4 = \frac{3}{32}2^4 = \frac{3}{2}.$



(i)



(ii)



(iii)

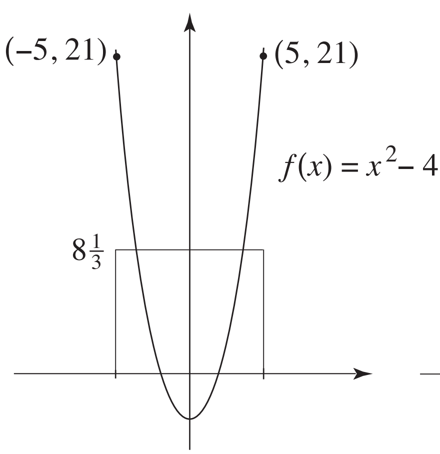
10.41. $\frac{1}{10-1}\int_1^{10} x^{-1}\,dx = \frac{1}{9}\cdot\ln x\Big|_1^{10} = \frac{1}{9}(\ln 10 - \ln 1) = \frac{1}{9}\ln 10 \approx \frac{1}{9}(2.3026) \approx 0.2558.$

10.42. **i.** $\frac{1}{5-(-5)}\int_{-5}^{5}(x^2-4)\,dx = \frac{1}{10}\left(\frac{1}{3}x^3 - 4x\right)\Big|_{-5}^{5} = \frac{1}{10}\left(\frac{1}{3}5^3 - 20) - (-\frac{1}{3}5^3 + 20\right) = \frac{2}{30}5^3 = \frac{25}{3}.$

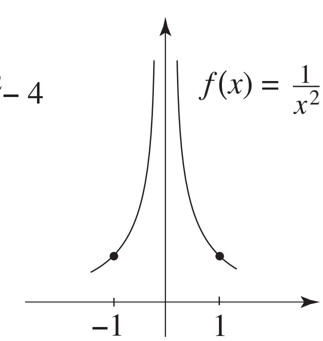
ii. $\frac{1}{1-(-1)}\int_{-1}^{1} x^{-2}\,dx = \frac{1}{2}(-x^{-1})\Big|_{-1}^{1} = \frac{1}{2}(-\frac{1}{1} - (\frac{1}{-1})) = 0$. So the average value of $f(x) = \frac{1}{x^2}$ over $[-1, 1]$ is zero? Even though the function is positive throughout?

iii. By applying Formula 16 of Section 9.11, we get

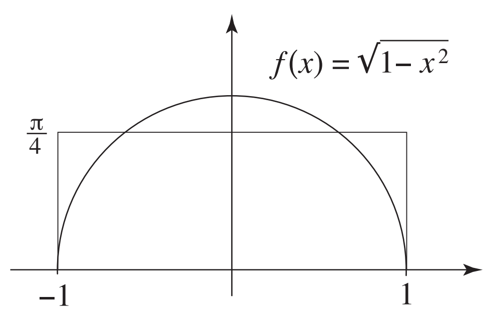
$\frac{1}{1-(-1)}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right)\Big|_{-1}^{1} = \frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot(-\frac{\pi}{2})\right) = \frac{\pi}{4}.$



(i)

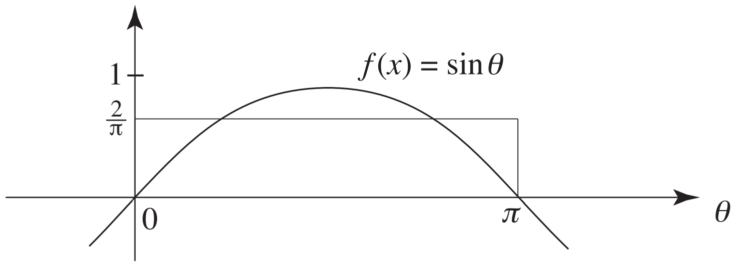


(ii)



(iii)

10.43. $\frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^\pi = \frac{1}{\pi}(-(-1)-(-1)) = \frac{2}{\pi}$.



