Solutions to Problems and Projects for Chapter 12

We are given a polar coordinate system in the coordinates r and θ along with a superimposed rectangular (or Cartesian) coordinate system with coordinates x and y. The transformation equations $x = r\cos\theta$ and $y = r\sin\theta$ as well as $r = \pm\sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$ that relate the two coordinate systems are relevant throughout.

12.1. **i.** The equation $2x + 3y = 4$ transforms to $2r\cos\theta + 3r\sin\theta = 4$ and hence to $r(2\cos\theta + 3\sin\theta) = 4$. Since $2\cos\theta + 3\sin\theta$ cannot be zero, $r = f(\theta) = \frac{4}{2\cos\theta+3\sin\theta}$.

ii. Since $x^2 + y^2 = 4y$ transforms to $r^2\cos^2\theta + r^2\sin^2\theta = 4r\sin\theta$ and therefore to $r^2(\cos^2\theta + \sin^2\theta) = 4r\sin\theta$, we get $r^2 = 4r\sin\theta$. For $r \neq 0$, this equivalent to the equation $r = 4\sin\theta$ which defines the function $r = f(\theta) = 4\sin\theta$. So the only difference between the equations $r^2 = 4r\sin\theta$ and $r = 4\sin\theta$ involves the origin O. The first equation is satisfied by any representation $(0,\theta)$ of O. The second equation on the other hand is only satisfied by representations $(0,\theta)$ for which $\sin\theta = 0$. In this case, $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

iii. This equation transforms to

$$r^2 = r\cos\theta(r^2\cos^2\theta - 3r^2\sin^2\theta) = r^3\cos\theta(\cos^2\theta - 2\sin^2\theta).$$

If $r \neq 0$, then $r\cos\theta(cos^2\theta - 3\sin^2\theta) = 1$. Because $\cos\theta(\cos^2\theta - 3\sin^2\theta)$ cannot be zero,

$$r = f(\theta) = \frac{1}{\cos\theta(\cos^2\theta - 3\sin^2\theta)}.$$

The graph of this function does not include the origin because r cannot be zero. In all other respects, its graph is the same as that of the Cartesian equation $x^2 + y^2 = x(3x^2 - 3y^2)$.

12.2. **i.** Since $r > 0$, the equation $r = 5$ transforms to $\sqrt{x^2+y^2} = 5$. The distance formula tells us that the graph of this Cartesian equation is the circle of radius 5.

ii. Observe first that the only difference between the equations $r = 3\cos\theta$ and $r^2 = 3r\cos\theta$ occurs when $r = 0$. Since $\cos\frac{\pi}{2} = 0$, the origin is on both graphs. What happens at the origin is analogous to what was observed in the solution of Problem 12.1ii. The first equation is only satisfied by representations $(0,\theta)$ for which $\cos\theta = 0$. The second equation is satisfied by any representation $(0,\theta)$ of O. The polar equation $r^2 = 3r\cos\theta$ transforms directly to the Cartesian (or rectangular) equation $x^2 + y^2 = 3x$.

iii. $\tan\theta = 6$ becomes $\frac{y}{x} = 6$.

iv. Consider the equation $r^2 = 2r\sin\theta\tan\theta$ and note that the only difference between it and $r = 2\sin\theta\tan\theta$ occurs when $r = 0$, hence at the origin. Because $(r,\theta) = (0,0)$ satisfies $r = 2\sin\theta\tan\theta$, its graph is the same as the graph of $r^2 = 2r\sin\theta\tan\theta$. This last equation transforms to $x^2 + y^2 = 2y\cdot\frac{y}{x}$. The graph of this equation does not include the y-axis, namely the line $x = 0$. Any set of polar coordinates of any point on this line (except the origin) involves one of the angles $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. Since these are all angles for which $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is not defined, $r^2 = 2r\sin\theta\tan\theta$ is not defined for any point on this line (except the origin) either.

12.3. **i.** The graph is the set of all (r, θ) with $r = -6$ and θ completely free. Since the distance of any point $(-6, \theta)$ from the origin is 6, all such points lie on the circle of radius 6 centered at the origin. The fact that any point $(6, \theta)$ on this circle can be expressed as $(-6, -\theta)$ means that the graph of the equation $r = -6$ is the entire circle.

ii. The graph is the set of all points of the form $(r, -\frac{8\pi}{6})$ with r arbitrary. The angle $\theta = -\frac{8\pi}{6} = -\frac{4\pi}{3}$ is equal to $-\frac{4\pi}{3} \cdot 180 = -4 \cdot 60 = -240°$. Consider the ray $\theta = -\frac{8\pi}{6}$. Any point on the ray has coordinates $(r, -\frac{8\pi}{6})$ with $r \geq 0$ and any point on the ray in the opposite direction has coordinates $(r, -\frac{8\pi}{6})$ with $r \leq 0$. So the graph of the equation $\theta = -\frac{8\pi}{6}$ is the line that the angle $-\frac{8\pi}{6}$ determines. The Cartesian equation of this line is given by $\tan(-\frac{8\pi}{6}) = \frac{y}{x}$. Because $\tan(-\frac{8\pi}{6}) = -\tan(\frac{4\pi}{3}) = -\tan\frac{\pi}{3} = -\sqrt{3}$, the Cartesian equation is $y = -\sqrt{3}x$.

iii. The graph of $r = 4\sin\theta$ is the same as that of $r^2 = 4r\sin\theta$. Any point (r, θ) with $r \neq 0$ that satisfies one of the equation satisfies the other too. Since $\sin 0 = 0$ the origin is on both graphs as well. Working with $r^2 = 4r\sin\theta$ is easier since it transforms to the Cartesian equation $x^2 + y^2 = 4y$. This equation can be analyzed by completing the square. Because, $x^2 + y^2 - 4y + 2^2 = 2^2 = 4$, we get

$$x^2 + (y-2)^2 = 4.$$

It follows that the graph of $r = 4\sin\theta$ is the circle with center the Cartesian point $(0, 2)$ and radius 2.

iv. The equation $r(\sin\theta + \cos\theta) = 1$ transforms to the Cartesian equation $y + x = 1$, or $y = -x + 1$. This is the line with slope -1 and y-intercept 1.

12.4. As θ varies from $\theta = \pi$ to $\theta = \frac{3\pi}{2}$, the ray that θ determines rotates counterclockwise through the third quarter. Since $r = \sin\theta$ varies from 0 to -1, the points (r, θ) move in the first quadrant from the origin to the Cartesian point $(0, 1)$. So it seems that the right half of the circle of Figure 12.4 is traced out again. As θ varies from $\theta = \frac{3\pi}{2}$ to $\theta = 2\pi$, the ray that θ determines rotates counterclockwise through the fourth quarter. Since $r = \sin\theta$ varies from -1 to 0, the points (r, θ) move in the second quadrant from the Cartesian point $(0, 1)$ to $(-1, 0)$. So it seems that the left half of the circle of Figure 12.4 is traced out once more. The same thing is true for columns $5, 6, 7$ and 8. The fact that any point (r, θ) satisfying $r = \sin\theta$ lies on the graph of the Cartesian equation $x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$ tells us that what "seems" is actually the case.

12.5. The table is drawn in below. As θ varies from 0 to $\frac{\pi}{2}$, $r = \cos\theta$ varies from 1 to 0, so the point (r, θ) traces out a curve from $(1, 0)$ to the origin $(0, \frac{\pi}{2})$. As θ varies from $\frac{\pi}{2}$ to π, $r = \cos\theta$ varies from 0 to -1, so the polar point (r, θ) traces out a curve in the fourth quadrant from the origin back to $(-1, \pi) = (1, 0)$. So as θ varies from 0 to π,

1	2	3	4	5	6	7	8
$0 \le \theta \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \theta \le \pi$	$\pi \le \theta \le \frac{3\pi}{2}$	$\frac{3\pi}{2} \le \theta \le 2\pi$	$0 \ge \theta \ge -\frac{\pi}{2}$	$-\frac{\pi}{2} \ge \theta \ge -\pi$	$-\pi \ge \theta \ge -\frac{3\pi}{2}$	$-\frac{3\pi}{2} \ge \theta \ge -2\pi$
$1 \xrightarrow{\cos\theta} 0$	$0 \xrightarrow{\cos\theta} -1$	$-1 \xrightarrow{\cos\theta} 0$	$0 \xrightarrow{\cos\theta} 1$	$1 \xrightarrow{\cos\theta} 0$	$0 \xrightarrow{\cos\theta} -1$	$-1 \xrightarrow{\cos\theta} 0$	$0 \xrightarrow{\cos\theta} 1$

the point (r, θ) traces out a loop to the right of the y-axis. What loop is this exactly? Since the graph of $r = \cos\theta$ includes the origin, it is identical to the graph of $r^2 = r\cos\theta$. In Cartesian coordinates this is the equation is $x^2 + y^2 = x$. Completing the square for $x^2 - x + y^2 = 0$, we get $x^2 - x + (\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$ and hence $(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$. It follows that the graph of $r = \cos\theta$ is the circle of radius $\frac{1}{2}$ centered at the Cartesian point $(\frac{1}{2}, 0)$.

