Solutions to Problems and Projects for Chapter 9

9.1. This is a warm up exercise using a basic result from Section 6.2.

i. $F(x) = 2\cdot\frac{1}{4}x^4 + C = \frac{1}{2}x^4 + C$

ii. $F(x) = 5\cdot\frac{3}{4}x^{\frac{4}{3}} + C = \frac{15}{4}x^{\frac{4}{3}} + C$

iii. $F(x) = 3\cdot\frac{1}{6}x^6 + \frac{1}{4}\cdot\frac{7}{9}x^{\frac{9}{7}} + C = \frac{1}{2}x^6 + \frac{7}{36}x^{\frac{9}{7}} + C$

iv. $F(x) = 6\cdot\frac{1}{5}x^5 - \frac{3}{8}\cdot\frac{3}{8}x^{\frac{8}{3}} + C = \frac{6}{5}x^5 - \frac{9}{64}x^{\frac{8}{3}} + C$

9.2. By antidifferentiating term by term,

$$\int (1 - 3x^2 + 2x^{-\frac{1}{2}})\,dx = x - x^3 + 4x^{\frac{1}{2}} + C$$

$$\int (-\tfrac{1}{3}x^{-2} + 8x^{\frac{1}{3}})\,dx = \tfrac{1}{3}x^{-1} + 6x^{\frac{4}{3}} + C$$

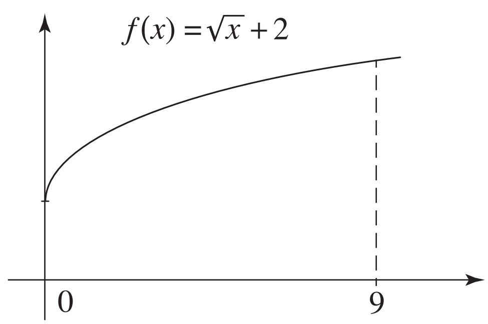
$$\int (-4 + 3x^{-2} + 7x^{\frac{1}{2}})\,dx = -4x - 3x^{-1} + \tfrac{14}{3}x^{\frac{3}{2}} + C$$

9.3. **i.** $\displaystyle\int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = 9 - 0 = 9$

ii. $\displaystyle\int_{-8}^{-2} x^{-2}\,dx = -x^{-1}\Big|_{-8}^{-2} = -\frac{1}{x}\Big|_{-8}^{-2} = -\frac{1}{-2} - (-\frac{1}{-8}) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$

iii. $\displaystyle\int_3^{12} x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_3^{12} = \frac{2}{3}\big(\sqrt{12}^3 - \sqrt{3}^3\big) = \frac{2}{3}\big(2^3\sqrt{3}^3 - \sqrt{3}^3\big) = \frac{2}{3}(7\cdot 3\sqrt{3}) = 14\sqrt{3}$

9.4. With the antiderivative $F(x) = \frac{2}{3}x^{\frac{3}{2}} + 2x$, we get $\displaystyle\int_0^9 (x^{\frac{1}{2}} + 2)\,dx = \frac{2}{3}x^{\frac{3}{2}} + 2x\Big|_0^9 = \frac{2}{3}(3^3) + 18 = 36$.



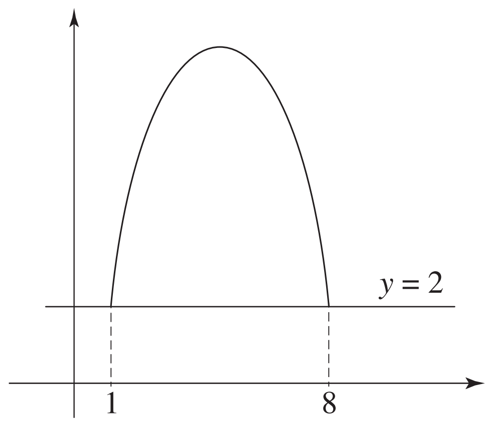
9.5. The graph of $y = x^3$ is sketched in Figure 5.11(c). By raising this graph by 1 unit, the graph of $f(x) = x^3 + 1$ is obtained. The area is given by $\displaystyle\int_0^4 (x^3 + 1)\,dx = \frac{1}{4}x^4 + x\Big|_0^4 = 64 + 4 = 68$.

9.6. At its highest point, the parabola $y = -3x^2 + 9x + 1$ has a horizontal tangent. At the x-coordinate of this point the derivative $\frac{dy}{dx} = -6x + 9$ is equal to 0, so that $x = \frac{9}{6} = \frac{3}{2}$. The y-coordinate of the corresponding point on the parabola is $y = -3(\frac{3}{2})^2 + 9(\frac{3}{2}) + 1 = \frac{27}{4} + 1 = \frac{31}{4}$. The x-coordinates of the points of intersection of the parabola with the x-axis satisfy $-3x^2 + 9x + 1 = 0$, so that $x = \frac{-9\pm\sqrt{9^2 - 4(-3)(1)}}{-6} = \frac{9\pm\sqrt{93}}{6} \approx -0.11$ or 3.11. We can now conclude that the area in question lies above the x-axis. It is equal to

$$\int_{\frac{9-\sqrt{93}}{6}}^{\frac{9+\sqrt{93}}{6}} (-3x^2+9x+1)\,dx = -x^3+\tfrac{9}{2}x^2+x\,\Big|_{\frac{9-\sqrt{93}}{6}}^{\frac{9+\sqrt{93}}{6}}$$

$$= -\left(\tfrac{9+\sqrt{93}}{6}\right)^3+\tfrac{9}{2}\left(\tfrac{9+\sqrt{93}}{6}\right)^2+\tfrac{9+\sqrt{93}}{6}-\left[-\left(\tfrac{9-\sqrt{93}}{6}\right)^3+\tfrac{9}{2}\left(\tfrac{9-\sqrt{93}}{6}\right)^2+\tfrac{9-\sqrt{93}}{6}\right] \approx 16.554+0.054 \approx 16.62.$$

9.7. Let's first get a sense for the shape and location of this area. At its highest point, the parabola $y=-x^2+9x-6$ has a horizontal tangent. Since $\frac{dy}{dx}=-2x+9$, this occurs for $x=\frac{9}{2}$. The corresponding y-coordinate is $-(\frac{9}{2})^2+9\cdot\frac{9}{2}-6=-\frac{81}{4}+\frac{81}{2}-6=\frac{81-24}{4}=\frac{57}{4}$. The parabola

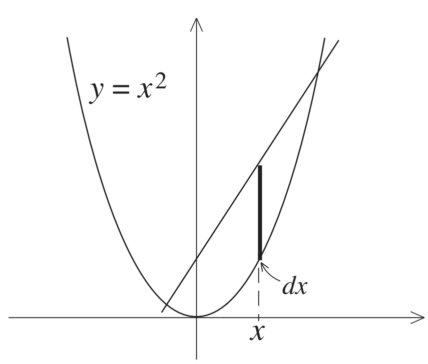


intersects the line $y=2$ for x satisfying $-x^2+9x-6=2$ or $x^2-9x+8=0$ and hence for $x=\frac{9\pm\sqrt{9^2-4(1)(8)}}{2}=\frac{9\pm\sqrt{49}}{2}=1$ or 8. The required area is given by

$$\int_1^8(-x^2+9x-6)\,dx-2(8-1)=-\tfrac{1}{3}x^3+\tfrac{9}{2}x^2-6x\,\Big|_1^8-14=-\tfrac{1}{3}8^3+\tfrac{9}{2}8^2-6\cdot8-(-\tfrac{1}{3}+\tfrac{9}{2}-6)-14$$

$$=8(-\tfrac{128}{6}+\tfrac{216}{6}-\tfrac{36}{6})-(-\tfrac{2}{6}+\tfrac{27}{6}-\tfrac{36}{6})-14=8\cdot\tfrac{52}{6}+\tfrac{11}{6}-\tfrac{84}{6}=\tfrac{343}{6}=57\tfrac{1}{6}.$$

9.8. The first thing we need to do is to find the x-coordinates of the points of intersection of the parabola and the line. They are obtained by solving $x^2-5x-8=0$ for x. By the quadratic formula $x=\frac{5\pm\sqrt{25-4(1)(-8)}}{2}=\frac{5\pm\sqrt{57}}{2}\approx-1.275$ or 6.275. We'll use the "strip" strategy of Section 9.2 to compute volumes. So let x be any coordinate with $\frac{5-\sqrt{57}}{2}\le x\le\frac{5+\sqrt{57}}{2}$ and



place a strip of thickness dx as shown in the figure. The upper end of the strip has y-coordinate $y=5x+8$ and the lower end of the strip has coordinate $y=x^2$. So the length of the strip is

$5x+8-x^2$. Since its width is dx, its area is $(5x+8-x^2)\,dx$. The sum of all the areas of all the strips with x varying from $x=\frac{5-\sqrt{57}}{2}$ to $x=\frac{5+\sqrt{57}}{2}$ adds up to the area between the line $y=5x+8$ and the parabola $y=x^2$. On the other hand, this sum is the integral

$$\int_{\frac{5-\sqrt{57}}{2}}^{\frac{5+\sqrt{57}}{2}} (5x+8-x^2)\,dx = \tfrac{5}{2}x^2+8x-\tfrac{1}{3}x^3\,\Big|_{\frac{5-\sqrt{57}}{2}}^{\frac{5+\sqrt{57}}{2}}$$
$$= \tfrac{5}{2}(\tfrac{5+\sqrt{57}}{2})^2+8(\tfrac{5+\sqrt{57}}{2})-\tfrac{1}{3}(\tfrac{5+\sqrt{57}}{2})^3-\big[\tfrac{5}{2}(\tfrac{5-\sqrt{57}}{2})^2+8(\tfrac{5-\sqrt{57}}{2})-\tfrac{1}{3}(\tfrac{5-\sqrt{57}}{2})^3\big]$$
$$\approx 66.278+5.445=71.723.$$

9.9. Solving $x^2+y^2=4$ for y gives $y=\pm\sqrt{4-x^2}$. The graph of $y=\sqrt{4-x^2}$ is the upper half of the circle and the graph of $y=-\sqrt{4-x^2}$ is the lower half. Notice that the definite integral

$$\int_0^2 \sqrt{4-x^2}\;dx$$

represents the area of one quarter of the circle of radius 2. So $\displaystyle\int_0^2 \sqrt{4-x^2}\;dx = \tfrac{1}{4}\pi(2)^2=\pi$. The same argument with the circle $x^2+y^2=a^2$ of radius a tells us that $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx=\tfrac{1}{4}\pi a^2$.

9.10. Since $\displaystyle\int_0^5 \tfrac{5}{2}\sqrt{5^2-x^2}\;dx = \tfrac{5}{2}\int_0^5 \sqrt{5^2-x^2}\;dx$, we get applying the conclusion of Problem 9.9 that $\displaystyle\int_0^5 \tfrac{5}{2}\sqrt{5^2-x^2}\;dx=\tfrac{5}{2}\cdot\tfrac{1}{4}\pi 5^2=\tfrac{125}{8}\pi$.

9.11. A look at Figure 4.24 tells us that this area is $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=\sin x\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=1-(-1)=2$. Not surprisingly, this is equal to the area under one loop of the sine curve. See Section 9.13.

9.12. **i.** $\displaystyle\int_2^8 \tfrac{1}{x}\,dx=\ln 8-\ln 2=\ln\tfrac{8}{2}=\ln 2^2=2\ln 2\approx 1.386.$

ii. $\displaystyle\int_{-1}^4 e^x\,dx=e^x\big|_{-1}^4=e^4-\tfrac{1}{e}\approx 54.23.$

9.13. $\displaystyle\int_{\ln 2}^{\ln 5} e^x\,dx=e^x\big|_{\ln 2}^{\ln 5}=e^{\ln 5}-e^{\ln 2}=5-2=3.$

9.14. $\displaystyle\int_3^7 \tfrac{1}{x}\,dx=\ln x\,\big|_3^7=\ln 7-\ln 3=\ln\tfrac{7}{3}\approx 0.847.$

The next four problems illustrate what has been described in Section 9.1 in the abstract. Similar computations were already carried out in Section 5.6.