The next item on the agenda is to consider the identity $\cos\alpha(t) = \frac{\cos\beta(t)-\varepsilon}{1-\varepsilon\cos\beta(t)}$ of Section 10.4.1 and to solve it for $\cos\beta(t)$. Since $\cos\alpha(t)(1-\varepsilon\cos\beta(t)) = \cos\beta(t)-\varepsilon$, we get $\cos\alpha(t) = \varepsilon\cos\alpha(t)\cos\beta(t)+\cos\beta(t)-\varepsilon$. So

$$\varepsilon + \cos\alpha(t) = (\varepsilon\cos\alpha(t)+1)\cos\beta(t)$$

and therefore $\cos\beta(t) = \frac{\varepsilon+\cos\alpha(t)}{1+\varepsilon\cos\alpha(t)}$. Combining this with $r(t) = a(1-\varepsilon\cos\beta(t))$, we get

$$r(t) = a(1-\varepsilon\tfrac{\varepsilon+\cos\alpha(t)}{1+\varepsilon\cos\alpha(t)}) = a(\tfrac{1+\varepsilon\cos\alpha(t)-\varepsilon(\varepsilon+\cos\alpha(t))}{1+\varepsilon\cos\alpha(t)}) = \tfrac{a(1-\varepsilon^2)}{1+\varepsilon\cos\alpha(t)}.$$

10.44. The average value of $r(\beta) = a(1-\varepsilon\cos\beta)$ over $[0,\pi]$ is

$$\tfrac{1}{\pi}\int_0^\pi r(\beta)\,d\beta = \tfrac{a}{\pi}\int_0^\pi (1-\varepsilon\cos\beta)\,d\beta = \tfrac{a}{\pi}(\beta-\varepsilon\sin\beta)\Big|_0^\pi = \tfrac{a}{\pi}(\pi-0) = a.$$

10.45. We'll provide the details for the argument that the average value of the function $r(\alpha)$ over $[0,\pi]$ is

$$\tfrac{1}{\pi}\int_0^\pi r(\alpha)\,d\alpha \;=\; \tfrac{a(1-\varepsilon^2)}{\pi}\int_0^\pi \tfrac{1}{1+\varepsilon\cos\alpha}\,d\alpha \;=\; b.$$

i. Let $u = \tan\frac{\alpha}{2}$. Solving the identity $u^2 = \tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$ for $\cos\alpha$, we get $u^2(1+\cos\alpha) = 1-\cos\alpha$ and hence $\cos\alpha(u^2+1) = 1-u^2$. So $\cos\alpha = \frac{1-u^2}{1+u^2}$. Turning to $\sin\alpha$, we get $\sin\alpha = \tan\frac{\alpha}{2}(1+\cos\alpha) = u(1+\frac{1-u^2}{1+u^2}) = u(\frac{1+u^2+1-u^2}{1+u^2}) = \frac{2u}{1+u^2}$. By Example 7.29, $\frac{du}{d\alpha} = \frac{1}{2}\sec^2\frac{\alpha}{2}$, so that $d\alpha = \frac{2\,du}{\sec^2\frac{\alpha}{2}} = 2\cos^2\frac{\alpha}{2}\,du$. By another conclusion of Problem 1.23, $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$ so that $d\alpha = (1+\frac{1-u^2}{1+u^2})\,du = \frac{1+u^2+1-u^2}{1+u^2}\,du = \frac{2}{1+u^2}\,du$. Therefore

$$\tfrac{1}{1+\varepsilon\cos\alpha}\,d\alpha = \tfrac{1}{1+\varepsilon\frac{1-u^2}{1+u^2}}\tfrac{2}{1+u^2}\,du = \tfrac{2}{1+u^2+\varepsilon(1-u^2)}\,du = \tfrac{2}{1+\varepsilon+(1-\varepsilon)u^2}\,du = \tfrac{2}{1+\varepsilon}\tfrac{1}{(1+\frac{1-\varepsilon}{1+\varepsilon}u^2)}\,du$$

and hence

$$\int \tfrac{1}{1+\varepsilon\cos\alpha}\,d\alpha = \tfrac{2}{1+\varepsilon}\int \tfrac{1}{1+\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}u\right)^2}\,du.$$