12.6. For $ax + by + c = 0$ to represent a line, it has to be assumed that one of a or b is not zero. (If both are zero, then c is zero as well, so that any point (x, y) satisfies the equation and the graph is the entire plane.)

The polar version of the equation $ax + by + c = 0$ is $ar\cos\theta + br\sin\theta + c = 0$. So $r(a\cos\theta + b\sin\theta) = -c$. If $c \ne 0$, then $a\cos\theta + b\sin\theta$ is never 0 and

$$r = f(\theta) = \tfrac{-c}{a\cos\theta + b\sin\theta}$$

is a polar function with graph the given line.

Let's assume that $c = 0$. In this case, the line is not the graph of a polar function. Consider the case $b \ne 0$ and hence the line $y = -\frac{a}{b}x$. Let θ_0 with $-\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$ be the angle the line makes with the polar axis. Note that $\tan\theta_0 = \frac{y}{x} = -\frac{a}{b}$ is the slope of the line. For any point (r, θ) on the line (that is not the origin), θ must be one of the angles $\theta_0, \theta_0 \pm \pi, \theta_0 \pm 2\pi, \dots$. Assume, if possible, that the line is the graph of a polar function $r = f(\theta)$. Since any point on the line is on the graph, there must be for any real number r a θ such that $r = f(\theta)$. But any such θ must be one of the angles $\theta_0, \theta_0 \pm \pi, \theta_0 \pm 2\pi, \dots$, so that there are not enough θs to provide all real values r.

12.7. Each equation has the form $r = \frac{d}{1+\varepsilon\cos\theta}$ with $d > 0$ and $\varepsilon \ge 0$. So the graphs are conic sections.

i. In these two cases $\varepsilon = 1$ and the graphs are both parabolas with focal point the origin O and a vertical directrix $x = d$. The directrix in the first case is the line $x = 4$ and in the second case it is $x = 8$. Given these facts the parabolas are easily sketched. The Cartesian equation is $\sqrt{x^2 + y^2} + x = d$. Since $x^2 + y^2 = d^2 - 2dx + x^2$ and hence $y^2 = d^2 - 2dx$, a more transparent form of the equation is $y^2 = -2d(x - \frac{d}{2})$.

ii. Here $\varepsilon < 1$ so that each graph is an ellipse with eccentricity ε, semimajor axis $a = \frac{d}{1-\varepsilon^2}$, and semiminor axis $b = \frac{d}{\sqrt{1-\varepsilon^2}}$. In each case, the focal points are the origin and $(-2\varepsilon a, 0)$ (in either polar or Cartesian coordinates) and the graph is obtained by shifting the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by εa units to the left.

In the first case, $\varepsilon = \frac{1}{5}, d = 2$,

$$a = \frac{2}{1-(\frac{1}{5})^2} = \frac{2}{\frac{24}{25}} = \frac{25}{12},\ b = \frac{2}{\sqrt{1-(\frac{1}{5})^2}} = \frac{2}{\sqrt{\frac{24}{25}}} = \frac{10}{\sqrt{24}} = \frac{5}{\sqrt{6}},$$

and the focal points are $(0,0)$ and $(-\frac{5}{6}, 0)$. In the second case, $\varepsilon = \frac{1}{2}, d = 5$,

$$a = \frac{5}{1-(\frac{1}{2})^2} = \frac{5}{\frac{3}{4}} = \frac{20}{3},\ b = \frac{5}{\sqrt{1-(\frac{1}{2})^2}} = \frac{5}{\sqrt{\frac{3}{4}}} = \frac{10}{\sqrt{3}},$$

and the focal points are $(0,0)$ and $(-\frac{20}{3}, 0)$.

iii. Now $\varepsilon > 1$ so that each graph is a hyperbola with eccentricity ε, semimajor axis $a = \frac{d}{\varepsilon^2-1}$, and semiminor axis $b = \frac{d}{\sqrt{\varepsilon^2-1}}$. In each case, the focal points are the origin and $(-2\varepsilon a, 0)$ and the graph is obtained by shifting the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by εa units to the left.

In the first case, $\varepsilon = 3, d = 3, a = \frac{3}{3^2-1} = \frac{3}{8}, b = \frac{3}{\sqrt{3^2-1}} = \frac{3}{\sqrt{8}}$, and the focal points are $(0,0)$ and $(-\frac{9}{4}, 0)$. In the second case, $\varepsilon = 5, d = \frac{1}{2}, a = \frac{\frac{1}{2}}{5^2-1} = \frac{1}{48}, b = \frac{\frac{1}{2}}{\sqrt{5^2-1}} = \frac{1}{2\sqrt{24}} = \frac{1}{4\sqrt{6}}$, and the focal points are $(0,0)$ and $(-\frac{5}{24}, 0)$.

12.8. We'll start with the equation $r = \frac{d}{1+\varepsilon\cos\theta}$ and determine ε and d in each case.

i. By Section 12.2 part (i), the directrix of the parabola is the vertical line $x = d$ and the focus is the origin. So $d = 7$ and $\varepsilon = 1$. Therefore the equation is $r = \frac{7}{1+\cos\theta}$.

ii. Since the semimajor and semiminor axes of the ellipse are $a = 6$ and $b = 4$ respectively, we know from Section 12.2 part (ii) that $\frac{d}{1-\varepsilon^2} = 6$ and $\frac{d}{\sqrt{1-\varepsilon^2}} = 4$. Since $\frac{d^2}{1-\varepsilon^2} = 16$, it follows that $d = \frac{d^2}{1-\varepsilon^2} \cdot \frac{1-\varepsilon^2}{d} = \frac{16}{6} = \frac{8}{3}$. Since $\frac{d}{1-\varepsilon^2} = 6$, we get $1 - \varepsilon^2 = \frac{d}{6} = \frac{8}{18} = \frac{4}{9}$. So $\varepsilon^2 = \frac{5}{9}$ and $\varepsilon = \frac{\sqrt{5}}{3}$. Therefore the equation that we are looking for is $r = \frac{\frac{8}{3}}{1+\frac{\sqrt{5}}{3}\cos\theta} = \frac{8}{3+\sqrt{5}\cos\theta}$.

iii. The fact that the semimajor and semiminor axes of the hyperbola are $a = 6$ and $b = 4$ respectively, together with the information in Section 12.2 part (iii) tell us that $\frac{d}{\varepsilon^2-1} = 6$ and $\frac{d}{\sqrt{\varepsilon^2-1}} = 4$. Since $\frac{d^2}{\varepsilon^2-1} = 16$, we get $d = \frac{d^2}{\varepsilon^2-1} \cdot \frac{\varepsilon^2-1}{d} = \frac{16}{6} = \frac{8}{3}$. Since $\frac{d}{\varepsilon^2-1} = 6, \varepsilon^2 - 1 = \frac{d}{6} = \frac{8}{18} = \frac{4}{9}$. Therefore $\varepsilon^2 = \frac{13}{9}$ and $\varepsilon = \frac{\sqrt{13}}{3}$. So the equation that we are looking for is $r = \frac{\frac{8}{3}}{1+\frac{\sqrt{13}}{3}\cos\theta} = \frac{8}{3+\sqrt{13}\cos\theta}$.

12.9. The following observation solves the problem. Suppose that two parabolas A and B in the plane are related by the fact that the distances between the focal points and directrixes are the same. Then parabola A can be moved in the plane in such a way that the moved parabola A′ and the parabola B have the same focal point and directrix. So the parabolas A′ and B coincide. This means that A and B have the same shape.

12.10. The ellipse discussed in Section 12.2 part (ii) has semimajor axis $a = \frac{d}{1-\varepsilon^2}$ and semiminor axis $b = \frac{d}{\sqrt{1-\varepsilon^2}}$. Assuming that $a = 7$ and $b = 4$, we see that $\frac{d}{1-\varepsilon^2} = 7$ and $\frac{d}{\sqrt{1-\varepsilon^2}} = 4$. Therefore $\frac{d^2}{1-\varepsilon^2} = 16$ and it follows that $d = \frac{d^2}{1-\varepsilon^2} \cdot \frac{1-\varepsilon^2}{d} = \frac{16}{7}$. Since $\frac{d}{1-\varepsilon^2} = 7$, we see that $1 - \varepsilon^2 = \frac{d}{7} = \frac{16}{49}$. So $\varepsilon^2 = \frac{33}{49}$ and $\varepsilon = \frac{\sqrt{33}}{7}$. This means that the ellipse $(*)$ given by $r = \frac{\frac{16}{7}}{1+\frac{\sqrt{33}}{7}\cos\theta}$ has semimajor axis $a = 7$ and semiminor axis $b = 4$. The ellipse C and the ellipse $(*)$ can each be moved in the plane to coincide with the ellipse that has equation $\frac{x^2}{7^2} + \frac{y^2}{4^2} = 1$. Therefore the ellipses C and $(*)$ have the same shape.