9.15. **i.** $$\sum_{i=0}^{3} g(x_i)\Delta x_i = g(0)0.3+g(0.3)0.2+g(0.5)0.2+g(0.7)0.3$$
$$=(4-0^2)0.3+(4-0.3^2)0.2+(4-0.5^2)0.2+(4-0.7^2)0.3=3.785.$$

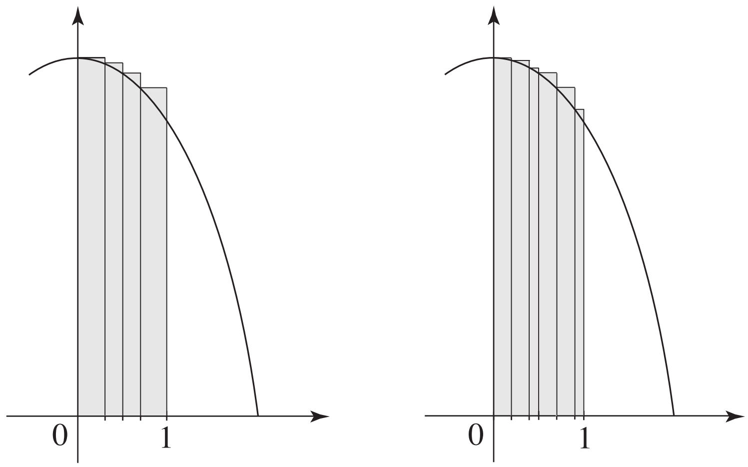
The corresponding graph is sketched in (a) below.

ii. $\sum_{i=0}^{5} g(x_i)\Delta x_i = g(0)0.2 + g(0.2)0.2 + g(0.4)0.1 + g(0.5)0.2 + g(0.7)0.2 + g(0.9)0.1$

$= (4-0^2)0.2+(4-0.2^2)0.2+(4-0.4^2)0.1+(4-0.5^2)0.2+(4-0.7^2)0.2+(4-0.9^2)0.1$

$= 3.747.$

The graph corresponding to this situation is sketched in (b) below.



(a) (b)

iii. The partitions of both (i) and (ii) are coarse relative to the length 1 of the given interval and therefore each of the approximations of the area under the graph are rough. Using the fundamental theorem of calculus we see that the exact value of the area under the graph is $\int_0^1 (4-x^2)\,dx = 4x - \frac{1}{3}x^3\big|_0^1 = 4 - \frac{1}{3} = 3\frac{2}{3} \approx 3.67$. The second partition is somewhat tighter and provides the better approximation.

9.16. **i.** Since $\frac{1}{3} < \frac{1}{2} < 1 < 2 < 2\frac{1}{3} < 3$ (note the correction $x_4 = 2\frac{1}{3}$), we get

$\Delta x_0 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}, \Delta x_1 = 1-\frac{1}{2} = \frac{1}{2}, \Delta x_2 = 2-1 = 1, \Delta x_3 = 2\frac{1}{3}-2 = \frac{1}{3}, \Delta x_4 = 3-2\frac{1}{3} = \frac{2}{3},$

so that

$$\sum_{i=0}^{4} f(x_i)\Delta x_i = f(\tfrac{1}{3})\tfrac{1}{6} + f(\tfrac{1}{2})\tfrac{1}{2} + f(1)1 + f(2)\tfrac{1}{3} + f(2\tfrac{1}{3})\tfrac{2}{3}.$$

$$= 3\cdot\tfrac{1}{6} + 2\cdot\tfrac{1}{2} + 1\cdot 1 + \tfrac{1}{2}\cdot\tfrac{1}{3} + \tfrac{3}{7}\cdot\tfrac{2}{3} = 2\tfrac{20}{21} \approx 2.952.$$

ii. Now $\frac{1}{3} < \frac{1}{2} < \frac{2}{3} < 1 < \frac{3}{2} < 2 < 2\frac{1}{3} < 3$ and hence

$$\Delta x_0 = \tfrac{1}{2}-\tfrac{1}{3} = \tfrac{1}{6}, \Delta x_1 = \tfrac{2}{3}-\tfrac{1}{2} = \tfrac{1}{6}, \Delta x_2 = 1-\tfrac{2}{3} = \tfrac{1}{3}, \Delta x_3 = \tfrac{3}{2}-1 = \tfrac{1}{2},$$

$$\Delta x_4 = 2-\tfrac{3}{2} = \tfrac{1}{2}, \Delta x_5 = 2\tfrac{1}{3}-2 = \tfrac{1}{3}, \Delta x_6 = 3-2\tfrac{1}{3} = \tfrac{2}{3},$$

$$\sum_{i=0}^{6} f(x_i)\Delta x_i = f(\tfrac{1}{3})\tfrac{1}{6} + f(\tfrac{1}{2})\tfrac{1}{6} + f(\tfrac{2}{3})\tfrac{1}{3} + f(1)\tfrac{1}{2} + f(\tfrac{3}{2})\tfrac{1}{2} + f(2)\tfrac{1}{3} + f(2\tfrac{1}{3})\tfrac{2}{3}.$$

$$= 3\cdot\tfrac{1}{6} + 2\cdot\tfrac{1}{6} + \tfrac{3}{2}\cdot\tfrac{1}{3} + 1\cdot\tfrac{1}{2} + \tfrac{2}{3}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{3} + \tfrac{3}{7}\cdot\tfrac{2}{3} = 2\tfrac{13}{21} \approx 2.619.$$

iii. The fact that $\frac{d}{dx}\ln x = \frac{1}{x}$ (see Section 7.11) and the fundamental theorem of calculus together imply that $\int_{\frac{1}{3}}^{3} \frac{1}{x}\,dx = \ln x\,\big|_{\frac{1}{3}}^{3} = \ln 3 - \ln 3^{-1} = 2\ln 3 \approx 2.197.$

9.17. Proceeding from left to right and adding as before, we get

$$\tfrac{1}{2}(0.3)+\tfrac{1}{2.3}(0.2)+\tfrac{1}{2.5}(0.4)+\tfrac{1}{2.9}(0.5)+\tfrac{1}{3.4}(0.2)+\tfrac{1}{3.6}(0.4)$$
$$=0.150+0.087+0.160+0.172+0.059+0.111=0.739.$$

This is a rough approximation of the area under the graph of $y=\frac{1}{x}$ over the interval from 2 to 4 on the x-axis. The precise value of this area is $\int_2^4 \frac{1}{x}\,dx=\ln x\,\big|_2^4=\ln 4-\ln 2=\ln 2\approx 0.693$. This computation involves facts from Section 7.11.

9.18. Proceeding from left to right and adding, we get

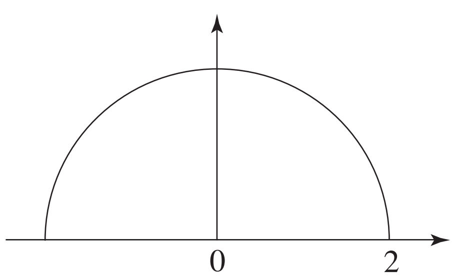
$$0\cdot\tfrac{1}{9}+\sqrt{\tfrac{1}{9}}\cdot\tfrac{1}{9}+\sqrt{\tfrac{2}{9}}\cdot\tfrac{2}{9}+\sqrt{\tfrac{4}{9}}\cdot\tfrac{3}{9}+\sqrt{\tfrac{7}{9}}\cdot\tfrac{4}{9}+\sqrt{\tfrac{11}{9}}\cdot\tfrac{5}{9}+\sqrt{\tfrac{16}{9}}\cdot\tfrac{2}{9}$$
$$=\left(\tfrac{1}{3}+\tfrac{2\sqrt{2}}{3}+2+\tfrac{4\sqrt{7}}{3}+\tfrac{5\sqrt{11}}{3}+\tfrac{8}{3}\right)\tfrac{1}{9}\approx 1.67.$$

This is a rough approximation of the area under the graph of $y=\sqrt{x}$ from 0 to 2. The precise value is $\int_0^2\sqrt{x}\;dx=\int_0^2 x^{\frac{1}{2}}dx=\frac{2}{3}x^{\frac{3}{2}}\big|_0^2\;=\frac{2}{3}(\sqrt{2})^3\approx 1.89$.

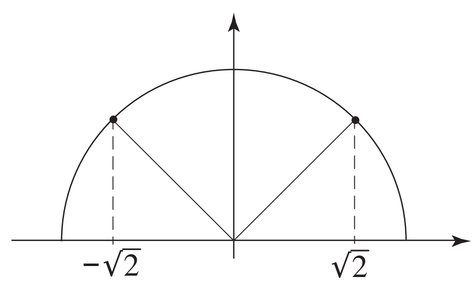
9.19. The relevant formula is $\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\big[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\big]+C$ with $a=2$. Applying it twice, we get

$$\int_0^2\sqrt{4-x^2}\,dx=\tfrac{1}{2}\big[x\sqrt{4-x^2}+4\sin^{-1}\tfrac{x}{2}\big]\big|_0^2=\tfrac{1}{2}(0+4\cdot\tfrac{\pi}{2})-0=\pi\quad\text{and}$$
$$\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{4-x^2}\,dx=\tfrac{1}{2}\big[x\sqrt{4-x^2}+4\sin^{-1}\tfrac{x}{2}\big]\big|_{-\sqrt{2}}^{\sqrt{2}}$$
$$=\tfrac{1}{2}\big[(\sqrt{2}\cdot\sqrt{2}+4\sin^{-1}\tfrac{\sqrt{2}}{2})-((-\sqrt{2})\cdot\sqrt{2}+4\sin^{-1}\tfrac{-\sqrt{2}}{2})\big]$$
$$=\tfrac{1}{2}(2+4\cdot\tfrac{\pi}{4}+2-4(-\tfrac{\pi}{4})\pi)=2+\pi.$$

The circle in question has center the origin and radius 2. To interpret the two integrals as parts of the area of this circle refer to the figure below. Figure (a) tells us that the first integral is equal to one-fourth of the area of a circle of radius 2. This is $\frac{1}{4}\cdot 2^2\pi=\pi$ and confirms our earlier result. Now turn to Figure (b). Notice that the points singled out are



(a)



(b)

$(-\sqrt{2},\sqrt{2})$ and $(\sqrt{2},\sqrt{2})$. Since the segments connecting them to the origin lie on the lines $y=-x$ and $y=x$, respectively, it follows that these segments are perpendicular to each other. In view of the figure, the area that the second integral represents consists of one quarter of the circle plus two triangles with base $\sqrt{2}$ and height $\sqrt{2}$. So this area is equal to $\frac{1}{4}2^2\pi + 2(\frac{1}{2}(\sqrt{2}\cdot\sqrt{2})) = \pi + 2$.

9.20. The fourth term is $4(\frac{3}{10{,}000})^3\frac{1}{10{,}000}$ and the next to last term is $4(5+\frac{9{,}{,}998}{10{,}000})^3\frac{1}{10{,}000}$. Since $\frac{1}{n} = dx = \frac{1}{10{,}000}$ we see that $n = 10{,}000$. The partition $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$ is

$$0,\ \tfrac{1}{10{,}000},\ \tfrac{2}{10{,}000},\ \ldots,\ 5+\tfrac{9{,}{,}999}{10{,}000},\ 5+\tfrac{10{,}{,}000}{10{,}000} = 6,$$

$f(x) = 4x^3$ and the integral that the sum approximates is $\displaystyle\int_0^6 4x^3 = x^4\,\Big|_0^6 = 6^4 = 1296.$

9.21. Since $dx = \frac{1}{10{,}000}$, it follows that $n = 10{,}000$. The partition starts with $a = x_0 = 5$ and jumps by $\frac{1}{10{,}000}$ at each step. So $x_1 = 5+\frac{1}{10{,}000}, x_2 = 5+\frac{2}{10{,}000}, \cdots, x_{n-1} = 5+\frac{9{,}999}{10{,}000}$, and $x_n = b = 6$. Taking $f(x) = \sqrt{x}$, and substituting into $\displaystyle\sum_{i=0}^{n-1} f(x_i)\Delta x_i$, we get the sum above. It follows that the sum is approximated by the integral

$$\int_5^6 \sqrt{x}\,dx = \tfrac{2}{3}(x)^{\frac{3}{2}}\,\Big|_5^6 = \tfrac{2}{3}(6^{\frac{3}{2}} - 5^{\frac{3}{2}}) \approx 2.34.$$

Note the correction of the answer in the text. Alternatively, with $dx = \frac{1}{n} = \frac{1}{10{,}000}$, the choices $a = x_0 = 0, x_n = b = 1$, and $f(x) = \sqrt{5+x}$ work as well. The integral $\displaystyle\int_0^1 \sqrt{5+x}\,dx = \frac{2}{3}(5+x)^{\frac{3}{2}}\,\Big|_0^1$ provides the same result. (This integral reduces to the first with the substitution $u = x+5$.)