ii. We next let $z = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\,u$. So $dz = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\,du$ and hence $du = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,dz$. Changing variables to z and then using integral Formula (10) of Section 9.11 transforms the last integral into

$$\frac{2}{1+\varepsilon}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\int \frac{dz}{1+z^2} = \frac{2}{(\sqrt{1+\varepsilon})^2}\frac{\sqrt{1+\varepsilon}}{\sqrt{1-\varepsilon}}\int \frac{dz}{1+z^2} = \frac{2}{\sqrt{1-\varepsilon^2}}\int \frac{dz}{1+z^2} = \frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1} z + C$$
$$= \frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\big(\sqrt{\tfrac{1-\varepsilon}{1+\varepsilon}}\tan\tfrac{\alpha}{2}\big) + C.$$

iii. Note that the antiderivative that part (ii) provides is not defined for $\alpha = \pi$. This means that in order to complete the evaluation of the integral $\int_0^\pi \frac{1}{1+\varepsilon\cos\alpha}\, d\alpha$ we'll need to turn to the strategy of improper integrals (as this was illustrated in Section 10.3.1). Instead of integrating from 0 to π, we will integrate from 0 to φ with $0 \le \varphi < \pi$ and then—once this integration is completed—push φ to π:

$$\int_0^\pi \frac{1}{1+\varepsilon\cos\alpha}\, d\alpha = \lim_{\varphi\to\pi}\int_0^\varphi \frac{1}{1+\varepsilon\cos\alpha}\, d\alpha = \lim_{\varphi\to\pi}\frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\big(\sqrt{\tfrac{1-\varepsilon}{1+\varepsilon}}\,\tan\tfrac{\varphi}{2}\big).$$

The last part of the calculation requires two facts from Section 9.9.1 about the inverse tangent function $\tan^{-1} x$. First that it is differentiable and hence continuous (see the first part of Section 7.6). This allows us to move $\lim\limits_{\varphi\to\pi}$ past $\tan^{-1}$. The second fact is that $\lim\limits_{x\to+\infty}\tan^{-1} x = \frac{\pi}{2}$ (see Figure 9.33). It now follows that

$$\lim_{\varphi\to\pi}\frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\big(\sqrt{\tfrac{1-\varepsilon}{1+\varepsilon}}\,\tan\tfrac{\varphi}{2}\big) = \frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\big(\sqrt{\tfrac{1-\varepsilon}{1+\varepsilon}}\,\lim_{\varphi\to\pi}\tan\tfrac{\varphi}{2}\big) = \frac{\pi}{\sqrt{1-\varepsilon^2}}.$$

iv. We can therefore conclude that

$$\frac{1}{\pi}\int_0^\pi r(\alpha)\,d\alpha = \frac{a(1-\varepsilon^2)}{\pi}\int_0^\pi \frac{1}{1+\varepsilon\cos\alpha}\, d\alpha = \frac{a(1-\varepsilon^2)}{\pi}\frac{\pi}{\sqrt{1-\varepsilon^2}} = a\sqrt{1-\varepsilon^2} = b.$$

We turn finally to the average of r as function of time t with time t ranging over the interval $[0,T]$, where T is the period of the orbit.

10.46. This average is given by the integral $\frac{1}{T}\int_0^T r(t)\,dt$. The key to evaluating it is the equality $r(t) = a\big(1-\varepsilon\cos\beta(t)\big)$ and Kepler's equation $\beta(t) - \varepsilon\sin\beta(t) = \frac{2\pi t}{T}$.

i. Differentiating the equation $\beta(t) - \varepsilon\sin\beta(t) = \frac{2\pi t}{T}$ through with respect to t, we get $\beta'(t) - \varepsilon\cos\beta(t)\cdot\beta'(t) = \frac{2\pi}{T}$. So $\beta'(t)(1-\varepsilon\cos\beta(t)) = \frac{2\pi}{T}$. After using $r(t) = a\big(1-\varepsilon\cos\beta(t)\big)$, we get $\beta'(t)(\frac{r(t)}{a}) = \frac{2\pi}{T}$ and hence $\frac{T}{2a\pi}\beta'(t)r(t) = 1$.