12.11. The hyperbola studied in Section 12.2 part (iii) has semimajor axis $a = \frac{d}{\varepsilon^2-1}$ and semiminor axis is $b = \frac{d}{\sqrt{\varepsilon^2-1}}$. With $a = 5$ and $b = 3$, we see that $\frac{d}{\varepsilon^2-1} = 5$ and $\frac{d}{\sqrt{\varepsilon^2-1}} = 3$. Therefore $\frac{d^2}{\varepsilon^2-1} = 9$ and hence $d = \frac{d^2}{\varepsilon^2-1} \cdot \frac{\varepsilon^2-1}{d} = \frac{9}{5}$. Since $\frac{d}{\varepsilon^2-1} = 5$, it follows that $\varepsilon^2 - 1 = \frac{d}{5} = \frac{9}{25}$. So $\varepsilon^2 = \frac{34}{25}$ and $\varepsilon = \frac{\sqrt{34}}{5}$. Therefore the hyperbola $(*)$ given by $r = \frac{\frac{9}{5}}{1+\frac{\sqrt{34}}{5}\cos\theta}$ has semimajor axis $a = 5$ and semiminor axis $b = 3$. The hyperbola C and the hyperbola $(*)$ can each be moved to coincide with the hyperbola that has equation $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$. Therefore the hyperbolas C and $(*)$ have the same shape.

12.12. The ellipse with equation $r = \frac{d}{1+\varepsilon\cos\theta}$ and $\varepsilon < 1$ discussed in Section 12.2 part (ii) has semimajor axis $a = \frac{d}{1-\varepsilon^2}$ and semiminor axis $b = \frac{d}{\sqrt{1-\varepsilon^2}}$. Since $b^2 = \frac{d^2}{1-\varepsilon^2}$ we see that $d = \frac{d^2}{1-\varepsilon^2} \cdot \frac{1-\varepsilon^2}{d} = \frac{b^2}{a}$. Since $\frac{1-\varepsilon^2}{d} = \frac{1}{a}$, we get $1 - \varepsilon^2 = \frac{d}{a} = \frac{b^2}{a^2}$. Hence $\varepsilon^2 = 1 - \frac{b^2}{a^2} = \frac{a^2-b^2}{a^2}$ and $\varepsilon = \frac{\sqrt{a^2-b^2}}{a}$. It follows that the graph of the equation $r = \frac{\frac{b^2}{a}}{1+\frac{\sqrt{a^2-b^2}}{a}\cos\theta}$ is an ellipse with semimajor axis a and semiminor axis b. Since this ellipse and the ellipse C have the same semimajor and semiminor axes, they have the same shape.

12.13. The hyperbola with equation $r = \frac{d}{1+\varepsilon\cos\theta}$ and $\varepsilon > 1$ discussed in part (iii) of Section 12.2 has semimajor axis $a = \frac{d}{\varepsilon^2-1}$ and semiminor axis $b = \frac{d}{\sqrt{\varepsilon^2-1}}$. Since $b^2 = \frac{d^2}{\varepsilon^2-1}$ we get $d = \frac{d^2}{\varepsilon^2-1} \cdot \frac{\varepsilon^2-1}{d} = \frac{b^2}{a}$ and since $\frac{\varepsilon^2-1}{d} = \frac{1}{a}$, we see that $\varepsilon^2 - 1 = \frac{d}{a} = \frac{b^2}{a^2}$. Hence $\varepsilon^2 = 1 + \frac{b^2}{a^2} = \frac{a^2+b^2}{a^2}$ and $\varepsilon = \frac{\sqrt{a^2+b^2}}{a}$. So the graph of the equation $r = \frac{\frac{b^2}{a}}{1+\frac{\sqrt{a^2+b^2}}{a}\cos\theta}$ is a hyperbola with semimajor axis a and semiminor axis b. Because it has the same semimajor and semimajor axes as hyperbola C, it has the same shape as C.

12.14. The equation $r = \frac{d}{1+\varepsilon\sin\theta}$, where $d > 0$ and $\varepsilon \geq 0$, can be written as $r + \varepsilon r\sin\theta = d$. The corresponding Cartesian equation is $\pm\sqrt{x^2+y^2} + \varepsilon y = d$.

Let's begin with the case $\varepsilon = 1$. Since $\sqrt{x^2+y^2} \geq y$ and $d > 0$, it follows that the minus alternative does not occur and hence that $\sqrt{x^2+y^2} + y = d$. That this is an equation of the parabola with focal point the origin and directrix the horizontal line $y = d$ can be seen as follows. Let $P = (x, y)$ be any point in the on the parabola. Its distance from the origin is $\sqrt{x^2+y^2}$. Since $d > 0$ the directrix lies above the focal point. It follows that $y < d$ and that the distance from P to the line $y = d$ is $d - y$.

So $\sqrt{x^2+y^2} = d - y$ and hence $\sqrt{x^2+y^2} + y = d$. Since this is the equation derived earlier it follows that the graph of $r = \frac{d}{1+\sin\theta}$ is a parabola with focal point the origin and directrix the line $y = d$.

We'll now suppose that $\varepsilon \neq 1$. Squaring both sides of $\pm\sqrt{x^2+y^2} = d - \varepsilon y$, we get in successive steps (one of them a completion of a square)

$$x^2 + y^2 = d^2 - 2\varepsilon dy + \varepsilon^2 y^2$$
$$x^2 + (1-\varepsilon^2)y^2 + 2\varepsilon dy = d^2$$
$$\tfrac{x^2}{1-\varepsilon^2} + y^2 + \tfrac{2\varepsilon d}{1-\varepsilon^2}y = \tfrac{d^2}{1-\varepsilon^2}$$
$$\tfrac{x^2}{1-\varepsilon^2} + y^2 + \tfrac{2\varepsilon d}{1-\varepsilon^2}y + \tfrac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2} = \tfrac{d^2}{1-\varepsilon^2} + \tfrac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2}$$
$$\tfrac{x^2}{1-\varepsilon^2} + \big(y + \tfrac{\varepsilon d}{1-\varepsilon^2}\big)^2 = \tfrac{(1-\varepsilon^2)d^2 + \varepsilon d^2}{(1-\varepsilon^2)^2} = \big(\tfrac{d}{1-\varepsilon^2}\big)^2, \text{ and}$$
$$\frac{x^2}{\frac{d^2}{1-\varepsilon^2}} + \frac{(y+\frac{\varepsilon d}{1-\varepsilon^2})^2}{(\frac{d}{1-\varepsilon^2})^2} = 1.$$

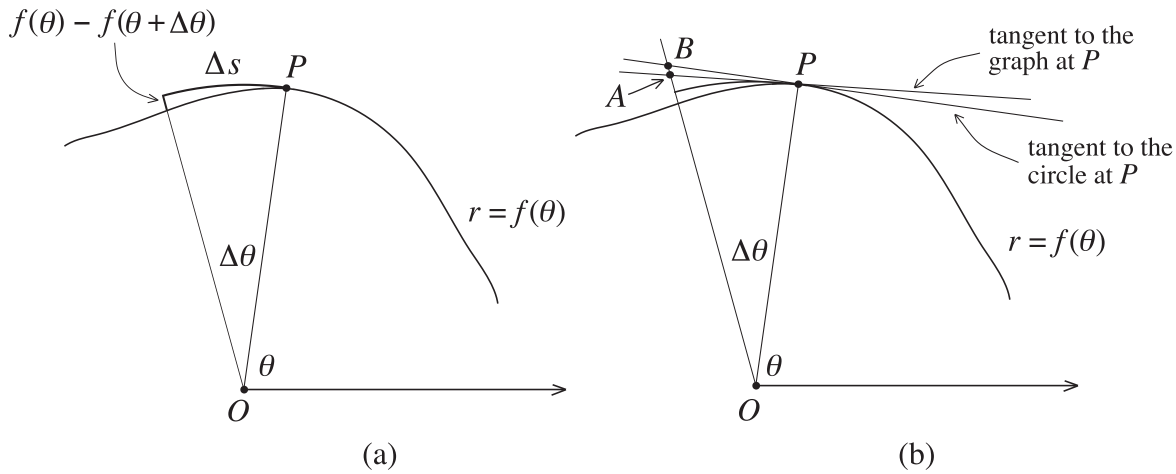
Now proceed as in cases (ii) and (iii) of Section 12.2 to show that the graph of this equation is an ellipse if $\varepsilon < 1$ and a hyperbola if $\varepsilon > 1$. In each case the y-axis is the focal axis and the origin is the upper focal point.