9.22. **i.** There is very little to do in this part except to realize that i is the only variable in the expression and that constants can be factored out. It follows that

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i = \sum_{i=0}^{n-1}\big(\tfrac{ib}{n}\big)^2\tfrac{b}{n} = \sum_{i=0}^{n-1}\tfrac{i^2b^2}{n^2}\tfrac{b}{n} = \big(\tfrac{b^3}{n^3}\big)\sum_{i=0}^{n-1} i^2.$$

ii. After reviewing the principle of mathematical induction from segment 3E of Section 3.8, let S_k be the statement $1^2+2^2+3^2+\cdots+(k-1)^2 = \frac{(k-1)k(2k-1)}{6}$. Note that the statement—whether true or not—makes sense for any $k \ge 2$. So with regard to your review, $m = 2$. Note first that S_2 is true because $1^2 = 1 = \frac{(2-1)(2)(2\cdot 2-1)}{6} = \frac{6}{6}$. We'll now prove for any $k \ge 2$, that the truth of S_k implies the truth of S_{k+1}. So we're assuming that

$$1^2+2^2+3^2+\cdots+(k-1)^2 = \tfrac{(k-1)k(2k-1)}{6}$$

is correct. After adding k^2 to both sides, we get

$$1^2+2^2+3^2+\cdots+(k-1)^2+k^2 = \tfrac{(k-1)k(2k-1)}{6} + k^2.$$

Working with the right side, we get

$$\tfrac{(k-1)k(2k-1)}{6} + k^2 = k[\tfrac{(k-1)(2k-1)}{6} + k] = k[\tfrac{(k-1)(2k-1)+6k}{6}] = k[\tfrac{2k^2+3k+1}{6}]$$
$$= k[\tfrac{(k+1)(2k+1)}{6}] = \tfrac{k(k+1)(2(k+1)-1)}{6}.$$

So

$$1^2+2^2+3^2+\cdots+(k-1)^2+k^2=\tfrac{k(k+1)(2(k+1)-1)}{6}.$$

Note therefore that S_{k+1} is true. We have verified for any $k\geq 2$ that the truth of S_k implies the truth of S_{k+1}. It follows from the principle of mathematical induction that S_k is true for all $k\geq 2$. So with n in place of k, S_n is true for all $n\geq 2$.

iii. We know that

$$\sum_{i=0}^{n-1} i^2 = 0+1^2+2^2+3^2+\cdots+(n-1)^2=\tfrac{(n-1)n(2n-1)}{6}$$

for any n and therefore that

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i = \left(\tfrac{b^3}{n^3}\right)\sum_{i=0}^{n-1} i^2 = \tfrac{b^3}{n^3}\tfrac{(n-1)n(2n-1)}{6} = \tfrac{b^3}{3}\tfrac{(n-1)n(2n-1)}{2n^3} = \tfrac{b^3}{3}\left(\tfrac{2n^3-3n^2+n}{2n^3}\right)$$
$$= \tfrac{b^3}{3}\left(1-\tfrac{3}{2n}+\tfrac{1}{2n^2}\right)$$

iv. If follows from the initial information for the problem that all the Δx_i are equal to $\frac{1}{n}$. By letting n go to infinity in part (iii), two things happen simultaneously: All the Δx_i go to zero, so that $\displaystyle\sum_{i=0}^{n-1} f(x_i)\Delta x_i$ closes in on $\displaystyle\int_0^b f(x)\,dx$ and at the same time, $\frac{b^3}{3}\left(1-\frac{3}{2n}+\frac{1}{2n^2}\right)$ gets pushed to $\frac{b^3}{3}$. The equality of part (iii) tells us that

$$\int_0^b x^2\,dx=\tfrac{b^3}{3}.$$

9.23. **i.** As in the previous problem,

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i = \left(\tfrac{b^4}{n^4}\right)\sum_{i=0}^{n-1} i^3$$

follows because i is the only variable in the expression and $\frac{b^4}{n^4}$ can be factored out.

ii. We'll let S_k be the statement $1^3+2^3+3^3+\cdots+(k-1)^3=\frac{1}{4}(k-1)^2k^2$ and follow the strategy of the principle of mathematical induction of segment 3E of Section 3.8. The statement S_k (whether true or not) makes sense for any $k\geq 2$. Note first that S_2 is true because $1^3=1=\frac{1}{4}(2-1)^2 2^2$. We'll now prove for any $k\geq 2$, that the truth of S_k implies the truth of S_{k+1}. So we're assuming that

$$1^3+2^3+3^3+\cdots+(k-1)^3=\tfrac{1}{4}(k-1)^2k^2$$

is correct. After adding k^3 to both sides, we get

$$1^3+2^3+3^3+\cdots+(k-1)^3+k^3=\tfrac{1}{4}(k-1)^2k^2+k^3.$$

Since

$$\tfrac{1}{4}(k-1)^2k^2+k^3=\tfrac{1}{4}k^2((k-1)^2+4k)=\tfrac{1}{4}k^2(k^2-2k+1+4k)=\tfrac{1}{4}k^2(k+1)^2$$

it follows that

$$1^3 + 2^3 + 3^3 + \cdots + (k-1)^3 + k^3 = \tfrac{1}{4}k^2(k+1)^2$$

and that hence S_{k+1} is true. We have verified for any $k \geq 2$ that the truth of S_k implies the truth of S_{k+1}. Therefore by the principle of mathematical induction, S_k is true for all $k \geq 2$. Changing notation from k to n tells us that S_n is true for any $n \geq 2$.

iii. We now know that

$$\sum_{i=0}^{n-1} i^3 = 0 + 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \tfrac{1}{4}(n-1)^2 n^2$$

and therefore that

$$\sum_{i=0}^{n-1} f(x_i)\Delta x_i = \left(\tfrac{b^4}{n^4}\right)\sum_{i=0}^{n-1} i^3 = \tfrac{b^4}{n^4}\tfrac{1}{4}(n-1)^2 n^2 = \tfrac{b^4}{4}\tfrac{n^4-2n^3+n^2}{n^4} = \tfrac{b^4}{4}\left(1 - \tfrac{2}{n} + \tfrac{1}{n^2}\right).$$

iv. From the given of the problem, all Δx_i are equal to $\frac{1}{n}$. So by pushing n go to infinity in part (iii), $\displaystyle\sum_{i=0}^{n-1} f(x_i)\Delta x_i$ closes in on $\displaystyle\int_0^b f(x)\,dx$ and simultaneously $\frac{b^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right)$ gets pushed to $\frac{b^4}{4}$. It follows that

$$\int_0^b x^3\,dx = \tfrac{b^4}{4}.$$

9.24. This problem is identical to Problem 5.58. Its solution is attended to in the solution set for Chapter 5.

9.25. By formula (V_1) of Section 9.2 and the fundamental theorem of calculus, this volume is

$$\pi\int_{-2}^{1}(e^x)^2\,dx = \pi\int_{-2}^{1} e^{2x}\,dx = \tfrac{1}{2}e^{2x}\,\Big|_{-2}^{1} = \tfrac{1}{2}(e^2 - e^{-4}) \approx 3.69.$$

9.26. We need an antiderivative of $\tan x = \frac{\sin x}{\cos x}$. Let $g(x) = \cos x$. Since $g'(x) = -\sin x$, we see that $\tan x = -\frac{g'(x)}{g(x)}$. By a formula of Section 7.11, $\frac{d}{dx}(-\ln|g(x)|) = -\frac{g'(x)}{g(x)}$ so that $\frac{d}{dx}(-\ln|\cos x|) = \tan x$. Using $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos 0 = 1$, and basic properties of the log function from Section 7.11,

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan x\,dx &= -\ln|\cos x|\,\Big|_0^{\frac{\pi}{4}} = -\ln|\cos\tfrac{\pi}{4}| - (-\ln|\cos 0|)\\ &= -\ln\tfrac{\sqrt{2}}{2} + \ln 1 = -\ln 2^{\frac{1}{2}} + \ln\ 2 + 0\\ &= -\tfrac{1}{2}\ln 2 + \ln 2 = \tfrac{1}{2}\ln 2 \approx 0.347.\end{aligned}$$

This is the area under the graph of $y = \tan x$ over the interval $[0, \frac{\pi}{4}]$. See Figure 4.26.

9.27. **i.** Since $\sqrt{1+4x^2} \geq 0$, the definite integral $\displaystyle\int_1^5 \sqrt{1+4x^2}\,dx$ is the area under the graph of the function $f(x) = \sqrt{1+4x^2}$ over the interval $1 \leq x \leq 5$.

ii. This integral is also the length of the graph of the function $f(x) = x^2$ from the point $(1,1)$ to the point $(5,25)$. This follows from the length of graphs formula in Section 9.3 and the observation that with $f(x) = x^2$, $f'(x) = 2x$ and $f'(x)^2 = 4x^2$.

9.28. **i.** Since $\sqrt{1+x} \geq 0$, the integral $\displaystyle\int_0^3 \sqrt{1+x}\,dx$ is the area under the graph of $f(x) = \sqrt{1+x}$ from $x = 0$ to $x = 3$.

ii. For $y = g(x) = \frac{1}{\sqrt{\pi}}(1+x)^{\frac{1}{4}}$, $\pi g(x)^2 = (1+x)^{\frac{1}{2}}$ so that by a formula in Section 9.2 this integral is also the volume obtained by rotating the region under the graph of $g(x)$ for $0 \leq x \leq 3$ one revolution about the x-axis.

iii. Finally for $h(x) = \frac{2}{3}x^{\frac{3}{2}}$, $h'(x)^2 = x$, so that by Section 9.3 the integral is the the length of the part of the graph of $h(x) = \frac{2}{3}x^{\frac{3}{2}}$ from $(0,0)$ to $(3, \frac{2}{3}3\sqrt{3}) = (3, 2\sqrt{3})$.

9.29. Since the circle with center $(r,0)$ and radius r has equation $(x-r)^2 + y^2 = r^2$, the upper half of this circle is the graph of the function $y = \sqrt{r^2 - (x-r)^2}$ with $0 \leq x \leq 2r$. The volume obtained by revolving the region under this graph around the x-axis is

$$\pi \int_0^{2r} (r^2 - (x-r)^2)\,dx = \pi \int_0^{2r} (-x^2 + 2rx)\,dx = \pi(-\tfrac{1}{3}x^3 + rx^2)\Big|_0^{2r} = \tfrac{4}{3}\pi r^3.$$

9.30. The segment from 0 to (h, r) in Figure 9.46 has slope $\frac{r}{h}$ and lies on the line $y = \frac{r}{h}x$. So the volume obtained by revolving the segment is

$$\pi \int_0^h \left(\tfrac{r}{h}\right)^2 x^2\,dx = \pi \cdot \tfrac{r^2}{3h^2}x^3\Big|_0^h = \tfrac{1}{3}\pi r^2 h.$$

9.31. This volume is given by

$$\pi \int_0^4 (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi \cdot \tfrac{1}{2}x^2\Big|_0^4 = 8\pi.$$

9.32. Check that

$$\pi \int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \pi \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \pi \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\,dx = \pi \int_0^{\frac{\pi}{2}} 1\,dx = \pi x\Big|_0^{\frac{\pi}{2}} = \tfrac{\pi^2}{2}.$$

Notice that $\pi \displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \pi \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ by studying the relevant areas under the graphs of Figure 4.23 and 4.24. It follows that both of these integrals are equal to $\frac{\pi^2}{4}$.

9.33. Since $\frac{dy}{dx} = 2x$, the length of is arc is given by the integral

$$\int_2^5 \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\,dx = \int_2^5 \sqrt{1+4x^2}\,dx.$$

This integral is also the area under the graph of $y = \sqrt{1+4x^2}$ from $x = 2$ to $x = 5$.

9.34. Refer to Figure 9.47. Since the distance from B to the origin is 2, the x-coordinate of B is equal to $x = 2\cos 60° = 1$ in case (a), $x = 2\cos 45° = 2\frac{\sqrt{2}}{2} = \sqrt{2}$ in case (b), and $x = 2\cos 30° = 2\frac{\sqrt{3}}{2} = \sqrt{3}$ in case (c). With $f(x) = (4-x^2)^{\frac{1}{2}}$, we get $f'(x) = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{4-x^2}}$. So $f'(x)^2 = \frac{x^2}{4-x^2}$ and hence

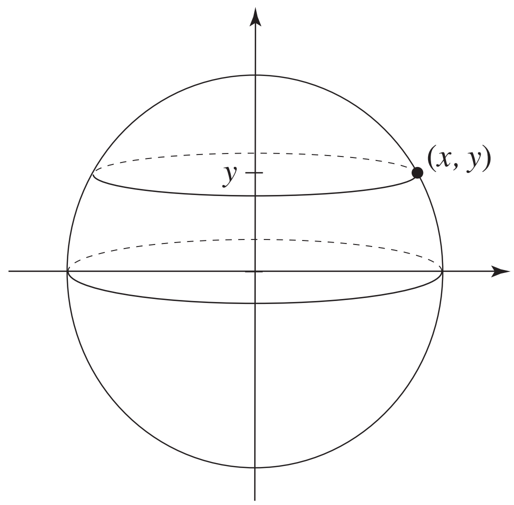
$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \tfrac{x^2}{4-x^2}} = \sqrt{\tfrac{4-x^2+x^2}{4-x^2}} = \sqrt{\tfrac{4}{4-x^2}} = \tfrac{2}{\sqrt{4-x^2}}.$$

Since the radius of the circle is 2, its upper half is 2π units long, so that the lengths of the circular arcs from A and B are $\frac{2\pi}{6}$, $\frac{2\pi}{4}$, and $\frac{2\pi}{3}$, respectively. So by the length formula of

Section 9.3, $\int_0^1 \frac{2}{\sqrt{4-x^2}}\,dx = \frac{2\pi}{6}$. This verifies the first equality. The other two follow in the same way.