ii. By inserting first $\frac{T}{2a\pi}\beta'(t)r(t)$ and then $r(t) = a\big(1-\varepsilon\cos\beta(t)\big)$ into the integrand,

$$\frac{1}{T}\int_0^T r(t)\,dt = \frac{1}{T}\int_0^T r(t)\big(\tfrac{T}{2a\pi}\beta'(t)r(t)\big)\,dt = \frac{1}{2a\pi}\int_0^T r(t)^2\beta'(t)\,dt$$
$$= \frac{a}{2\pi}\int_0^T \big(1-\varepsilon\cos\beta(t)\big)^2\beta'(t)\,dt.$$

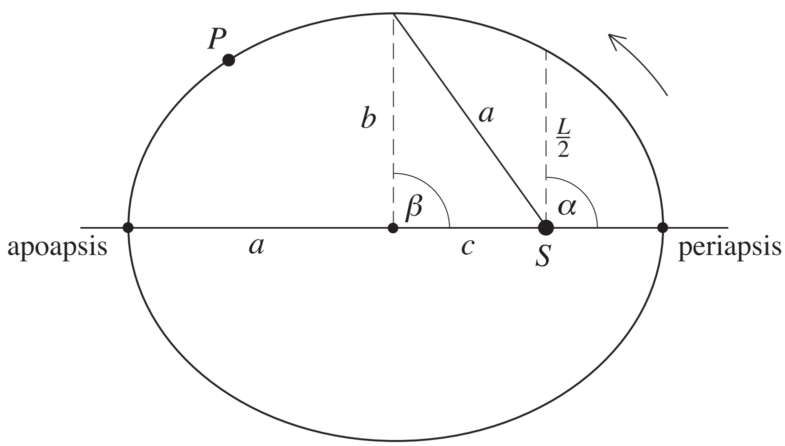
iii. With the substitutions $u = \beta(t)$ and $du = \beta'(t)\,dt$ this last integral becomes

$$\frac{a}{2\pi}\int_0^{2\pi}\big(1-\varepsilon\cos u\big)^2\,du = \frac{a}{2\pi}\left[\int_0^{2\pi} du - 2\varepsilon\int_0^{2\pi}\cos u\,du + \varepsilon^2\int_0^{2\pi}\cos^2 u\,du\right].$$

Evaluating these integrals (use the formula the formula $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$ from Problem 1.23 for the last one), we get

$$\tfrac{a}{2\pi}\big[2\pi - 2\varepsilon \cdot 0 + \tfrac{\varepsilon^2}{2}(2\pi + 0)\big] = a\big(1 + \tfrac{1}{2}\varepsilon^2\big).$$

10.47. We have determined the averages a, b and $a\big(1 + \frac{1}{2}\varepsilon^2\big)$ of the distance r between P and S as function of the angles β, α, as well as time t. Let's consider these averages in the context of the figure below. Notice that the distance r increases from $a(1-\varepsilon)$ at periapsis to $a(1+\varepsilon)$ at apoapsis. For $\beta = \frac{\pi}{2}$ and $\alpha = \frac{\pi}{2}$, the distances from P to S are $r = a$ and $r = \frac{L}{2} = \frac{b^2}{a} = \frac{b}{a}\, b \le b \le a$, respectively. Therefore as the angle ranges from 0 to $\frac{\pi}{2}$, r is larger as function of β than of α (unless $a = b$ and the orbit is a circle). It is not surprising therefore that the average a of r as a function of β is larger than the



average b of r as a function of α (again, unless $a = b$). The average $a(1 + \frac{1}{2}\varepsilon^2)$ of r as a function of time seems to reflect best what is going on. We know from Kepler's second law that the longer the segment PS is the more slowly it rotates. The example of the comet of Problem 10.37 is a good illustration of this. The first quarter of the orbit takes about 9% of its period, so that the second quarter takes about 41%. So the larger the distance r, the more time the comet "spends" at that distance. The larger average $a(1 + \frac{1}{2}\varepsilon^2)$ is a reflection of this.

10.48. The motion of P in the xy-plane of Figure 10.47a is determined by the equations $x(t) = \frac{5}{3}t^3$ and $y(t) = 4t^3 + 2$ with $t \ge 0$.