12.15. Over the interval $0 < \theta < \pi$, $r = f(\theta) = \sin\theta$ is positive and equal to the distance from (r,θ) to the origin. A look at the graph of Figure 12.4 shows that this distance increases from 0 to 1 as the ray determined by θ rotates from $\theta = 0$ to $\theta = \frac{\pi}{2}$ and decreases from 1 back to 0 as this ray moves from $\frac{\pi}{2}$ to π. The point traces out the circle in the process. Over the interval $\pi < \theta < 2\pi$, $r = f(\theta) = \sin\theta$ is negative so that the distance from (r,θ) to the origin is $-r = -f(\theta) = -\sin\theta$. As the ray determined by θ rotates from $\theta = \pi$ to $\theta = 2\pi$ the point (r,θ) traces out the circle again. In the process, its distance $-\sin\theta$ from the origin increases over $\pi \leq \theta \leq \frac{3\pi}{2}$ and decreases over $\frac{3\pi}{2} \leq \theta \leq 2\pi$. The derivative $f'(\theta) = \cos\theta$ reflects this behavior of the graph. It is positive over $0 \leq \theta \leq \frac{\pi}{2}$ and negative over $\frac{\pi}{2} < \theta < \pi$. Over $\pi < \theta < 2\pi$, $-\cos\theta$ is positive for $\pi < \theta < \frac{3\pi}{2}$ and negative over $\frac{3\pi}{2} < \theta < 2\pi$.

12.16. Let $P = (f(\theta), \theta)$ be any point on the graph of a polar function $f(\theta)$. Assume that P is not the origin O, so that $f(\theta) \neq 0$. Consider the point $(f(\theta+\Delta\theta), \theta+\Delta\theta)$ for a small positive $\Delta\theta$. Draw the segment from O to this point and put in the circular arc with center O and radius $f(\theta)$ between the rays determined by θ and $\theta+\Delta\theta$. Figure (a) below sketches a situation where the graph of the function lies below this circular arc. The curving triangle of the figure is the *beak* at P. With Δs the length of the circular arc the radian measure of $\Delta\theta$ is $\Delta\theta = \frac{\Delta s}{f(\theta)}$. So $\frac{1}{\Delta\theta} = \frac{1}{\Delta s}\cdot f(\theta)$. After a substitution,

$$\frac{f(\theta+\Delta\theta) - f(\theta)}{\Delta\theta} = \frac{f(\theta+\Delta\theta) - f(\theta)}{\Delta s} \cdot f(\theta).$$

Put in the tangent line *to the graph* of $r = f(\theta)$ at P, and let A be the point of



intersection of the tangent with the ray determined by $\theta + \Delta\theta$. Also put in the tangent line *to the circle* at P, and let B be the point of intersection of this tangent and the same ray. The two tangent lines and the ray form the triangle ΔAPB that we call the *triangle* at P. See Figure (b).

We'll now push $\Delta\theta$ to 0 and investigate $\lim\limits_{\Delta\theta\to 0} \dfrac{f(\theta+\Delta\theta)-f(\theta)}{\Delta s}$. As $\Delta\theta$ is pushed to 0, the segment OAB rotates toward the segment OP. Both the beak at P and the triangle at P shrink in the direction of their tips at P. The shrinking triangle approximates the shrinking beak better and better as the gap between OBA and OP closes. In the process, Δs gets closer to BP and $f(\theta)-f(\theta+\Delta\theta) = -(f(\theta+\Delta\theta)-f(\theta))$ to AB. Therefore, as $\Delta\theta$ is pushed to 0,

$$\frac{-(f(\theta+\Delta\theta)-f(\theta))}{\Delta s} \text{ closes in on the ratio } \frac{AB}{BP}.$$

Because the tangent line to a circle at a point is perpendicular to its radius to the point, we know that the angle at P between PO and PB is $\frac{\pi}{2}$. So as $\Delta\theta$ shrinks to 0, the angle $\angle PBA$ approaches $\frac{\pi}{2}$, and the triangle ΔAPB approaches a right triangle with right angle at B. It follows that the ratio $\frac{AB}{BP}$ closes in on the tangent of the angle $\angle APB$. Because $\angle APO = \gamma$ and $\angle BPO = \frac{\pi}{2}$, the angle $\angle APB = \frac{\pi}{2} - \gamma$. By putting it all together, we have demonstrated that as $\Delta\theta$ shrinks to 0

$$\frac{-(f(\theta+\Delta\theta)-f(\theta))}{\Delta s} \text{ closes in on } \frac{AB}{BP} \text{ and this in turn on } \tan(\tfrac{\pi}{2}-\gamma).$$

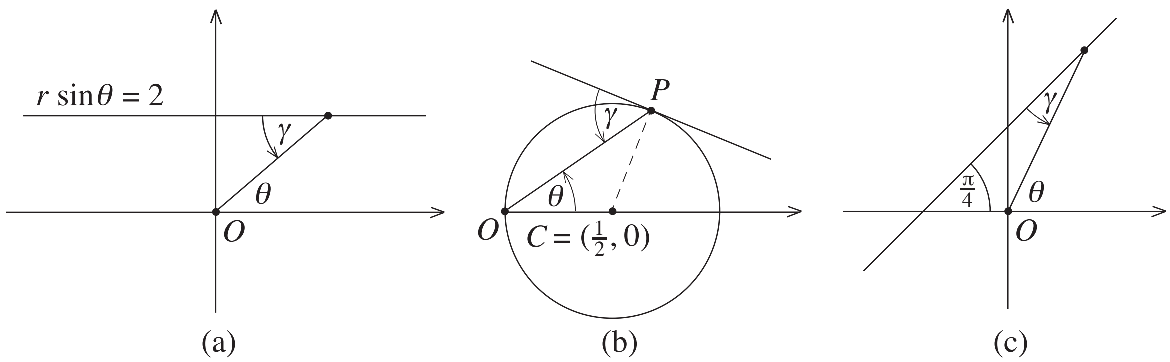
Since $\tan(\frac{\pi}{2}-\gamma) = -\tan(\gamma-\frac{\pi}{2})$ we have verified that

$$\lim_{\Delta\theta\to 0} \frac{f(\theta+\Delta\theta)-f(\theta)}{\Delta s} = \tan(\gamma - \tfrac{\pi}{2}).$$

We have now arrived at the conclusion $f'(\theta) = f(\theta)\cdot\tan(\gamma(\theta)-\frac{\pi}{2})$ in the case of Figure (a). The notation $\gamma(\theta)$ emphasizes the fact that γ depends on θ that is to say, $\gamma = \gamma(\theta)$ is a function of θ.

12.17. Consider the function $r = f(\theta) = \frac{2}{\sin\theta}$. Rewriting the equation as $r\sin\theta = 2$ gives us the Cartesian version $y = 2$. That $\gamma(\theta) = \theta$ follows from elementary geometry. See Figure (a) below. To confirm the equality $f'(\theta) = f(\theta)\cdot\tan(\theta - \frac{\pi}{2})$, observe first that $f'(\theta) = -2(\sin\theta)^{-2}\cos\theta = -2\frac{\cos\theta}{\sin^2\theta}$. Next, we'll rewrite $\tan(\theta - \frac{\pi}{2})$. Using a diagram similar to Figure 4.25 and the argument in Section 4.6 that establishes the identities $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta + \frac{\pi}{2}) = \cos\theta$, we get the identities $\cos(\theta - \frac{\pi}{2}) = \sin\theta$ and $\sin(\theta - \frac{\pi}{2}) = -\cos\theta$. It follows that $\tan(\theta - \frac{\pi}{2}) = \frac{\sin(\theta - \frac{\pi}{2})}{\cos(\theta - \frac{\pi}{2})} = \frac{-\cos\theta}{\sin\theta}$. Therefore $f(\theta)\cdot\tan(\theta - \frac{\pi}{2})$ is also equal to $-2\frac{\cos\theta}{\sin^2\theta}$.

12.18. Consider $f(\theta) = \cos\theta$. Its graph (described in Problem 12.5) is the circle with center the polar (and Cartesian) point $C = (\frac{1}{2}, 0)$ shown in Figure (b) below. Since the radius CP is perpendicular to the tangent at P, the angles γ and $\angle OPC$ add to $\frac{\pi}{2}$. Since the triangle ΔOCP is isosceles, $\angle OPC = \theta$. So $\gamma + \theta = \frac{\pi}{2}$ and $\gamma - \frac{\pi}{2} = -\theta$. It remains to



check that $-\sin\theta = \cos\theta\cdot\tan(-\theta)$. But this is so since $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin\theta}{\cos\theta}$.