9.35. The description of the process leading to the formula $V = \int_c^d A(y)\,dy$ is complete. It follows the script of Section 9.1. The only difference is the fact that the partition involves an interval of the y-axis rather than the x-axis. The argument is valid for any interval $[c,d]$ on the y-axis and not just those on the positive y-axis (as depicted in Figure 9.48).

9.36. The circle below depicts the cross of the sphere with the xy-plane. Let (x,y) be an arbitrary point on this circle with $x \geq 0$. Its coordinates satisfy the equation $x^2+y^2=r^2$, where r is



the radius of the sphere. The circle determined by cutting the sphere with a plane through its y-coordinate (x,y) perpendicular to the y-axis is also shown. Since its radius is $x=\sqrt{r^2-y^2}$, the area of this horizontal circle is $A(y)=\pi(r^2-y^2)$. It follows from the volume formula of Problem 9.35, that the volume of the sphere of radius r is

$$V = \int_{-r}^{r} \pi(r^2-y^2)\,dy = \pi(r^2y - \tfrac{y^3}{3})\Big|_{-r}^{r} = \pi\big((r^3-\tfrac{r^3}{3}) - (r^2(-r) - \tfrac{(-r)^3}{3})\big) = \tfrac{4}{3}\pi r^3.$$

9.37. By the surface area formula of Section 9.4 applied to $f(x)=x^{\frac{1}{2}}$, we get

$$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx = 2\pi\int_a^b x^{\frac{1}{2}}\sqrt{1+(\tfrac{1}{2}x^{-\frac{1}{2}})^2}\,dx = 2\pi\int_a^b \sqrt{x(1+\tfrac{x^{-1}}{4})}\,dx$$
$$= 2\pi\int_a^b \sqrt{x+\tfrac{1}{4}}\,dx.$$

To evaluate the integral, let $u = x+\frac{1}{4}$, to get

$$2\pi\int_{a+\frac{1}{4}}^{b+\frac{1}{4}} u^{\frac{1}{2}}\,du = 2\pi\Big[\tfrac{2}{3}u^{\frac{3}{2}}\Big|_{a+\frac{1}{4}}^{b+\frac{1}{4}}\Big] = \tfrac{4}{3}\pi\big[(b+\tfrac{1}{4})^{\frac{3}{2}} - (a+\tfrac{1}{4})^{\frac{3}{2}}\big].$$

9.38. **i.** The cone has height $h=x$ and radius $r=y$. Solving $(x-R)^2+y^2=R^2$ for y, we get $y=\sqrt{R^2-(x-R)^2}=\sqrt{2Rx-x^2}$. So by Problem 9.30, the volume of the cone is $V(x)=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi x(2Rx-x^2)$ with x restricted to $0\leq x\leq 2R$.

ii. By a formula of Section 9.4 the surface area of the cone that the segment from the origin to (x, y) generates is equal to πys. Since $y = \sqrt{2Rx - x^2}$ and $s = \sqrt{x^2 + y^2} = \sqrt{2Rx}$, we find that the surface area of the cone is

$$S(x) = \pi ys = \pi\sqrt{2Rx - x^2}\sqrt{2Rx} = \pi\sqrt{2Rx(2Rx - x^2)} = \pi(4R^2x^2 - 2Rx^3)^{\frac{1}{2}}$$

with x restricted to $0 \le x \le 2R$.

iii. By the product rule,

$$V'(x) = \tfrac{1}{3}\pi[(2Rx - x^2) + x(2R - 2x)] = \tfrac{1}{3}\pi(4Rx - 3x^2) = \tfrac{1}{3}\pi x(4R - 3x)$$

so that $V'(x) = 0$ only for $x = 0$ and $x = \frac{4}{3}R$. Since $V'(x) > 0$ for $x < \frac{4}{3}R$ and $V'(x) < 0$ for $x > \frac{4}{3}R$, it follows that $V(x)$ reaches its maximum value for $x = \frac{4}{3}R$. This maximum value is

$$V(\tfrac{4}{3}R) = \tfrac{1}{3}\pi(\tfrac{4}{3}R)\big(2R(\tfrac{4}{3}R) - (\tfrac{4}{3}R)^2\big) = (\tfrac{4\pi}{9}R)(\tfrac{24}{9}R^2 - \tfrac{16}{9}R^2) = \tfrac{32\pi}{81}R^3.$$

Since $S'(x) = \frac{1}{2}\pi(4R^2x^2 - 2Rx^3)^{-\frac{1}{2}}(8R^2x - 6Rx^2) = \frac{\pi}{2}\frac{2Rx(4R-3x)}{(2R)^{\frac{1}{2}}x(2R-x)^{\frac{1}{2}}}$, the critical points of the function $S(x)$ occur at x equal to $0, \frac{4}{3}R$, and $2R$. Since $S'(x) > 0$ for $0 < x < \frac{4}{3}R$ and $S'(x) < 0$ for $x > \frac{4}{3}R$, it follows that $S(x)$ reaches its maximum value for $x = \frac{4}{3}R$. This maximum value is

$$S(\tfrac{4}{3}R) = \pi\big(4R^2(\tfrac{4}{3}R)^2 - 2R(\tfrac{4}{3}R)^3\big)^{\frac{1}{2}} = \pi(\tfrac{64}{9}R^4 - \tfrac{128}{27}R^4)^{\frac{1}{2}} = \pi(\tfrac{192}{27} - \tfrac{128}{27})^{\frac{1}{2}}R^2 = \tfrac{8\pi}{3\sqrt{3}}R^2.$$

A related problem was considered by Archimedes. Consider the sphere of radius r inscribed in the cylinder with base the circle of radius r and height $2r$. Archimedes had derived the expressions $\frac{4}{3}\pi r^3$ and $4\pi r^2$ for the volume and surface area of a sphere of radius r. He knew that the volume and surface area of the cylinder (including its bases) are $(\pi r^2)2r = 2\pi r^3$ and $(2\pi r)(2r) + 2\pi r^2 = 6\pi r^2$ respectively. So he knew that the respective ratios of the volume and surface area of the sphere to the volume and surface area of the cylinder are both equal to $\frac{2}{3}$. Archimedes was evidently very proud of this discovery. According to the eye-witness report of the Roman statesman Cicero in 75 BC, the fraction $\frac{2}{3}$ and a figure of the cylinder and the inscribed sphere were etched on Archimedes's tomb. (The location of the tomb today is unknown.)

9.39. The area function $A(x)$ is an antiderivative of $f(x) = 1 + x + x^2$ that satisfies $A(a) = A(3) = 0$. So $A(x)$ has the form $A(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$. Since $0 = A(3) = 3 + \frac{9}{2} + 9 + C = 16\frac{1}{2} + C$, it follows that $C = -16\frac{1}{2}$. Therefore $A(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - 16\frac{1}{2}$.

9.40. Since $A(x)$ is an antiderivative of $y = \cos x$ satisfying $A(\frac{\pi}{2}) = 0$, we see that $A(x) = \sin x + C$ with $0 = A(\frac{\pi}{2}) = 1 + C$. Therefore $A(x) = \sin x - 1$.

9.41. This $A(x)$ is an antiderivative of $y = \sinh x$ satisfying $A(1) = 0$. So $A(x) = \cosh x + C$ with $0 = A(1) = \frac{e^1 + e^{-1}}{2} + C = \frac{e}{2} + \frac{1}{2e} + C$. Therefore $A(x) = \cosh x - (\frac{e}{2} + \frac{1}{2e})$.

9.42. Since the functions have the same domain $(-\infty, \infty)$ and since they assign the same value at each number of this domain (for example $f(c) = F(c) = \phi(c) = c^2 + 3c$), these three functions are identical.

9.43. $\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}8 = \frac{16}{3}$. In the same way $\int_0^4 \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}$ and again for the remaining two integrals. The bottom line is that the variable used for the integrand of a definite integral has no effect on the value of the integral. It is a so-called "dummy" variable.

9.44. In terms of the variables involved, the definite integral $\int_0^t \sqrt{x}\,dx$ is equal to a number that depends only on t. So the rule $t \longrightarrow \int_0^t \sqrt{x}\ dx$ defines a function. The value this function assigns to any t is the number

$$\int_0^t \sqrt{x}\,dx = \tfrac{2}{3}x^{\frac{3}{2}}\Big|_0^t = \tfrac{2}{3}t^{\frac{3}{2}}.$$

So the functions is $t \longrightarrow \frac{2}{3}t^{\frac{3}{2}}$. The numbers it assigns to $1, 4$, and 100 are $\frac{2}{3}, \frac{2}{3}4^{\frac{3}{2}} = \frac{16}{3}$, and $\frac{2}{3}(100)^{\frac{3}{2}} = \frac{2}{3}(1000)$, respectively.

9.45. By applying the fundamental theorem of calculus, we see that

$$F(x) = -t^{-1}\Big|_2^x = -x^{-1} + \tfrac{1}{2},\ G(x) = -z^{-1}\Big|_2^x = -x^{-1} + \tfrac{1}{2}, \text{ and } H(t) = -x^{-1}\Big|_2^t = -t^{-1} + \tfrac{1}{2}.$$

So $F(4) = \frac{1}{4}, G(4) = \frac{1}{4}$, and $H(4) = \frac{1}{4}$, and for any c, $F(c) = -\frac{1}{c} + \frac{1}{2}$, $G(c) = -\frac{1}{c} + \frac{1}{2}$, and $H(c) = -\frac{1}{c} + \frac{1}{2}$. It is clear that the three functions are identical. In terms of area, the functions $y = \frac{1}{t^2}, y = \frac{1}{z^2}$, and $y = \frac{1}{x^2}$ have identical graphs on their respective $(t, y), (z, y)$, and (x, y) coordinate planes. So the areas under these graphs from 2 to any fixed number $c \geq 2$ are also the same. Since

$$K(x) = \int_1^2 \tfrac{1}{t^2}\,dt + \int_2^x \tfrac{1}{t^2}\,dt = \big(-\,t^{-1}\big|_1^2\big) + F(x) = F(x) + \big(-\tfrac{1}{2} + 1\big) = F(x) + \tfrac{1}{2}.$$

So $K(x)$ and $F(x)$ differ by the constant $C = \frac{1}{2}$ equal to the area $\int_1^2 \frac{1}{t^2}\,dt$ under the graph of the function $y = \frac{1}{t^2}$ from $t = 1$ to $t = 2$.

9.46. The fundamental theorem of calculus tells us that

$$\int_3^x (t^2 + 5)\,dt = (\tfrac{1}{3}t^3 + 5t)\Big|_3^x = \tfrac{1}{3}x^3 + 5x - (\tfrac{1}{3}3^3 + 15) = \tfrac{1}{3}x^3 + 5x - 24.$$

So the function in question is $x \longrightarrow \frac{1}{3}x^3 + 5x - 24$. Its value at $x = 5$ is $\frac{125}{3} + 25 - 24 = 42\frac{2}{3}$. The problem with defining this function by $x \longrightarrow \int_3^x (x^2 + 5)\,dx$ is the double use of x for both the variable of the function and a limit of integration. (Under some circumstances this could lead to serious confusion.)

9.47. The equality highlighted in the box toward the end of Section 9.6 tells us that $F'(x) = x^2 + 3x$, $G'(x) = \frac{1}{x^2}$, and $K'(x) = \sqrt{x^3 + 5}$.

9.48. The integral $\int_1^x \frac{1}{t}\,dt$ defines an area function that has $f(x) = \frac{1}{x}$ as its derivative (apply the equality highlighted toward the end of Section 9.6 to see this). A look at Section 7.11 tells us that this area function is how the natural log $\ln x$ is defined for $x \geq 1$.

9.49. Such a definite integral is $G(x) = \int_0^x \sqrt{2t^2 + 4}\,dt$. By the equality highlighted toward the end of Section 9.6, $G'(x) = g(x)$.