- **i.** For $t = 0$, the coordinates of P are $x(0) = 0$ and $y(0) = 2$. So the point starts at $(0, 2)$. For $t = 2$, $x(2) = \frac{40}{3}$ and $y(2) = 34$ so that P is at $(\frac{40}{3}, 34)$. Since $t^3 = \frac{3}{5}x(t)$, $y(t) = \frac{12}{5}x(t) + 2$, so that P moves on the line $y = \frac{12}{5}x + 2$.
- **ii.** Since $x'(t) = 5t^2$ and $y'(t) = 12t^2$, the speed of P is $v(t) = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{25t^4 + 144t^4} = \sqrt{169t^4} = 13t^2$.
- **iii.** Since the mass of P is equal to 1, the horizontal and vertical components of the force acting on it are $x''(t) = 10t$ and $y''(t) = 24t$, respectively. By the

parallelogram law and the Pythagorean theorem, the magnitude of the force on P is $\sqrt{x''(t)^2+y''(t)^2} = \sqrt{100t^2+576t^2} = \sqrt{676}\,t = 26t$. Since both $x''(t)$ and $y''(t)$ are positive, the horizontal and vertical components of the force act to the right and upward respectively. It follows that the force pushes P up along the line.

iv. The equation $W = \displaystyle\int_a^b f(x)\,dx = \tfrac{1}{2}mv(t_1)^2 - \tfrac{1}{2}mv(t_0)^2$ of Section 10.5.1 connects work and kinetic energy. Taking $m = 1, t_0 = 0, t_1 = 2$ in the case of the point P, we get $mv(t_0)^2 = 0$ and $mv(t_1)^2 = 52^2 = 2704$. Therefore $W = \frac{1}{2}(2704) = 1352$.

v. The formula $J = \displaystyle\int_{t_0}^{t_1} F(t)\,dt = mv(t_1) - mv(t_0)$ of Section 10.5.2 relates impulse and momentum. With $m = 1, v(t_0) = v(0) = 0$ and $v(t_1) = v(2) = 52$, we get $J = 52$.

10.49. In this problem the point P moves on the z-axis of Figure 10.47b with position given by $z(t) = \frac{13}{3}t^3$ at any time $t \geq 0$.

i. Over the time interval $[0,2]$, the point moves from $z(0) = 0$ to $z(2) = \frac{13}{3}(2^3) = \frac{104}{3}$.

ii. The speed of the point at any time t is $v(t) = z'(t) = 13t^2$.

iii. Since force = mass × acceleration, the force $F(t)$ on the point is $F(t) = z''(t) = 26t$. Since $z(t) = \frac{13}{3}t^3$ and hence $t = (\frac{3}{13}z(t))^{\frac{1}{3}}$, we get

$$26t = 26(\tfrac{3}{13})^{\frac{1}{3}}z(t)^{\frac{1}{3}} = (26^3\tfrac{3}{13})^{\frac{1}{3}}z(t)^{\frac{1}{3}} = (2^3 13^3\tfrac{3}{13})^{\frac{1}{3}}z(t)^{\frac{1}{3}} = 2(3\cdot 13^2)^{\frac{1}{3}}z(t)^{\frac{1}{3}},$$

so that the force on the point as a function of the position z is $f(z) = 2(3\cdot 13^2)^{\frac{1}{3}}z^{\frac{1}{3}}$.

iv. By the fundamental theorem of calculus,

$$W = \int_0^{\frac{104}{3}} f(z)\,dz = \tfrac{3}{2}(3\cdot 13^2)^{\frac{1}{3}}z^{\frac{4}{3}}\Big|_0^{\frac{104}{3}} = \tfrac{3}{2}(3\cdot 13^2)^{\frac{1}{3}}(\tfrac{104}{3})^{\frac{4}{3}} = \tfrac{3}{2}(3\cdot 13^2)^{\frac{1}{3}}(\tfrac{2^3 13}{3})^{\frac{4}{3}}$$

$$= \tfrac{3}{2}(3\cdot 13^2)^{\frac{1}{3}}\frac{2^4(13^4)^{\frac{1}{3}}}{(3^4)^{\frac{1}{3}}} = 2^3 3\big(\tfrac{13^6}{3^3}\big)^{\frac{1}{3}} = 2^3 13^2 = 1352.$$

v. The impulse J of the force from $t = 0$ to $t = 2$ is $J = \displaystyle\int_0^2 26t\,dt = 13t^2\Big|_0^2 = 52$.