12.19. After $r = f(\theta) = \frac{1}{\sin\theta - \cos\theta}$ is rewritten as $r(\sin\theta - \cos\theta) = 1$, we recognize that the Cartesian version of this equation is $y = x + 1$. Its graph is the slanted line depicted in Figure (c) above. The triangle in the figure tells us that $\gamma + \frac{\pi}{4} + (\pi - \theta) = \pi$. It follows that $\gamma = \theta - \frac{\pi}{4}$ and hence that $\tan(\gamma - \frac{\pi}{2}) = \tan(\theta - \frac{3\pi}{4})$. By applying the addition formulas for the sine and cosine to $\sin(\theta - \frac{3\pi}{4})$ and $\cos(\theta - \frac{3\pi}{4})$ we get

$$\tan(\theta - \tfrac{3\pi}{4}) = \frac{\sin(\theta - \frac{3\pi}{4})}{\cos(\theta - \frac{3\pi}{4})} = \frac{\sin\theta\cos(-\frac{3\pi}{4}) + \cos\theta\sin(-\frac{3\pi}{4})}{\cos\theta\cos(-\frac{3\pi}{4}) - \sin\theta\sin(-\frac{3\pi}{4})}.$$

Since $\cos(-\frac{3\pi}{4}) = \cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$ and $\sin(-\frac{3\pi}{4}) = -\sin(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$, it follows that $\tan(\theta - \frac{3\pi}{4}) = \frac{-\frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta}{-\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$. Because $f(\theta) = (\sin\theta - \cos\theta)^{-1}$, we get

$$f'(\theta) = -(\sin\theta - \cos\theta)^{-2}(\cos\theta + \sin\theta) = -\frac{\cos\theta + \sin\theta}{(\sin\theta - \cos\theta)^2}.$$

The fact that $f(\theta)\tan(\theta - \frac{3\pi}{4}) = \frac{1}{\sin\theta - \cos\theta}\cdot\frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} = -\frac{\sin\theta + \cos\theta}{(\sin\theta - \cos\theta)^2}$ verifies the formula $f'(\theta) = f(\theta)\cdot\tan(\gamma - \frac{\pi}{2})$.

12.20. In this problem the angles γ and φ are restricted $0 \le \gamma < \pi$ and $0 \le \varphi < \pi$ (but there are no restrictions on θ). The concern is the determination of the slope $\tan\varphi$ of the graph (in the Cartesian context) in terms of polar data. Observe that if $\gamma(\theta) = 0$, then $\tan(\gamma(\theta) - \frac{\pi}{2}) = \tan(-\frac{\pi}{2})$. In this case both $\tan(\gamma(\theta) - \frac{\pi}{2})$ and $f'(\theta)$ are undefined.

i. If $f(\theta) \neq 0$ the formula $f'(\theta) = f(\theta) \cdot \tan(\gamma - \frac{\pi}{2})$ tells us that $\tan(\gamma - \frac{\pi}{2}) = \frac{f'(\theta)}{f(\theta)}$. The inverse tangent $\tan^{-1}\frac{f'(\theta)}{f(\theta)}$ is the angle ϕ with $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ such that $\tan\phi = \tan(\gamma - \frac{\pi}{2})$. The fact that γ satisfies $0 \leq \gamma < \pi$ and hence $-\frac{\pi}{2} \leq \gamma - \frac{\pi}{2} < \frac{\pi}{2}$ means that $\gamma - \frac{\pi}{2} = \phi$ except when $\gamma = 0$ when $\tan(\gamma - \frac{\pi}{2})$ is not defined. Therefore either $\gamma = \tan^{-1}\frac{f'(\theta)}{f(\theta)} + \frac{\pi}{2}$ or $\gamma = 0$. Let's turn to Figure 12.36.

Adding up the angles of the triangle ΔABP tells us that $\varphi + (\pi - \theta) + \gamma = \pi$ in case (a) and $\theta + (\pi - \varphi) + (\pi - \gamma) = \pi$ in case (b). So $\varphi = \theta - \gamma$ in case (a) and $\varphi = \theta - \gamma + \pi$ in case (b). The fact that $\sin(\phi \pm \pi) = -\sin\phi$ and $\cos(\phi \pm \pi) = -\cos\phi$ for any angle ϕ (see Example 4.18 for instance) tells us that $\tan(\phi \pm \pi) = \tan\phi$ and hence that the slope of the tangent line is

$$\tan\varphi = \tan(\theta - \gamma)$$

in either case. It is not hard to check that this equality also holds when ΔABP is "degenerate" (for instance if $\gamma = 0$, or if θ is π in case (a) or 0 in case (b)).

ii. Differentiating the equations $x = r\cos\theta$ and $y = r\sin\theta$ (using the product rule) we get $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$. Since $\frac{dy}{dx}$ is the slope of the graph in Cartesian terms it follows that

$$\tan\varphi = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \cdot \sin\theta + r\cos\theta}{\frac{dr}{d\theta} \cdot \cos\theta - r\sin\theta}.$$

If $r = f(\theta) = 0$ and $\frac{dr}{d\theta} = f'(\theta) \neq 0$ this simplifies to $\tan\varphi = \frac{\sin\theta}{\cos\theta} = \tan\theta$.

12.21. We'll start by computing the angle γ in all these cases. The values of $f'(\theta)/f(\theta) = -\frac{\sin\theta}{1+\cos\theta}$ for θ equal to $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$ are computed in the table below. Since $f'(\theta)$ is

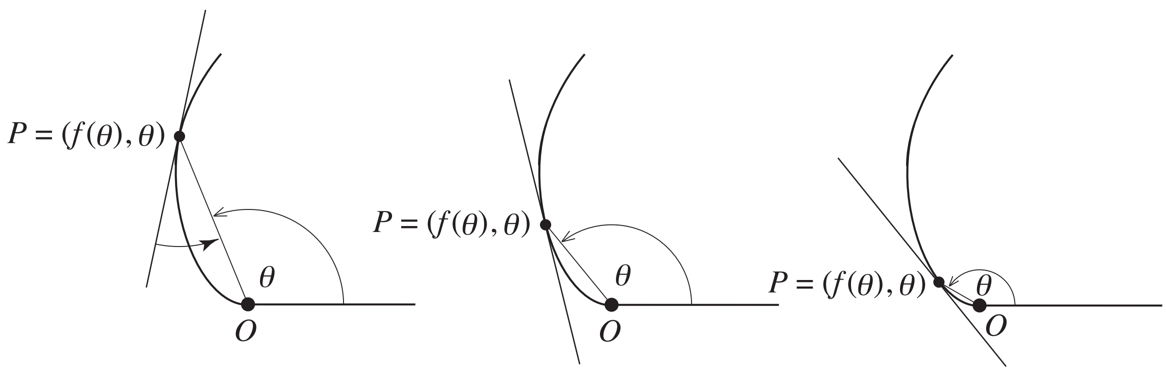
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f'(\theta)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$f(\theta)$	2	$1+\frac{\sqrt{3}}{2}$	$1+\frac{\sqrt{2}}{2}$	$1+\frac{1}{2}$	1
$f'(\theta)/f(\theta)$	0	$\frac{-1}{1+\sqrt{3}}$	$\frac{-1}{1+\sqrt{2}}$	$\frac{-1}{\sqrt{3}}$	-1

defined for these angles, $\gamma \neq 0$ and hence $\gamma = \tan^{-1}\frac{f'(\theta)}{f(\theta)} + \frac{\pi}{2}$ for each of them. The values of $\tan^{-1}\frac{f'(\theta)}{f(\theta)}$ for the numbers in the last row of the table are $0, -0.3509, -0.3927, -\frac{\pi}{6}$, and $-\frac{\pi}{4}$ with the second and third being approximations. (Use the inverse tan button of a calculator or the definition of tan). After adding $\frac{\pi}{2} \approx 1.5708$, we find that γ is equal to $\frac{\pi}{2} = 90°, 1.2199 \approx 69.90°, 1.1781 \approx 67.50°, \frac{\pi}{3} = 60°$, and $\frac{\pi}{4} = 45°$ for θ equal to $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$, respectively. Since $f(\theta) \neq 0$ for these angles, case (i) of Problem 12.20 applies to tell us that corresponding values of $\tan\varphi = \tan(\theta - \gamma)$ are the undefined $\tan(-\frac{\pi}{2})$, $\tan(\frac{\pi}{6} - 1.2199) \approx -0.84, \tan(\frac{\pi}{4} - 1.1781) \approx -0.41, \tan(\frac{\pi}{3} - \frac{\pi}{3}) = 0$ and $\tan(\frac{\pi}{2} - \frac{\pi}{4}) = 1$, respectively. These numbers are the slopes of the tangent lines to the graph of $f(\theta) = 1 + \cos\theta$ for θ equal to $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$, respectively. Check that they are consistent with the way the graph of $f(\theta) = 1 + \cos\theta$ in Figure 12.38 rises and falls. Notice that the tangent is vertical for $\theta = 0$ and horizontal for $\theta = \frac{\pi}{3}$.