9.50. The circle in question is $x^2 + y^2 = 1$. The upper half of the circle is the graph of the function $y = \sqrt{1 - x^2}$. For any x with $-1 \le x \le 1$, let $G(x)$ be the area under the upper half of the circle (and above the x-axis) from -1 to x. Observe (after changing variables from x to t) that $G(x) = \displaystyle\int_{-1}^{x} \sqrt{1 - t^2}\, dt$. Given the equality (in the box) toward the end of Section 9.6, it follows that $G'(x) = \sqrt{1 - x^2}$.

9.51. With $u = 4x - 5$, we get $\frac{du}{dx} = 4$ and hence $du = 4dx$ and $dx = \frac{du}{4}$. So

$$\begin{aligned}\int (4x-5)^{\frac{1}{2}} dx &= \int u^{\frac{1}{2}} \cdot \tfrac{du}{4} = \tfrac{1}{4}\int u^{\frac{1}{2}}\, du = \tfrac{1}{4}\left[\tfrac{2}{3}\, u^{\frac{3}{2}} + C'\right] \\ &= \tfrac{2}{12}(4x-5)^{\frac{3}{2}} + \tfrac{C'}{4} = \tfrac{1}{6}(4x-5)^{\frac{3}{2}} + C.\end{aligned}$$

9.52. After trying the other possibilities, you will probably settle on $u = 1 - 5x^2$. With this substitution, $\frac{du}{dx} = -10x$ and hence $du = -10x\, dx$. So

$$\begin{aligned}\int 10x(1-5x^2)^{\frac{2}{3}}\, dx &= \int u^{\frac{2}{3}}(-du) = -\int u^{\frac{2}{3}}\, du = -\left[\tfrac{3}{5}u^{\frac{5}{3}} + C'\right] \\ &= -\tfrac{3}{5}\left(1-5x^2\right)^{\frac{5}{3}} - C' = -\tfrac{3}{5}\left(1-5x^2\right)^{\frac{5}{3}} + C.\end{aligned}$$

9.53. With $u = x^2$ and $du = 2x\, dx$, we get

$$\begin{aligned}\int x\cos x^2\, dx &= \int (\cos u)(\tfrac{1}{2}\, du) = \tfrac{1}{2}\int \cos u\, du \\ &= \tfrac{1}{2}\left[\sin u + C'\right] = \tfrac{1}{2}\sin x^2 + C.\end{aligned}$$

Taking $u = \sin t$, we have $\frac{du}{dt} = \cos t$ and therefore

$$\int \sin^3 t\cos t\, dt = \int u^3\, du = \tfrac{u^4}{4} + C = \tfrac{1}{4}\sin^4 t + C.$$

9.54. With $u = x + 1$, the problem with the square root gets resolved. Note that $du = dx$ and $x = u - 1$. So

$$\begin{aligned}\int (x-1)(x+1)^{\frac{1}{2}}\, dx &= \int (u-2)\, u^{\frac{1}{2}}\, du = \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du \\ &= \tfrac{2}{5}u^{\frac{5}{2}} - 2\cdot\tfrac{2}{3}u^{\frac{3}{2}} + C \\ &= \tfrac{2}{5}(x+1)^{\frac{5}{2}} - \tfrac{4}{3}(x+1)^{\frac{3}{2}} + C.\end{aligned}$$

9.55. For the first integral, let $u = x + 3$. Because $dx = du$ and $x = u - 3$, we get

$$\begin{aligned}\int x^2(x+3)^{\frac{1}{2}}\, dx &= \int (u-3)^2 u^{\frac{1}{2}}\, du = \int (u^2 - 6u + 9)u^{\frac{1}{2}}\, du \\ &= \int \left(u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}}\right) du \\ &= \tfrac{2}{7}u^{\frac{7}{2}} - 6\cdot\tfrac{2}{5}u^{\frac{5}{2}} + 9\cdot\tfrac{2}{3}u^{\frac{3}{2}} + C \\ &= \tfrac{2}{7}(x+3)^{\frac{7}{2}} - \tfrac{12}{5}(x+3)^{\frac{5}{2}} + 6(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

For the second integral, let $u = x - 2$. Since $dx = du$ and $x = u + 2$, we see that

$$\begin{aligned}
\int \tfrac{x^2}{(x-2)^3}\,dx &= \int \tfrac{(u+2)^2}{u^3}\,du = \int \tfrac{u^2+2u+4}{u^3}\,du \\
&= \int (u^{-1} + 2u^{-2} + 4u^{-3})\,du \\
&= \ln u - 2u^{-1} - 2u^{-2} + C \\
&= \ln(x-2) - 2(x-2)^{-1} - 2(x-2)^{-2} + C.
\end{aligned}$$

9.56. Let's try $u = \tan\varphi$ with $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Note that $\frac{du}{d\varphi} = \sec^2\varphi$ and $du = \sec^2\varphi\,d\varphi$. Therefore

$$\int \tfrac{\sec^2\varphi}{\tan^2\varphi+1}\,d\varphi = \int \tfrac{du}{u^2+1}\,.$$

By Formula (10) from Section 9.11, $\int \frac{du}{u^2+1} = \tan^{-1} u + C$. So

$$\textstyle\int \frac{\sec^2\varphi}{\tan^2\varphi+1}\,d\varphi = \tan^{-1}(\tan\varphi) + C = \varphi + C.$$

There is a much simpler approach to this problem. Dividing the identity

$$\sin^2\varphi + \cos^2\varphi = 1$$

by $\cos^2\varphi$, gives us the identity $\tan^2\varphi + 1 = \sec^2\varphi$. So

$$\int \tfrac{\sec^2\varphi}{\tan^2\varphi+1}\,d\varphi = \int d\varphi = \varphi + C.$$

9.57. With $u = x^7 + 9$, $\frac{du}{dx} = 7x^6$. Hence $du = 7x^6\,dx$ and (looking ahead), $x^6dx = \frac{1}{7}du$. So

$$\begin{aligned}
\int \tfrac{5x^6}{x^7+9}\,dx &= \int \tfrac{5(\frac{1}{7}du)}{u} = \int \tfrac{5}{7}u^{-1}du = \tfrac{5}{7}\ln|u| + C \\
&= \tfrac{5}{7}\ln|x^7+9| + C.
\end{aligned}$$

9.58. Try $u = 1 + 2x + 4x^2$. So $\frac{du}{dx} = 2 + 8x = 2(1+4x)$. Hence $du = 2(1+4x)dx$. Therefore,

$$\begin{aligned}
\int (1+4x)(1+2x+4x^2)^{\frac{1}{2}}dx &= \int u^{\frac{1}{2}}\tfrac{du}{2} = \int \tfrac{1}{2}u^{\frac{1}{2}}\,du \\
&= \tfrac{1}{2}\cdot\tfrac{2}{3}u^{\frac{3}{2}} + C = \tfrac{1}{3}(1+2x+4x^2)^{\frac{3}{2}} + C.
\end{aligned}$$

9.59. With $u = \cos x$, $\frac{du}{dx} = -\sin x$ and $du = -\sin x\,dx$. Therefore

$$\int \tan x\,dx = \int \tfrac{\sin x}{\cos x}\,dx = \int -\tfrac{1}{u}\,du = \ln|u| + C = \ln|\cos x| + C.$$

9.60. With $u = e^z + 1, du = e^z dz$ and $e^z = u - 1$. For the first integral we get

$$\int (e^z+1)^{\frac{1}{2}}e^z dz = \int u^{\frac{1}{2}}\,du = \tfrac{2}{3}u^{\frac{3}{2}} + C = \tfrac{2}{3}(e^z+1)^{\frac{3}{2}} + C.$$

For the second one,

$$\begin{aligned}\int (e^z+1)^{\frac{1}{2}} e^{2z} dz &= \int (e^z+1)^{\frac{1}{2}} e^z \cdot e^z dz \\ &= \int u^{\frac{1}{2}}(u-1)du \;=\; \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du \\ &= \tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}} + C \\ &= \tfrac{2}{5}(e^z+1)^{\frac{5}{2}} - \tfrac{2}{3}(e^z+1)^{\frac{3}{2}} + C.\end{aligned}$$

9.61. For the first integral, let $u = 1 - x^2$. So $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}du$. For both $x = -1$ and $x = 1$, $u = 0$. It follows that

$$\int_{-1}^{1} \sqrt{1-x^2}x\,dx = \int_0^0 -\tfrac{1}{2}u^{\frac{1}{2}}\,du = 0.$$

Alternatively,

$$\int \sqrt{1-x^2}x\,dx = \int -\tfrac{1}{2}u^{\frac{1}{2}}\,du = -\tfrac{1}{2}\cdot\tfrac{2}{3}u^{\frac{3}{2}} + C = -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$$

An application of the fundamental theorem of calculus confirms the earlier result.

For the second integral, let $u = 3 + 5x^3$. So $du = 15x^2\,dx$. It follows that

$$\int_{-1}^{3} (3+5x^3)^{\frac{3}{2}} x^2\,dx = \int_{-2}^{138} \tfrac{1}{15}\,u^{\frac{3}{2}}\,du = \tfrac{1}{15}\cdot\tfrac{2}{5}\,u^{\frac{5}{2}}\Big|_{-2}^{138} = \tfrac{2}{75}\big(138^{\frac{5}{2}} - (-2)^{\frac{5}{2}}\big).$$

Now we see that there is a problem because $(-2)^{\frac{5}{2}} = ((-2)^{\frac{1}{2}})^5$ is not defined. For the integrand $(3+5x^3)^{\frac{3}{2}}x^2$ to be defined we need $3 + 5x^3 \geq 0$, hence $x^3 \geq -\frac{3}{5}$ and $x \geq -(\frac{3}{5})^{\frac{1}{3}} \approx -0.84$. Since the integrand is not defined for $-1 \leq x < -(\frac{3}{5})^{\frac{1}{3}}$ the integral does not make sense.

9.62. Let $u = 1 + 4x^{\frac{1}{3}}$. So $\frac{du}{dx} = \frac{4}{3}x^{-\frac{2}{3}}$ and hence $x^{-\frac{2}{3}}dx = \frac{3}{4}du$. When $x = 1$ and $x = 8$, $u = 5$ and $u = 9$, respectively. Therefore,

$$\begin{aligned}\int_1^8 x^{-\frac{2}{3}}\sqrt{1+4x^{\frac{1}{3}}}dx &= \int_5^9 u^{\frac{1}{2}}\cdot\tfrac{3}{4}\,du = \int_5^9 \tfrac{3}{4}u^{\frac{1}{2}}\,du \\ &= \big[\tfrac{3}{4}\cdot\tfrac{2}{3}u^{\frac{3}{2}}\big|_5^9\big] = \tfrac{1}{2}9^{\frac{3}{2}} - \tfrac{1}{2}5^{\frac{3}{2}} \\ &= \tfrac{1}{2}\cdot 3^3 - \tfrac{1}{2}\big(\sqrt{5}\big)^3 = \tfrac{1}{2}\big(27 - 5\sqrt{5}\big).\end{aligned}$$

9.63. Let $u = \ln x$. So $\frac{du}{dx} = \frac{1}{x}$ and $du = \frac{dx}{x}$. For $x = 1$ and 3, respectively $u = 0$ and $\ln 3$. Therefore,

$$\begin{aligned}\int_3^1 \tfrac{(\ln x)^2}{x}\,dx &= -\int_1^3 \tfrac{(\ln x)^2}{x}\,dx = -\int_0^{\ln 3} u^2\,du \\ &= -\tfrac{1}{3}u^3\,\Big|_0^{\ln 3} = -\tfrac{1}{3}(\ln 3)^3.\end{aligned}$$

9.64. Starting with $u = x$ and $dv = \cos x\,dx$, we get $du = dx$ and $v = \sin x$. So

$$\begin{aligned}\int x\cos x\,dx &= \int u\,dv \;=\; uv - \int v\,du = x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C.\end{aligned}$$

9.65. With $u = x$ and $dv = e^{5x}\,dx$, we get $du = dx$ and $v = \frac{1}{5}e^{5x}$. So

$$\begin{aligned} \int xe^{5x}\,dx &= \int u\,dv = uv - \int v\,du = x \cdot \tfrac{1}{5}e^{5x} - \int \tfrac{1}{5}e^{5x}\,dx \\ &= \tfrac{x}{5}e^{5x} - \tfrac{1}{5} \cdot \tfrac{1}{5}e^{5x} + C = \tfrac{x}{5}e^{5x} - \tfrac{1}{25}e^{5x} + C. \end{aligned}$$