10.50. Both points move on a line starting at time $t = 0$ from rest. For any time $t \geq 0$ they have the same velocity $v(t) = 13t^2$. Their accelerations are both $a(t) = 26t$. It follows that the motions of the two points on their respective lines are identical.

The next set of problems involve applications of Pappus's Theorems A and B.

10.51. The length of the semicircle of radius r is πr. Let c be the distance from the centroid C to the axis of revolution. The path described by C is a circle of radius c so that the distance C travels is $2\pi c$. Since (by Section 9.4) the surface area of a sphere of radius r is $4\pi r^2$, it follows by Pappus's Theorem A that $(\pi r)(2\pi c) = 4\pi r^2$. So $\pi c = 2r$ and hence $c = \frac{2}{\pi}r$.

10.52. The circumference of the circle of radius r is $2\pi r$ and the distance traveled by its centroid (the center C) is $2\pi R$. It follows from Pappus's Theorem A that the surface area of the perfect geometric donut is $4\pi^2 rR$.

10.53. The centroid of the slanting side of the triangle is its midpoint. It follows by using similar triangles that its distance from the axis of revolution is $\frac{r}{2}$. So the centroid travels a distance of $2\pi(\frac{r}{2}) = \pi r$ and by Pappus's Theorem A, the surface area of the cone is $s \cdot \pi r = \pi rs$.

10.54. Let c be the distance from the centroid C of the region to the axis of revolution. Since the area of the semicircle is $\frac{1}{2}\pi r^2$, Pappus's Theorem B tells us that $\frac{4}{3}\pi r^3 = (\frac{1}{2}\pi r^2)(2\pi c) = \pi^2 r^2 c$. It follows that $c = \frac{4}{3\pi} r$.

10.55. The area of the circle is πr^2 and its centroid (the center C) travels a distance $2\pi R$. By Pappus's Theorem B the volume of the perfect geometric donut that is generated is $(\pi r^2)(2\pi R) = 2\pi^2 r^2 R$. This is the same answer as that of Example 9.9.

10.56. By Archimedes's result in Section 2.5, the centroid C of the triangle with base r and side length s satisfies the condition that ratio of CB' over BB' is $\frac{1}{3}$. Since the right triangles with hypotenuse CB' and BB' are similar, the distance c from C to the axis of revolution satisfies $\frac{c}{r} = \frac{1}{3}$. So $c = \frac{r}{3}$. Since the area of the triangle with base r and height h is $\frac{1}{2}rh$, it follows that by Pappus's Theorem B that the volume of the cone with base radius r and height h is $(\frac{1}{2}rh)(2\pi c) = (\frac{1}{2}rh)(\frac{2}{3}\pi r) = \frac{1}{3}\pi r^2 h$.

The discussion of the connection between torque and center of mass illustrates the essential role that integral calculus plays when it comes to the explanation of the basic physics of force and rotation. The example as well as the two problems that follow study beams that are homogeneous, in other words beams of constant density $\rho(x) = c$. As the discussion and Figure 10.32a demonstrate, for such a beam $\bar{x} = \frac{b}{2}$, so that the center of mass $C = (\bar{x}, \bar{y})$ is the geometric center of the beam. In addition, the torque of the beam of is $W \cdot \frac{b}{2}$, where W is the weight of the beam. If the homogeneous beam is in horizontal position, then b is equal to the beam's length L and the torque generated is $W \cdot \frac{L}{2}$.