Finally to the slope of the graph at the origin $(0, \pi)$. For $\theta < \pi$ and close to π the graph lies above the polar axis and we'll study how it flows into the point $(0, \pi)$ with a look at $\lim\limits_{\theta\to\pi^-}$. Since $\sin\theta > 0$ for $\theta < \pi$ and close to π, we know that

$$\lim_{\theta\to\pi^-} \frac{f'(\theta)}{f(\theta)} = \lim_{\theta\to\pi^-} \left(\frac{-\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta}\right) = \lim_{\theta\to\pi^-} \frac{\cos\theta - 1}{\sin\theta} = -\infty.$$

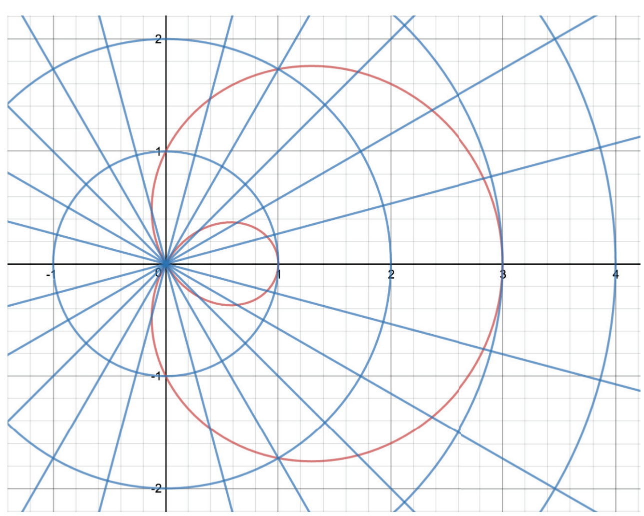
A look at Figure 9.33 tells us that $\lim\limits_{\theta\to\pi^-} \tan^{-1}\frac{f'(\theta)}{f(\theta)} = -\frac{\pi}{2}$ and hence that $\lim\limits_{\theta\to\pi^-} \gamma(\theta) = 0$. The figure below illustrates what we can conclude. As θ is pushed to π the segment from the origin O to the point $P = (f(\theta), \theta)$ flows into horizontal position and the angle



between the tangent at P and the segment OP goes to zero. It follows that the tangent at O is the horizontal line $\theta = \pi$. (An analysis of $\lim\limits_{\theta\to\pi^+}$ shows that this is also the case for the graph below the polar axis.)

12.22. The graph of $r = 1 + 2\cos\theta$ is sketched below with

https://www.desmos.com/calculator/ms3eghkkgz (polar graphing calculator)



$r = 1 + 2\cos\theta$

12.23. Since $f(\theta) = \sin 2\theta$ and $f'(\theta) = 2\cos 2\theta$, we get the table

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f'(\theta)$	2	1	0	-1	-2
$f(\theta)$	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0
$f'(\theta)/f(\theta)$	undefined	$\frac{2}{\sqrt{3}}$	0	$-\frac{2}{\sqrt{3}}$	undefined

We'll study $\gamma(\theta) = \tan^{-1}\frac{f'(\theta)}{f(\theta)} + \frac{\pi}{2}$ and $\tan(\theta - \gamma(\theta))$ for the specified angles θ.

Start with $\theta = 0$. Since $\frac{f'(\theta)}{f(\theta)}$ is not defined, we'll consider $\lim\limits_{\theta\to 0^+} \tan^{-1}\frac{f'(\theta)}{f(\theta)}$. The fact that $\lim\limits_{\theta\to 0^+} \frac{\cos 2\theta}{\sin 2\theta} = +\infty$ in combination with Figure 9.33 tells us that

$$\lim_{\theta\to 0^+} \tan^{-1}\tfrac{f'(\theta)}{f(\theta)} = \lim_{\theta\to 0^+} \tan^{-1}\left(\tfrac{\cos 2\theta}{\sin 2\theta}\right) = \tfrac{\pi}{2}$$

and hence that $\lim\limits_{\theta\to 0^+} \gamma(\theta) = \lim\limits_{\theta\to 0^+} \tan^{-1}\frac{f'(\theta)}{f(\theta)} + \frac{\pi}{2} = \pi$. So $\lim\limits_{\theta\to 0^+} \tan(\theta - \gamma(\theta)) = 0$ and it follows that the tangent to the graph at $(0,0)$ is horizontal. This is in agreement with the depiction of the graph in Figure 12.40.

We take $\theta = \frac{\pi}{6}$ next. A calculator tells us that $\tan^{-1}(\frac{2}{\sqrt{3}}) \approx 0.8571$, so that $\gamma(\frac{\pi}{6}) \approx 0.8571 + 1.5708 = 2.4279$ and $\tan(\frac{\pi}{6} - \gamma(\frac{\pi}{6})) \approx \tan(-1.9043) \approx 2.8865$. Moving to $\theta = \frac{\pi}{4}$, we get $\tan^{-1}(0) = 0$ so that $\gamma(\frac{\pi}{4}) = 0 + \frac{\pi}{2}$ and hence that $\tan(\frac{\pi}{4} - \gamma(\frac{\pi}{4})) = \tan(\frac{\pi}{4} - \frac{\pi}{2}) = \tan(\frac{-\pi}{4}) = -1$. For $\theta = \frac{\pi}{3}$ we see that $\tan^{-1}(-\frac{2}{\sqrt{3}}) \approx -0.8571$ so that $\gamma(\frac{\pi}{3}) \approx -0.8571 + 1.5708 = 0.7137$ and $\tan(\frac{\pi}{3} - \gamma(\frac{\pi}{3})) \approx \tan 0.3335 \approx 0.3464$.

Finally to $\theta = \frac{\pi}{2}$. Here too $\frac{f'(\theta)}{f(\theta)}$ is undefined, so that we'll consider $\lim\limits_{\theta\to \frac{\pi}{2}^-} \tan^{-1}\frac{f'(\theta)}{f(\theta)}$. The limit $\lim\limits_{\theta\to \frac{\pi}{2}^-} \frac{\cos 2\theta}{\sin 2\theta} = -\infty$ and the graph of Figure 9.33 inform us that

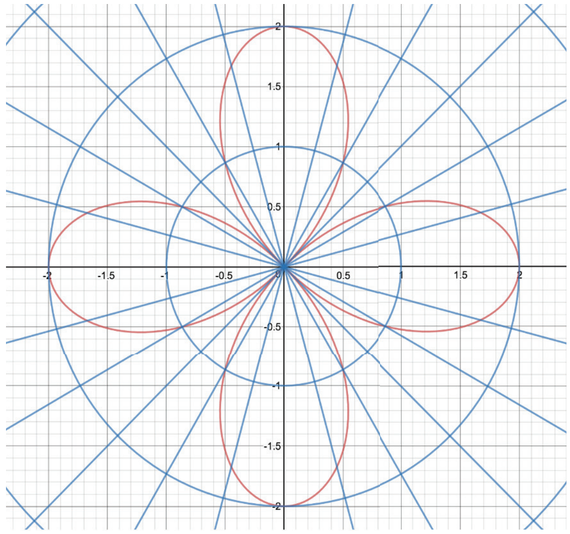
$$\lim_{\theta\to \frac{\pi}{2}^-} \tan^{-1}\tfrac{f'(\theta)}{f(\theta)} = \lim_{\theta\to \frac{\pi}{2}^-} \tan^{-1}\left(\tfrac{\cos 2\theta}{\sin 2\theta}\right) = -\tfrac{\pi}{2}$$

and hence that $\lim\limits_{\theta\to \frac{\pi}{2}^-} \gamma(\theta) = \lim\limits_{\theta\to \frac{\pi}{2}^-} \tan^{-1}\frac{f'(\theta)}{f(\theta)} + \frac{\pi}{2} = 0$. So $\lim\limits_{\theta\to \frac{\pi}{2}^-} \tan(\theta - \gamma(\theta)) = +\infty$. This time the tangent to the graph at $(0,0)$ is vertical just as the graph of Figure 12.40 indicates. Check the values for the slopes of the graph at the points corresponding to the angles $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$ against Figure 12.40.

12.24. The double angle formula tells us that $r = \sin 2\theta = 2\sin\theta\cos\theta$. So $r^3 = r^2 \sin 2\theta = 2(r\sin\theta)(r\cos\theta) = 2xy$. Since $r = \pm\sqrt{x^2+y^2}$, it follows that $(x^2+y^2)^{\frac{3}{2}} = \pm 2xy$.

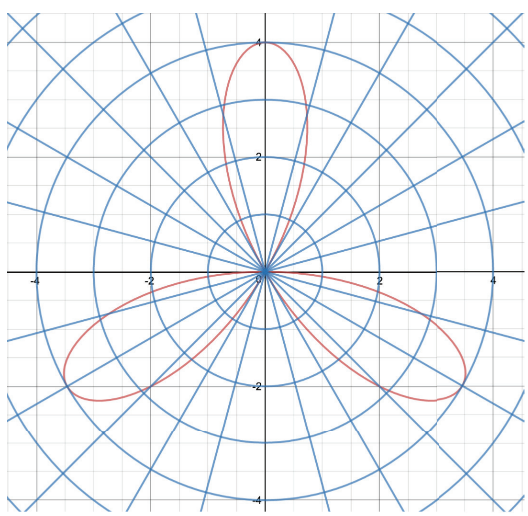
12.25. The graphs below were drawn by https://www.desmos.com/calculator/ms3eghkkgz.

i. The graph of $r = 2\cos 2\theta$ is similar to the graph of $r = \sin 2\theta$. It is obtained by the same strategy and is also a four-leaf rose. The coefficient 2 stretches the leaves by a factor of 2. The fact that the Cartesian graph of $r = \cos 2\theta$ is gotten by shifting the Cartesian graph of $r = \sin 2\theta$ by $\frac{\pi}{4}$ units to the left explains the fact that the rose of $r = 2\cos 2\theta$ is obtained by rotating the rose of the sine by $\frac{\pi}{4}$.