Turning to the integral $\int x^2 e^{5x} dx$, let's check whether $u = x^2$ and $dv = e^{5x}dx$ accomplishes anything. Since $du = 2x\,dx$ and $v = \frac{1}{5}e^{5x}$, we get

$$\begin{aligned} \int x^2 e^{5x} dx &= \int u\,dv = uv - \int v\,du = x^2 \cdot \tfrac{1}{5}e^{5x} - \int 2x \cdot \tfrac{1}{5}e^{5x} dx \\ &= \tfrac{x^2}{5}e^{5x} - \tfrac{2}{5}\int xe^{5x} dx. \end{aligned}$$

Notice that the integral has been reduced to the one already solved earlier. So

$$\int x^2 e^{5x}\,dx = \tfrac{x^2}{5}e^{5x} - \tfrac{2}{5}\left[\tfrac{x}{5}e^{5x} - \tfrac{1}{25}e^{5x} + C'\right] = \tfrac{x^2}{5}e^{5x} - \tfrac{2x}{25}e^{5x} + \tfrac{2}{125}e^{5x} + C.$$

9.66. Proceeding as suggested, we have $u = \ln x, dv = x\,dx$, as well as $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$. So

$$\begin{aligned} \int x \ln x\,dx &= \int u\,dv \;=\; uv - \int v\,du \\ &= \tfrac{x^2}{2}\ln x - \int \tfrac{x^2}{2} \cdot \tfrac{1}{x}\,dx \;=\; \tfrac{x^2}{2}\ln x - \tfrac{1}{2}\int x\,dx \\ &= \tfrac{x^2}{2}\ln x - \tfrac{1}{4}x^2 + C. \end{aligned}$$

The suggestion is to let $u = \ln x$ and hence $dv = x^2\,dx$. So $du = \frac{1}{x}\,dx, v = \frac{x^3}{3}$, and we get

$$\begin{aligned} \int x^2 \ln x\,dx &= \int u\,dv \;=\; uv - \int v\,du = \tfrac{x^3}{3}\ln x - \int \tfrac{x^3}{3} \cdot \tfrac{1}{x}\,dx \\ &= \tfrac{x^3}{3}\ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{x^3}{3}\ln x - \tfrac{1}{9}x^3 + C. \end{aligned}$$

9.67.

i. Integrating by parts with $u = \ln(x^2+1), dv = dx$, and hence $du = \frac{2x}{x^2+1}$ and $v = x$ transforms $\int \ln(x^2+1)\,dx = \int u\,dv$ into $uv - \int v\,du = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx$.

ii. By a polynomial division $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$ or by noticing that $1 - \frac{1}{x^2+1} = \frac{x^2+1-1}{x^2+1} = \frac{x^2}{x^2+1}$, we see that $\int \frac{x^2}{x^2+1}\,dx = \int (1 - \frac{1}{x^2+1})\,dx = x - \int \frac{1}{x^2+1}\,dx$. Therefore the integral of (i) is equal to $x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx = x\ln(x^2+1) - 2x + 2\int \frac{1}{x^2+1}\,dx$.

iii. By a result of Section 9.9.1, $\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}$. So from (ii),

$$\int \ln(x^2+1)dx = x\ln(x^2+1) - 2x + 2\tan^{-1}x + C.$$

iv. Differentiating the function $x\ln(x^2+1) - 2x + 2\tan^{-1}x$, we get

$$\ln(x^2+1) + x \cdot \tfrac{2x}{x^2+1} - 2 + \tfrac{2}{x^2+1} = \ln(x^2+1) + \tfrac{2x^2 - 2(x^2+1) + 2}{x^2+1} = \ln(x^2+1),$$

so that this function is the antiderivative we need.

9.68. Let $z = t^{\frac{1}{2}}$. So $dz = \frac{1}{2}t^{-\frac{1}{2}}\,dt$ and $\int \cos t^{\frac{1}{2}}\,dt = \int (\cos z)\cdot 2z\,dz = 2\int z\cos z\,dz$. Now let $u = z$ and $dv = \cos z\,dz$. So $du = dz$ and $v = \sin z$. Therefore,

$$\begin{aligned}\int z\cos z\,dz &= \int u\,dv = uv - \int v\,du\\ &= z\sin z - \int \sin z\,dz \;=\; z\sin z + \cos z + C'.\end{aligned}$$

It follows that

$$\int \cos t^{\frac{1}{2}}\,dt = 2\left[z\sin z + \cos z + C'\right] = 2t^{\frac{1}{2}}\sin t^{\frac{1}{2}} + 2\cos t^{\frac{1}{2}} + C.$$

9.69. By applying the partial fractions maneuver,

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{(A+B)x-(3A+2B)}{(x-2)(x-3)}$$

so that $A + B = 0$ and $3A + 2B = -1$. Since $B = -A$ and $3A - 2A = -1$, we get $A = -1$ and $B = 1$. Combining this computation with the fact (from Section 7.11) that $\ln|g(x)|$ is an antiderivative of $\frac{g'(x)}{g(x)}$ for any differentiable function $y = g(x)$, we get

$$\int \tfrac{1}{(x-2)(x-3)}\,dx = \int \tfrac{-1}{x-2}\,dx + \int \tfrac{1}{x-3}\,dx = -\ln|x-2| + \ln|x-3| + C.$$

9.70. By the partial fractions maneuver once more,

$$\frac{x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} = \frac{A(x-3)+B(x+2)}{(x+2)(x-3)} = \frac{(A+B)x-(3A-2B)}{(x+2)(x-3)}$$

and hence $A + B = 1$ and $3A - 2B = -1$. Adding $2A + 2B = 2$ and $3A - 2B = -1$, we get $A = \frac{1}{5}$ and hence $B = \frac{4}{5}$. So $\frac{x+1}{(x+2)(x-3)} = \frac{\frac{1}{5}}{x+2} + \frac{\frac{4}{5}}{x-3}$. Turning to the fact from Section 7.11 that $\ln|g(x)|$ is an antiderivative of $\frac{g'(x)}{g(x)}$ for any differentiable function $y = g(x)$, we get

$$\int \tfrac{x+1}{(x+2)(x-3)}\,dx = \tfrac{1}{5}\int \tfrac{1}{x+2}\,dx + \tfrac{4}{5}\int \tfrac{1}{x-3}\,dx = \tfrac{1}{5}\ln|x+2| + \tfrac{4}{5}\ln|x-3| + C.$$

9.71. Since $\tanh^{-1}x$ is involved, we will assume that $-1 < x < 1$. To see why, consider Figure 9.38 and the discussion that precedes it. The equality $\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$ is the case $s = 1$ of an equality verified in Section 9.7.3. It follows that

$$\int \tfrac{1}{1-x^2}\,dx = -\tfrac{1}{2}\ln\left|\tfrac{x-1}{x+1}\right| + C = \tfrac{1}{2}\ln\left|\tfrac{x-1}{x+1}\right|^{-1} + C = \tfrac{1}{2}\ln\left|\tfrac{x+1}{x-1}\right| + C.$$

Since $-1 < x < 1$, it follows that $1 > -x > -1$. Hence both $1 + x > 0$ and $1 - x > 0$. So $\frac{x+1}{x-1} > 0$ and hence by fact 3 of Section 9.9.2,

$$\int \tfrac{1}{1-x^2}\,dx = \tfrac{1}{2}\ln\left(\tfrac{1+x}{1-x}\right) + C = \tanh^{-1}x + C.$$

The formula $\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$ is valid for all x except $x = \pm 1$ and has much greater applicability than the formula involving $\tanh^{-1}x + C$ as it requires that $-1 < x < 1$.

9.72. Review the procedure described in Section 9.8 that leads from the graph of any increasing or decreasing function $y = f(x)$ to the graph of its inverse $y = f^{-1}(x)$. Applying it to the graph of $y = f(x) = \sqrt{r^2 - x^2}$ for $0 \le x \le r$ we see that the graph of the inverse is the same as the graph of the function. Two functions that have identical graphs must be the same function. It follows that $y = f^{-1}(x) = \sqrt{r^2 - x^2}$ with $0 \le x \le r$. To verify this explicitly, let's solve $y = \sqrt{r^2 - x^2}$ for x under the assumption that $0 \le x \le r$. Doing this, we get $y^2 = r^2 - x^2$, hence $x^2 = r^2 - y^2$, and therefore $x = \sqrt{r^2 - y^2}$. So the rule for the inverse is $f^{-1}(y) = \sqrt{r^2 - y^2}$. Noting that $0 \le y \le r$ and relabeling the variable, we get $f^{-1}(x) = \sqrt{r^2 - x^2}$. Therefore $f^{-1}(x) = f(x)$ for all x with $0 \le x \le r$.

9.73. By applying the formula $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ to the function $f(x) = \cos x$ with $0 \le x \le \pi$, we get $\frac{d}{dx} \cos^{-1}(x) = \frac{1}{-\sin(\cos^{-1}(x))}$. Since $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = 1 - \cos^2 x$ and hence $\sin x = \sqrt{1 - \cos^2 x}$ (note that $\sin x \ge 0$ over $0 \le x \le \pi$). It follows that

$$\frac{d}{dx} \cos^{-1}(x) = \frac{1}{-\sin(\cos^{-1}(x))} = \frac{1}{-\sqrt{1 - (\cos(\cos^{-1}(x)))^2}} = -\frac{1}{\sqrt{1-x^2}}.$$

Since the derivative of $\sin^{-1} x$ is equal to $\frac{1}{\sqrt{1-x^2}}$, the derivatives of functions $\cos^{-1}(x)$ and $-\sin^{-1}(x)$ are equal. It follows that $\cos^{-1}(x) = -\sin^{-1}(x) + C$ for some constant C (and all x with $-1 \le x \le 1$.) Since $\cos 0 = 1$ and $\sin \frac{\pi}{2} = 1$, $\cos^{-1}(1) = 0$ and $\sin^{-1}(1) = \frac{\pi}{2}$. Evaluating $\cos^{-1}(x) = -\sin^{-1}(x) + C$ at $x = 1$, we se that $C = \frac{\pi}{2}$, and hence that $\cos^{-1}(x) = -\sin^{-1}(x) + \frac{\pi}{2}$. Notice that some minus signs were erroneously omitted in the formulation of Problem 9.73.

9.74. The graph of the inverse $y = f^{-1}(x)$ of a function $y = f(x)$ is gotten by reflecting the graph of the function by reflecting it across the line $y = x$. So the two graphs have the same shape and differ only in the way they are positioned. Figure 9.53 depicts the example under consideration. It follows from the observation just made that the area bounded by the graph of the function $f(x) = x^2$ for $x \ge 0$, the y-axis, and the horizontal dotted segment is equal to the area bounded by the graph of $f^{-1}(x) = \sqrt{x}$, the x-axis, and the vertical dotted segment. Therefore the area under the graph of $f^{-1}(x) = \sqrt{x}$ over the interval $0 \le x \le c$ is equal to the area cd of the rectangle determined by the point (c, d) minus the area under the graph of $f(x) = x^2$ over the interval $0 \le x \le d$. Translated, this is the equality

$$\int_0^d \sqrt{x}\, dx = cd - \int_0^c x^2\, dx.$$

By the fundamental theorem of calculus, the two integrals are equal to $\frac{2}{3} x^{\frac{3}{2}} \Big|_0^d = \frac{2}{3} d\sqrt{d}$ and $\frac{1}{3} x^3 \Big|_0^c = \frac{1}{3} c^3$, respectively. Since $c = \sqrt{d}$, $c^2 = d$ and $\frac{2}{3} c^3 = c^3 - \frac{1}{3} c^3$, it follows that $\frac{2}{3} d\sqrt{d} = cd - \frac{1}{3} c^3$ as we needed to show.

Now to the second equality. Since $f'(x)^2 = 4x^2$, the term $\sqrt{1 + 4x^2}$ suggests that this time the lengths of the two graphs might be involved. It follows from the way the two graphs are related, that the length of the graph of $f^{-1}(x) = \sqrt{x}$ between the points $(0,0)$ and (d, c) is the same as that of $f(x) = x^2$ between $(0,0)$ and (c, d). Since $\frac{d}{dx} f^{-1}(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}$ and $\frac{d}{dx} f(x) = 2x$, the length formula of Section 9.3 tells us that

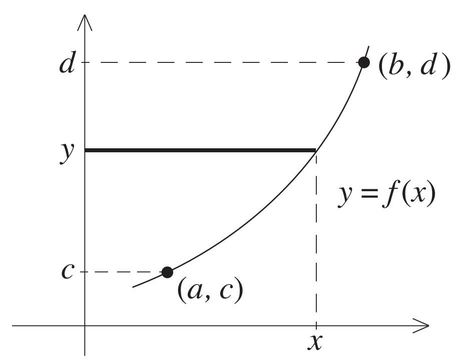
$$\int_0^d \sqrt{1+\tfrac{1}{4x}}\,dx = \int_0^c \sqrt{1+4x^2}\,dx\,.$$

The second integral can be solved by letting $u = 2x$ and $du = 2dx$ and applying Formula (17) of Section 9.11. Doing so, we get that this integral is equal to

$$\begin{aligned}\tfrac{1}{2}\int_0^{2c} \sqrt{1+u^2}\,du &= \tfrac{1}{4}\big[u\sqrt{1+u^2} + \ln(u+\sqrt{1+u^2})\big]\Big|_0^{2c} \\ &= \tfrac{1}{4}\big[2c\sqrt{1+4c^2} + \ln(2c+\sqrt{1+4c^2})\big].\end{aligned}$$

The solution of the integral $\int_0^d \sqrt{1+\frac{1}{4x}}\,dx = \int_0^d \sqrt{\frac{1}{4}(\frac{1}{x}+4)}\,dx = \frac{1}{2}\int_0^d \sqrt{\frac{1}{x}+4}\,dx$ relies on the substitution $u = \sqrt{\frac{1}{x}+4}$ that transforms the integral into one to which the algebraic maneuver of partial fractions can be applied (the maneuver that deals with quadratic factors). We defer to the site http://www.integral-calculator.com/# for the details. The site displays not only the solution, but also the steps involved. Type `sqrt(1+1/(4x))` into the box containing `e^(x/2)* sin(ax)` and then click Go!.