10.57. Let m_D be the mass placed at D. With the clown standing at the end of the beam to the left of B, the counterclockwise torques around B are the $784.8x$ N-m of the clown with $x = 3\,\mathrm{m}$ plus the 176.58 N-m of the beam. Clockwise on the right, we have the torque of 706.32 N-m of the beam plus the $(9.81m_D \cdot 6)$ N-m generated by the mass m_D at D. The mass m_D will balance the beam if $(784.8)(3) + 176.58 = 706.58 + (9.81)(6)m_D$. It follows that the smallest mass placed on the beam at the point D that will allow the clown to walk all the way to the end of the beam is $m_D = \frac{(784.8)(3)+176.58}{706.58+(9.81)(6)} \approx 3.31\,\mathrm{kg}$.

10.58. The weight of the clown is now $(9.81)(65) = 637.65\,\mathrm{N}$ so that with the clown a distance x from B, the counterclockwise torque around B to the left is $637.65x + 176.58$ N-m.

The mass of the 9 meters of beam to the right of B is $4 \cdot 9 = 36\,\text{kg}$ so that its weight is $(9.81)(36) = 353.16\,\text{N-m}$. So the 9 meters of the beam to right of B produce a clockwise torque of $(353.16)(\frac{9}{2}) = 1589.22\,\text{N-m}$ around B. The tipping point occurs when $637.65x + 176.58 = 1589.22$, or when $x = \frac{1589.22-176.58}{637.65} \approx 2.22$ meters.

Finally the discussion of the indexes of inertia of the disc and the sphere. The discussion proceeds in two steps: from the circle to the disc and from the disc to the sphere.

A disc is given by a circle and the region inside it. Consider a thin, homogeneous disc of radius r and mass m. Since the area of the disc is πr^2, the density of the disc is $\frac{m}{\pi r^2}$ per unit area.

10.59. **i.** The circular ring with inner radius x_i and thickness $\Delta x_i = x_{i+1} - x_i$ shown in Figure 10.54a is the difference between a disc of radius x_{i+1} and a smaller one of radius x_i. The area of this difference is

$$\begin{aligned}\pi x_{i+1}^2 - \pi x_i^2 &= \pi[(x_i + \Delta x_i)^2 - x_i^2] = \pi[x_i^2 + 2x_i\Delta x_i + (\Delta x_i)^2 - x_i^2] \\ &= \pi[2x_i\Delta x_i + (\Delta x_i)^2] \approx 2\pi x_i \Delta x_i.\end{aligned}$$

Since Δx_i is very small compared to x_i the term $(\Delta x_i)^2$ is much smaller yet (in the same way that $(0.0001)^2 = 0.0000001$ is much smaller than 0.0001) and can be ignored. It follows that the mass of the circular ring (area $\times$ density) is approximately $\frac{m}{\pi r^2}(2\pi x_i \Delta x_i) = \frac{2m}{r^2}x_i\Delta x_i$, so that its index of inertia (mass $\times$ radius^2) is approximately $\left(\frac{2m}{r^2}x_i\Delta x_i\right)x_i^2 = \frac{2m}{r^2}x_i^3\Delta x_i$.

ii. By summing up all these indices of inertia we get the approximation $\sum_{i}^{n-1} \frac{2m}{r^2}x_i^3\Delta x_i$ of the index of inertia of the homogeneous disc of radius r and mass m. By repeating this construction with partitions $\mathcal{P}$ of smaller and smaller norm, we see in the limit that the index of inertia of the homogeneous disc of radius r and mass m is

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{i=0}^{n-1} \frac{2m}{r^2}x_i^3\Delta x_i = \int_0^r \frac{2m}{r^2}x^3\,dx.$$

By the fundamental theorem of calculus this is equal to $\left(\frac{2m}{r^2}\frac{x^4}{4}\right)\Big|_0^r = \frac{1}{2}mr^2$.

Finally, there is the step from the disc to the sphere. The sphere in this context refers the solid that it encloses. Suppose that the sphere is homogeneous with radius r and mass m. Since the volume of the sphere is $\frac{4}{3}\pi r^3$, its density is $\frac{m}{\frac{4}{3}\pi r^3} = \frac{3}{4}\frac{m}{\pi r^3}$ per unit volume.

10.60. The ouline of the solution—this time using the more succinct dx notation introduced in Section 9.1—along with the many examples of the definite integral already discussed (in Sections 9.1 to 9.5 and above) should be sufficient to allow the industrious reader to fill in the details that the assertions in parts (i) and (ii) call for.