$r = 2\cos 2\theta$

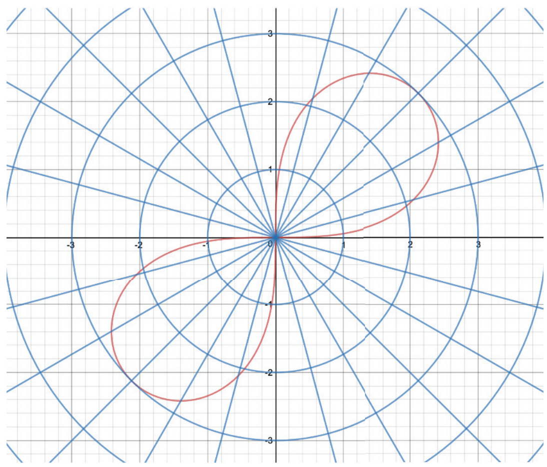
ii. We turn to $r = -4\sin 3\theta$. As the ray θ rotates from 0 to $\frac{\pi}{6}$, r slides from 0 to -4 and as θ goes from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r goes from -4 back to 0. The loop on the lower left of the graph below is traced out in the process. Similarly, as the ray θ rotates from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$, r slides from 0 to 4 (at $\theta = \frac{\pi}{2}$) and back to 0 tracing out the upper loop. Next, as the ray θ rotates from $\frac{2\pi}{3}$ to π, r slides from 0 to -4 (at $\theta = \frac{5\pi}{6}$) back to 0. This traces out the loop on the lower right. As the rotation of θ continues the graph we already have is repeated. For example, as θ moves from 0 to $-\frac{\pi}{3}$, the



$r = -4\sin 3\theta$

loop on the lower right is traced out again. As θ moves from π to $\frac{4\pi}{3}$, the loop on the lower left is repeated.

iii. Finally to the polar graph of $r^2 = 9\sin 2\theta$. Note that $9\sin 2\theta \geq 0$, so that θ must fall into one of the intervals $[0, \frac{\pi}{2}], [\pi, \frac{3\pi}{2}], [2\pi, \frac{5\pi}{2}], \ldots$ or $[-\frac{\pi}{2}, -\pi], [-\frac{3\pi}{2}, -2\pi], \ldots$. As the ray θ rotates from 0 to $\frac{\pi}{2}$, $r = +3\sqrt{\sin 2\theta}$ slides from 0 to 3 (at $\theta = \frac{\pi}{4}$) and back to 0. In the process, the loop on the upper right of the graph below is traced out. But $r = -3\sqrt{\sin 2\theta}$ is also possible. This time as θ varies from 0 to $\frac{\pi}{2}$ the loop on the lower left is traced out. Similar considerations show that as θ varies from π to $\frac{3\pi}{2}$, $r = +3\sqrt{\sin 2\theta}$ traces out the loop on the lower left again and $r = -3\sqrt{\sin 2\theta}$ repeats the loop on the upper right. As θ varies from $-\frac{\pi}{2}$ to $-\pi$, $r = +3\sqrt{\sin 2\theta}$ and $r = -3\sqrt{\sin 2\theta}$ trace these loops once more. The same



$r^2 = 9\sin 2\theta$

is the case as θ sweeps from $-\frac{3\pi}{2}$ to -2π. The fact that the values of $\sin 2\theta$ repeat tells us that the graph of $r^2 = 9\sin 2\theta$ is complete as sketched.

12.26. The relevant formulas are $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$. By taking $\phi = 2\theta$ in the last formula, we get

$$\sin 3\theta = \sin\theta(\cos^2\theta - \sin^2\theta) + \cos\theta(2\sin\theta\cos\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta.$$

i. For $r = 2\cos 2\theta$ we get $r = 2(\cos^2\theta - \sin^2\theta)$ and hence

$$r^3 = 2(r^2\cos^2\theta - r^2\sin^2\theta) = 2(x^2 - y^2).$$

Therefore $\pm(x^2+y^2)^{\frac{3}{2}} = 2(x^2 - y^2)$.

ii. For $r = -4\sin 3\theta$ we get $r = -4(3\sin\theta\cos^2\theta - \sin^3\theta)$. So

$$r^4 = -4((3r\sin\theta)(r^2\cos^2\theta) - r^3\sin^3\theta) = -4(3yx^2 - y^3) = 4(y^3 - 3yx^2).$$

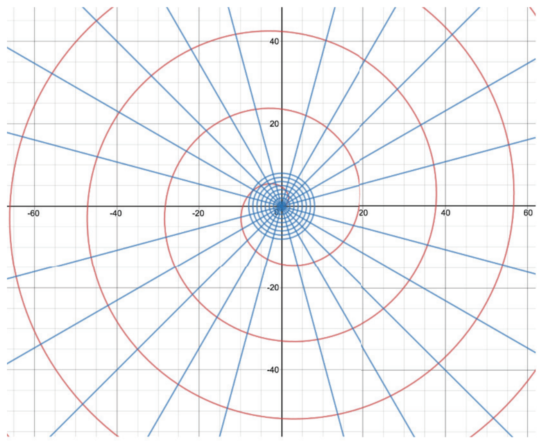
Therefore $(x^2+y^2)^2 = 4(y^3 - 3yx^2)$.

iii. Since $\sin 2\theta = 2\sin\theta\cos\theta$, we get $r^2 = 9(2\sin\theta\cos\theta)$, and hence

$$r^4 = 9r^2(2\sin\theta\cos\theta) = 18(r\sin\theta)(r\cos\theta) = 18yx.$$

Therefore $(x^2+y^2)^2 = 18yx$.

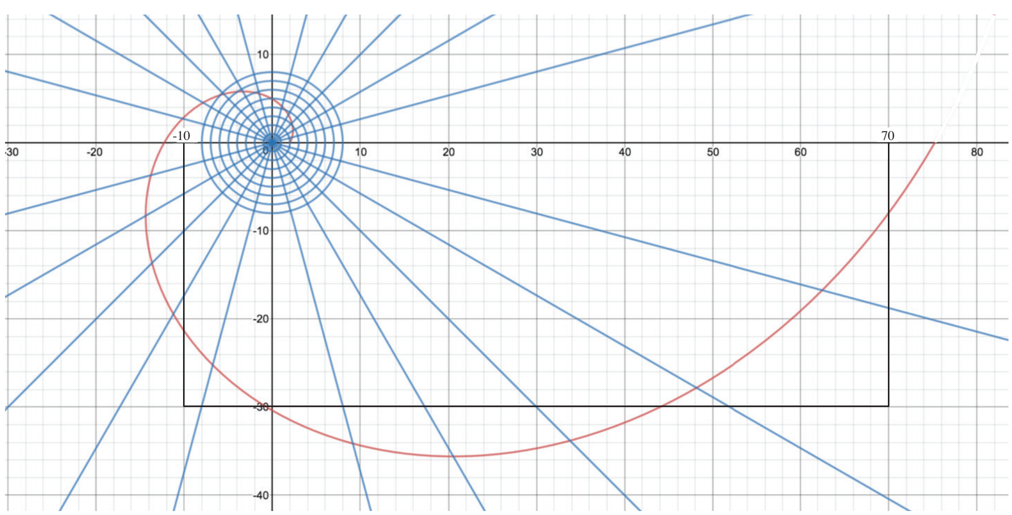
12.27. The *Archimedean spiral* $r = f(\theta) = 3\theta$ satisfies $\frac{f'(\theta)}{f(\theta)} = \frac{1}{\theta}$. Since $f'(\theta) > 0$, $r = f(\theta) = 3\theta$ is an increasing function of θ. So $\frac{\pi}{2} < \gamma(\theta)$ throughout. It follows that $0 < (\gamma(\theta) - \frac{\pi}{2}) < \frac{\pi}{2}$ throughout. For θ small and positive $\frac{f'(\theta)}{f(\theta)} = \tan(\gamma(\theta) - \frac{\pi}{2})$ is large and positive. So the graph of the tangent tells us (see Figure 9.32) that $\gamma(\theta) - \frac{\pi}{2}$ is close to $\frac{\pi}{2}$. So $\gamma(\theta)$ is close to π and hence (see Figure 12.9) the expansion of the spiral is rapid. For θ large and positive $\gamma(\theta) - \frac{\pi}{2}$ is close to 0. By the same argument $\gamma(\theta)$ is



$r = 3\theta$

close to $\frac{\pi}{2}$ and the expansion of the spiral is slow. A look at the graph of $r = f(\theta) = 3\theta$ confirms what we have observed.