9.75. We'll consider the situation of an increasing function (that if a decreasing function is dealt with in the same way). The figure below depicts a typical situation. The relevant area element is shown. Its thickness is dy and its length is x. The corresponding value $y = f(x)$ is its y-coordinate. When it is rotated one complete revolution about the y-axis this thin strip generates a disc of volume $\pi x^2\,dy$. The definition of the inverse implies that $x = f^{-1}(y)$ so that this



volume is $\pi f^{-1}(y)^2\,dy$. The volume obtained by revolving the region bounded by the graph of $y = f(x)$ and the lines $y = c$ and $y = d$ around the y-axis is therefore equal to

$$V = \int_c^d \pi(f^{-1}(y))^2\,dy.$$

Since the cross-sectional area of the solid at y is $\pi f^{-1}(y)^2$ the conclusion of Problem 9.35 provides the same result.

9.76. **i.** Since the derivative of $\tan^{-1} x$ is $\frac{1}{x^2+1}$, we get by the chain rule that

$$\begin{aligned}f'(x) &= \tfrac{d}{dx}\tan^{-1}\big(\tfrac{x-1}{x+1}\big) = \tfrac{1}{(\frac{x-1}{x+1})^2+1}\,\tfrac{d}{dx}\big(\tfrac{x-1}{x+1}\big) = \tfrac{1}{\frac{(x^2-2x+1)+(x^2+2x+1)}{(x+1)^2}}\cdot\tfrac{(x+1)-(x-1)}{(x+1)^2} \\ &= \tfrac{(x+1)^2}{2(x^2+1)}\cdot\tfrac{2}{(x+1)^2} = \tfrac{1}{x^2+1}.\end{aligned}$$

ii. Because $f(x)$ and $\tan^{-1}x$ have the same derivative, we should be able to conclude that $\tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}x + C$ for some constant C. Let's see how this conclusion holds up.

iii. Since $\tan(-\frac{\pi}{4}) = -1$, $\tan^{-1}(-1) = -\frac{\pi}{4}$. Plugging $x = 0$ into the equality of (ii) we get $-\frac{\pi}{4} = \tan^{-1}(-1) = \tan^{-1}(0) + C$, and hence that $C = -\frac{\pi}{4}$. Therefore

$$\tan^{-1}\left(\tfrac{x-1}{x+1}\right) = \tan^{-1}x - \tfrac{\pi}{4}.$$

iv. From Figure 9.33 we know that $\lim\limits_{x\to+\infty} \tan^{-1}x = \frac{\pi}{2}$ and $\lim\limits_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2}$. Combined with the equality derived in (iii) this implies that

$$\lim_{x\to+\infty} \tan^{-1}\left(\tfrac{x-1}{x+1}\right) = \tfrac{\pi}{4} \quad\text{and}\quad \lim_{x\to-\infty} \tan^{-1}\left(\tfrac{x-1}{x+1}\right) = -\tfrac{3\pi}{4}.$$

The fact that $\frac{x-1}{x+1} = \frac{x(1-\frac{1}{x})}{x(1+\frac{1}{x})} = \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$ tells us that $\lim\limits_{x\to\pm\infty} \frac{x-1}{x+1} = 1$. Since $\tan\frac{\pi}{4} = 1$, we know that $\tan^{-1}(1) = \frac{\pi}{4}$. It follows from this that $\lim\limits_{x\to+\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right)$ and $\lim\limits_{x\to-\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right)$ are both equal to $\frac{\pi}{4}$.

v. Clearly $\lim\limits_{x\to-\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right) = -\frac{3\pi}{4}$ and $\lim\limits_{x\to-\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4}$ cannot both hold. What has gone wrong? The problem is that while $y = \tan^{-1}x$ is differentiable for all x, this is not the case for the function $y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$ at $x = -1$ where there is a discontinuity. It is for this reason that the conclusion of part (ii) is wrong. It is the case that $\tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}x + C$ over each of the intervals $(-\infty, -1)$ and $(-1, +\infty)$. However the two constants C are different.

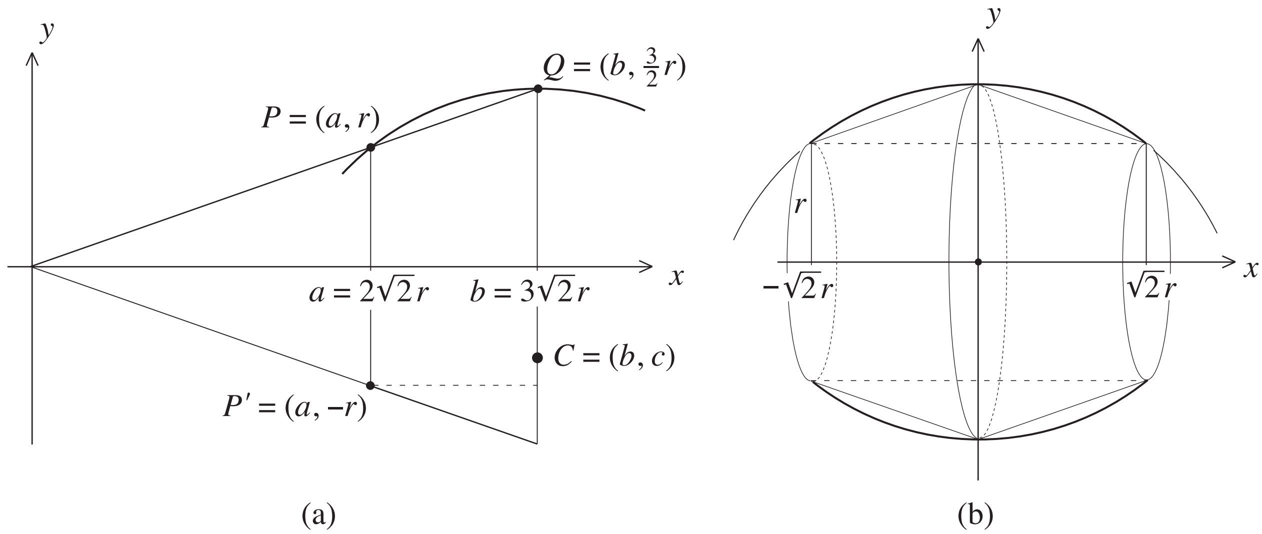
9.77. Consider the parabola $f(x) = x^2$. Let (c, c^2) with $c \geq 0$ be a point on the parabola, and let $L(c)$ be the length of the parabola from the origin $(0,0)$ to (c, c^2). See Figure 9.54. By the length formula $L(c) = \displaystyle\int_0^c \sqrt{1+4x^2}\,dx$. To determine $L(c)$ we'll use the substitutions $u = 2x$ and $du = 2\,dx$ along with the formula (from Section 9.10 or Section 9.11)

$$\int \sqrt{1+u^2}\,du = \tfrac{1}{2}[u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2})] + C$$

and the fact that $\ln 1 = 0$ to get

$$\begin{aligned}\int_0^c \sqrt{1+4x^2}\,dx &= \tfrac{1}{2}\int_0^{2c} \sqrt{1+u^2}\,du = \tfrac{1}{2}[u\sqrt{1+u^2} + \ln(u+\sqrt{1+u^2})]\Big|_0^{2c} \\ &= \tfrac{1}{4}[2c\sqrt{1+(2c)^2} + \ln(2c + \sqrt{1+(2c)^2}\,)].\end{aligned}$$

9.78. Project. As already pointed out this study relies on Kepler's analysis in Sections 5.5 and 5.9 and in particular on Figure 5.39. Figure 9.55 below summarizes much relevant information and includes the circular arc that the two slanting segments of Figure 5.39a determine. By Problem 1.9 the center C of the circle on which the arc lies is at the intersection of the perpendicular bisectors of the two slanting segments of Figure 9.55b. The symmetry of the situation places C on the vertical line $x = b$ as Figure 9.55a shows. The angulation of the perpendicular bisector suggests—and our computation will confirm—that the coordinate c is



negative.

i. Let's start with a look at Figure 9.55a. Since $C = (b, c)$ is the center of the circle and $\frac{3}{2}r - c$ is its radius, the equation of the circle is $(x-b)^2 + (y-c)^2 = (\frac{3}{2}r - c)^2$. Since $P = (a, r)$ is on the circle, $(a-b)^2 + (r-c)^2 = (\frac{3}{2}r - c)^2$. It follows that

$$(\sqrt{2}r)^2 + r^2 - 2rc + c^2 = \tfrac{9}{4}r^2 - 3rc + c^2$$

and hence that $rc = \frac{9}{4}r^2 - 3r^2$. So $c = (\frac{9}{4} - \frac{12}{4})r = -\frac{3}{4}r$ and $\frac{3}{2}r - c = \frac{3}{2}r + \frac{3}{4}r = \frac{9}{4}r$. It follows that the equation of the circle on which the arc of Figure 9.55a lies is $(x-b)^2 + (y + \frac{3}{4}r)^2 = (\frac{9}{4}r)^2$.

ii. We now shift this circle b units to the left into the position shown in Figure 9.55b. The center of this shifted circle is the point $(0, c) = (0, -\frac{3}{4}r)$ on the y-axis and its equation is $x^2 + (y + \frac{3}{4}r)^2 = (\frac{9}{4}r)^2$. Solving it for y, we get $y + \frac{3}{4}r = \pm\sqrt{(\frac{9}{4}r)^2 - x^2}$. So the upper half of this shifted circle is the graph of the function $f(x) = \sqrt{\left(\frac{9}{4}r\right)^2 - x^2} - \frac{3}{4}r$.

iii. Continue to focus on Figure 9.55b and consider the barrel-shaped solid obtained by revolving the region bounded by the graph of $f(x) = \sqrt{\left(\frac{9}{4}r\right)^2 - x^2} - \frac{3}{4}r$, the x-axis, and the lines $x = -\sqrt{2}r$ and $x = \sqrt{2}r$ once around the x-axis. By appealing to symmetry and the formula (V_1) of Section 9.2, we see that the volume V of this barrel-shape is

$$V = 2\pi\int_0^{\sqrt{2}r}\Big(\sqrt{(\tfrac{9}{4}r)^2 - x^2} - \tfrac{3}{4}r\Big)^2 dx = 2\pi\int_0^{\sqrt{2}r}\big(\tfrac{81}{16}r^2 - x^2 - \tfrac{3}{2}r\sqrt{(\tfrac{9}{4}r)^2 - x^2} + \tfrac{9}{16}r^2\big)\,dx =$$

$$2\pi\int_0^{\sqrt{2}r}\big(\tfrac{90}{16}r^2 - x^2 - \tfrac{3}{2}r\sqrt{(\tfrac{9}{4}r)^2 - x^2}\,\big)\,dx = 2\pi\int_0^{\sqrt{2}r}\big(\tfrac{90}{16}r^2 - x^2\big)\,dx - 3\pi r\int_0^{\sqrt{2}r}\sqrt{(\tfrac{9}{4}r)^2 - x^2}\,dx.$$

By the fundamental theorem of calculus and the integral formula preceding Example 9.31 in Section 9.10 with $a = \frac{9}{4}r$,

$$\begin{aligned} V &= \pi\Big(\tfrac{45}{4}r^2x - \tfrac{2}{3}x^3 - \tfrac{3}{2}r\big[x\sqrt{(\tfrac{9}{4}r)^2 - x^2} + (\tfrac{9}{4}r)^2\sin^{-1}\tfrac{x}{\frac{9}{4}r}\big]\Big)\Big|_0^{\sqrt{2}r} \\ &= \pi\Big(\tfrac{45\sqrt{2}}{4}r^3 - \tfrac{4\sqrt{2}}{3}r^3 - \tfrac{3}{2}r\big[\sqrt{2}r\sqrt{(\tfrac{9}{4}r)^2 - (\sqrt{2}r)^2} + (\tfrac{9}{4}r)^2\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big]\Big). \end{aligned}$$

The rest consists of algebraic simplifications:

$$\begin{aligned}
&= \pi r^3\big(\tfrac{45\sqrt{2}}{4} - \tfrac{4\sqrt{2}}{3} - \tfrac{3}{2}\big[\sqrt{2}\sqrt{(\tfrac{9}{4})^2 - (\sqrt{2})^2} + (\tfrac{9}{4})^2\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big]\big)\\
&= \pi r^3\big(\tfrac{270\sqrt{2}}{24} - \tfrac{32\sqrt{2}}{24} - \tfrac{3}{2}\big[\sqrt{2}\sqrt{\tfrac{49}{16}} + (\tfrac{9}{4})^2\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big]\big)\\
&= \pi r^3\big(\tfrac{238\sqrt{2}}{24} - \tfrac{3}{2}\big[\tfrac{7}{4}\sqrt{2} + \tfrac{81}{16}\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big]\big)\\
&= \pi r^3\big(\tfrac{238\sqrt{2}}{24} - \tfrac{3}{2}\tfrac{7}{4}\sqrt{2} - \tfrac{3}{2}\tfrac{81}{16}\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big)\\
&= \pi r^3\big(\tfrac{238\sqrt{2}}{24} - \tfrac{63\sqrt{2}}{24} - \tfrac{243}{32}\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big)\\
&= \pi r^3\big(\tfrac{175\sqrt{2}}{24} - \tfrac{243}{32}\sin^{-1}(\tfrac{4\sqrt{2}}{9})\big) \approx 16.18r^3.
\end{aligned}$$

We have shown that the barrel with circular sides depicted in Figure 9.55b has volume V closely approximated by $16.18r^3$ where r is the radius of the barrel's circular ends. Inscribed in this barrel is Kepler's model of the Austrian barrel. It has the same circular ends and volume $\frac{19}{6}\sqrt{2}\pi r^3 \approx 14.07r^3$.

iv. Let s be the length of the slanting diagonal on which the Austrian wine merchants based their measurement of the volumes of barrels. By applying the distance formula to the points Q and P' of Figure 9.55a, we get $s^2 = (QP')^2 = (b-a)^2 + (\frac{3}{2}r + r)^2 = ((\frac{5}{2}r)^2 + 2r^2) = (\frac{25}{4} + \frac{8}{4})r^2 = \frac{33}{4}r^2$ and hence $s = \frac{\sqrt{33}}{2}r$. Since $r = \frac{2}{\sqrt{33}}s$, the volume $V \approx 16.18r^3 \approx 16.18(\frac{2}{\sqrt{33}}s)^3 \approx 0.68s^3$. A comparison of Figures 5.40 and 9.55b tells us that the s for the barrel with the circular sides is the same as the s for Kepler's Austrian barrel inscribed in it. Since the wine merchants' rule for determining the volume of a barrel is $V_{\text{rule}} = 0.6s^3$ the price for a full barrel of wine for the circular barrel and Kepler's Austrian barrel will be the same, even though the volume of the latter (approximately $0.59s^3$) is about 10% less.

9.79. **i.** Let $x = \tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Note that $dx = \sec^2\theta\, d\theta$. Therefore

$$\int \tfrac{1}{\sqrt{x^2+1}}\, dx = \int \tfrac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^{\frac{1}{2}}}.$$

Recall that $\tan^2\theta + 1 = \sec^2\theta$. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, both $\cos\theta \geq 0$ and $\sec\theta \geq 0$. So $\sec\theta = (1+\tan^2\theta)^{\frac{1}{2}}$. By applying the integral formula in Section 9.10 for $\sec\theta$, we get

$$\int \tfrac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^{\frac{1}{2}}} = \int \sec\theta\, d\theta \;=\; \ln|\sec\theta + \tan\theta| + C.$$

ii. Since $\tan\theta = x$ and $\sec\theta = (1+\tan^2\theta)^{\frac{1}{2}} = \sqrt{1+x^2}$, it follows that

$$\int \tfrac{1}{\sqrt{x^2+1}}\, dx = \ln|x + \sqrt{1+x^2}| + C.$$

9.80. **i.** Taking $x = \sec\theta$ with $0 \leq \theta < \frac{\pi}{2}$, we get $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$ and since $\tan\theta > 0$ over this range of θ that $\sqrt{x^2-1} = \tan\theta$. Since $dx = \sec\theta\tan\theta\, d\theta$, we get

$$\int \tfrac{x^2}{\sqrt{x^2-1}}\, dx \;=\; \int \tfrac{\sec^2\theta}{\tan\theta}\sec\theta\tan\theta\, d\theta \;=\; \int \sec^3\theta\, d\theta.$$

ii. Making use of an equality from Section 9.10,

$$\int \sec^3\theta\, d\theta = \tfrac{1}{2}\big[\sec\theta\cdot\tan\theta + \ln|\sec\theta+\tan\theta|\big] + C.$$

After substituting $\sec\theta = x$ and $\tan\theta = \sqrt{x^2-1}$ into this expression, we get

$$\int \tfrac{x^2}{\sqrt{x^2-1}}\,dx = \tfrac{1}{2}\big[x\sqrt{x^2-1} + \ln(x+\sqrt{x^2-1})\big] + C.$$

9.81. **i.** Consider the substitution $x = \sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So $dx = \cos\theta\, d\theta$. Since $1 - x^2 = 1-\sin^2\theta = \cos^2\theta$ and $\cos\theta \ge 0$, we get

$$\begin{aligned}\int \tfrac{\sqrt{1-x^2}}{x}\,dx &= \int \tfrac{\cos\theta}{\sin\theta}\cdot\cos\theta\,d\theta = \int \tfrac{\cos^2\theta}{\sin\theta}\,d\theta\\ &= \int \tfrac{1-\sin^2\theta}{\sin\theta}\,d\theta = \int \tfrac{1}{\sin\theta}\,d\theta - \int \sin\theta\,d\theta.\\ &= \int \tfrac{1}{\sin\theta}\,d\theta + \cos\theta.\end{aligned}$$

The integral $\int \frac{1}{\sin\theta}\,d\theta$ can be evaluated with the same trick that led to the solution

$$\int \tfrac{1}{\cos\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C,$$

except that $\csc\theta = \frac{1}{\sin\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$ take the place of $\sec\theta$ and $\tan\theta$.

ii. The trick can be avoided by use of the substitution $x = \cos\theta$ with $0 \le \theta \le \pi$ and $dx = -\sin\theta\,d\theta$. Since $\sin^2\theta + \cos^2\theta = 1$ and $\sin\theta \ge 0$ over this range of θ, $\sin\theta = \sqrt{1-\cos^2\theta}$. Using this and the formula $\int \sec\, d\theta = \ln|\sec\theta+\tan\theta| + C$, we get

$$\begin{aligned}\int \tfrac{\sqrt{1-x^2}}{x}\,dx &= -\int \tfrac{\sin\theta}{\cos\theta}\cdot\sin\theta\,d\theta = \int \tfrac{\cos^2\theta-1}{\cos\theta}\,d\theta = \int \cos\theta\,d\theta - \int \sec\theta\,d\theta\\ &= \sin\theta - \ln|\sec\theta+\tan\theta| + C = \sqrt{1-x^2} - \ln\Big|\tfrac{1+\sqrt{1-x^2}}{x}\Big| + C.\end{aligned}$$

9.82. Solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y, we get $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$. So $y^2 = \frac{b^2}{a^2}(a^2-x^2)$ and $y = \pm\frac{b}{a}\sqrt{a^2-x^2}$. The $+$ option provides the function $f(x) = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$ that has the upper half of the ellipse as its graph. Since

$$f'(x) = \tfrac{1}{2}\cdot\tfrac{b}{a}(a^2-x^2)^{-\frac{1}{2}}(-2x) = -\tfrac{b}{a}x(a^2-x^2)^{-\frac{1}{2}} = -\tfrac{b}{a}\tfrac{x}{(a^2-x^2)^{\frac{1}{2}}},$$

the length of the upper half of the ellipse is given by the integral

$$\begin{aligned}\int_{-a}^{a}\sqrt{1+\tfrac{b^2}{a^2}\tfrac{x^2}{a^2-x^2}}\,dx &= \int_{-a}^{a}\sqrt{\tfrac{a^2(a^2-x^2)+b^2x^2}{a^2(a^2-x^2)}}\,dx = \int_{-a}^{a}\sqrt{\tfrac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}\,dx\\ &= \int_{-a}^{a}\sqrt{\tfrac{a^4-c^2x^2}{a^2(a^2-x^2)}}\,dx = \int_{-a}^{a}\sqrt{\tfrac{a^2-\varepsilon^2x^2}{a^2-x^2}}\,dx.\end{aligned}$$

The trig substitution $x = a\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $dx = a\cos\theta\,d\theta$ transforms this integral to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\tfrac{a^2-\varepsilon^2a^2\sin^2\theta}{a^2-a^2\sin^2\theta}}(a\cos\theta)\,d\theta = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\tfrac{1-\varepsilon^2\sin^2\theta}{1-\sin^2\theta}}(\cos\theta)\,d\theta = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\sin^2\theta}\,d\theta.$$

9.83. With $a = 1$ and $b = \frac{1}{\sqrt{2}}$, $c^2 = a^2 - b^2 = 1 - \frac{1}{2} = \frac{1}{2}$ and $c = \frac{1}{\sqrt{2}}$. Therefore, $\varepsilon = \frac{c}{a} = \frac{1}{\sqrt{2}}$. So $\sqrt{1 - \varepsilon^2 \sin^2\theta} = \sqrt{1 - \frac{1}{2}\sin^2\theta} = \sqrt{\frac{1}{2} + \frac{1}{2}(1 - \sin^2\theta)} = \frac{1}{\sqrt{2}}\sqrt{1 + \cos^2\theta}$ and therefore

$$\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \sin^2\theta}\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \cos^2\theta}\, d\theta.$$

Since the length of the sine curve between the points $(-\frac{\pi}{2}, -1)$ and $(\frac{\pi}{2}, 1)$ is equal to its length between $(0, 0)$ and $(\pi, 0)$, it follows from Section 9.13 that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \cos^2\theta}\, d\theta$ is the length of one loop of the sine curve.

9.84. Consider $\int_0^3 \cosh x^2\, dx$. Put into the four boxes of the trapezoidal-rule-calculator above: Enter function: = cosh(x^2), $a = 0$, $b = 3$, and successively $n = 10$, 20, 50, 100, and 200. (The trapezoidal and Simpson rules calculators referred to have a limit of 200 on the number of intervals allowed.) Then (possibly after dealing with annoying advertisements) push the *CALCULATE* to get the trapezoidal approximations:

i. $T_{10} \approx 894.6303$, $T_{20} \approx 767.5961$, $T_{50} \approx 729.9901$, $T_{100} \approx 724.5377$, and $T_{200} \approx 723.1714$.

Then do the same thing with the simpsons-rule-calculator to get

ii. $S_{10} \approx 752.6349$, $S_{20} \approx 725.2513$, $S_{50} \approx 722.7877$, $S_{100} \approx 722.7202$, and $S_{200} \approx 722.7160$.

The close agreement between S_{50}, S_{100}, and S_{200} suggests that $S_{200} \approx 722.7160$ is a close approximation of the value of the integral. The convergence of the trapezoidal approximations, on the other hand, is a bit sluggish.

9.85. Type 1/x into the Enter function: = box and repeat this with $\int_1^{100} \frac{1}{x}\, dx$, to get

i. $T_{100} \approx 4.6809$ and $T_{200} \approx 4.6251$.

ii. $S_{100} \approx 4.6176$ and $S_{200} \approx 4.6025$.

iii. We know that $\int_1^x \frac{1}{t}\, dt$ defines the function $\ln x$ for $x \geq 1$. Therefore the actual value of the integral is $\ln 100$, and this is approximated by 4.6052 with accuracy to four decimal places. Once again, the Simpson rule wins the accuracy race.

Finally turn to

http://www.integral-calculator.com/#

and have the site solve the integrals by replacing `e^(x/2) * sin(ax)` in the box by `1/(9+x^3)` for the first integral, `sin(sqrt(x))`, for the second, `e^(-x^2)` for the third, `cosh(x^2)` for the fourth, and `sqrt(1-tan^2(x))` for the last. In each case click on Go! and go to Show steps for the details. What has been achieved for `e^(-x^2)`?