12.28. For the equiangular spiral $f(\theta) = 2e^{\frac{\theta}{\sqrt{3}}}$ we see that $\tan(\gamma(\theta) - \frac{\pi}{2}) = \frac{f'(\theta)}{f(\theta)} = \frac{1}{\sqrt{3}}$. It follows that $\gamma(\theta) - \frac{\pi}{2} = \frac{\pi}{6}$ and hence that $\gamma(\theta) = \frac{2\pi}{3}$. The graph of this spiral is



$r = 2e^{\frac{\theta}{\sqrt{3}}}$

sketched above. Since $f'(\theta) = \frac{2}{\sqrt{3}}e^{\frac{\theta}{\sqrt{3}}}$ we see that the length of the spiral from $\theta = 0$ to $\theta = 2\pi$ is given by

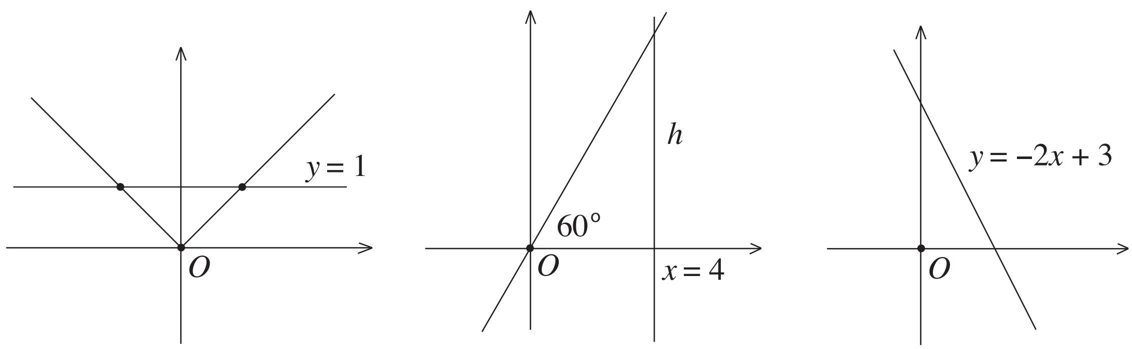
$$\int_0^{2\pi}\sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{4e^{\frac{2}{\sqrt{3}}\theta}+\tfrac{4}{3}e^{\frac{2}{\sqrt{3}}\theta}}\,d\theta=\sqrt{4+\tfrac{4}{3}}\int_0^{2\pi}\sqrt{e^{\frac{2}{\sqrt{3}}\theta}}\,d\theta$$
$$=\sqrt{\tfrac{16}{3}}\int_0^{2\pi}e^{\frac{1}{\sqrt{3}}\theta}\,d\theta=\tfrac{4}{\sqrt{3}}\big(\sqrt{3}e^{\frac{1}{\sqrt{3}}\theta}\,\big|_0^{2\pi}\big)=4\big(e^{\frac{2\pi}{\sqrt{3}}}-1\big)\approx 146.$$

The area that the spiral (along with the polar axis) encloses is

$$\int_0^{2\pi}\tfrac{1}{2}f(\theta)^2\,d\theta=\int_0^{2\pi}\tfrac{1}{2}\big(4e^{\frac{2}{\sqrt{3}}\theta}\big)\,d\theta=\int_0^{2\pi}2e^{\frac{2}{\sqrt{3}}\theta}\,d\theta=\big(\sqrt{3}e^{\frac{2}{\sqrt{3}}\theta}\big)\Big|_0^{2\pi}=\sqrt{3}(e^{\frac{4\pi}{\sqrt{3}}}-1)\approx 2450.$$

The fact that the two shorter sides plus the bottom side of the rectangle determined by the intervals $[-10,70]$ on the x-axis and $[0,-30]$ on the y-axis (see the figure) add to $30+80+30=140$ and that its area is $80\cdot 30=2400$ confirms the reasonableness of the two answers.

12.29. Since $y=r\sin\theta$, the graph of the polar function $r=f(\theta)=\frac{1}{\sin\theta}$ is the line $y=1$. It follows that the integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{2}\frac{1}{\sin^2\theta}\,d\theta$ is the area of the triangle of Figure (a) below.



(a) (b) (c)

Since this area is equal to $\frac{1}{2}(2\cdot 1)=1$, it follows that $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{\sin^2\theta}\,d\theta=2$.

12.30. Since $x=r\cos\theta$, the graph of the function $r=f(\theta)=\frac{4}{\cos\theta}$ is the line $x=4$. It follows that $\int_0^{\frac{\pi}{3}}\frac{1}{2}\big(\frac{4}{\cos\theta}\big)^2\,d\theta$ is the area of the triangle with base 4 and height h shown in Figure (b) above. Since $\frac{h}{4}=\tan 60°=\sqrt{3}$, $h=4\sqrt{3}$ and $\int_0^{\frac{\pi}{3}}\frac{8}{\cos^2\theta}\,d\theta=\frac{1}{2}4\cdot 4\sqrt{3}=8\sqrt{3}$.

12.31. After writing $r=f(\theta)=\frac{3}{\sin\theta+2\cos\theta}$ as $r\sin\theta+2r\cos\theta=3$, we see that the graph of this polar function is the line $y=-2x+3$ with slope -2 and y-intercept 3 sketched in Figure (c) above. The integral $\int_0^{\frac{\pi}{2}}\frac{1}{2}f(\theta)^2\,d\theta$ is the area of the right triangle bounded by the graph of $y=-2x+3$ and the x- and y-axes. It follows that

$$\int_0^{\frac{\pi}{2}}\tfrac{1}{2}f(\theta)^2\,d\theta=\tfrac{1}{2}(\tfrac{3}{2}\cdot 3)=\tfrac{9}{4}.$$

Similarly, $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$ is the length of the hypotenuse of this triangle. So by the Pythagorean theorem,

$$\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \sqrt{(\tfrac{3}{2})^2 + 3^2} = \sqrt{\tfrac{45}{4}} = \tfrac{3}{2}\sqrt{5}.$$

In terms of the particulars it turns out that these area and length integrals are essentially the same. From the area integral we get $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{3^2}{(\sin\theta + 2\cos\theta)^2}\, d\theta = \frac{9}{4}$ and hence that

$$\int_0^{\frac{\pi}{2}} \tfrac{1}{(\sin\theta+2\cos\theta)^2}\, d\theta = \tfrac{1}{2}.$$

Let's turn to the length integral. Since $f(\theta) = 3(\sin\theta + 2\cos\theta)^{-1}$, we see that

$$f'(\theta) = -3((\sin\theta + 2\cos\theta)^{-2}(\cos\theta - 2\sin\theta) = \tfrac{3(2\sin\theta - \cos\theta)}{(\sin\theta+2\cos\theta)^2}.$$

Hence

$$\begin{aligned} f(\theta)^2 + f'(\theta)^2 &= \tfrac{3^2}{(\sin\theta+2\cos\theta)^2} + \tfrac{3^2(2\sin\theta-\cos\theta)^2}{(\sin\theta+2\cos\theta)^4} = \tfrac{3^2[(\sin\theta+2\cos\theta)^2+(2\sin\theta-\cos\theta)^2]}{(\sin\theta+2\cos\theta)^4} \\ &= \tfrac{3^2(5\sin^2\theta+5\cos^2\theta)}{(\sin\theta+2\cos\theta)^4} = \tfrac{3^2 5}{(\sin\theta+2\cos\theta)^4}. \end{aligned}$$

Therefore $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{5}}{(\sin\theta+2\cos\theta)^2}\, d\theta$. Because this integral is equal to $\frac{3}{2}\sqrt{5}$, we see again that

$$\int_0^{\frac{\pi}{2}} \tfrac{1}{(\sin\theta+2\cos\theta)^2}\, d\theta = \tfrac{1}{2}.$$

12.32. With $f(\theta) = \sin\theta$, we get $f'(\theta) = \cos\theta$, so that $\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$. It follows that these lengths are equal to

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1\, d\theta \; = (\tfrac{3\pi}{4} - \tfrac{\pi}{4}) = \tfrac{\pi}{2}, \;\; \int_0^{\pi} 1\, d\theta \; = \pi, \text{ and } \int_0^{2\pi} 1\, dx \; = 2\pi,$$

respectively. By Figure 12.4 the graph of $r = \sin\theta$ in is a circle of radius $\frac{1}{2}$. A look at the limits of integration tells us that the first, second, and third integral are equal to one-half the circumference of this circe, the full circumference of this circle, and twice the circumference of this circle, respectively. These lengths are $\frac{1}{2}(2\pi(\frac{1}{2})) = \frac{\pi}{2}, 2\pi(\frac{1}{2}) = \pi$, and $2(2\pi(\frac{1}{2})) = 2\pi$ respectively, as before.

12.33. We see from the graph of $r = f(\theta) = \sin\theta$ of Figure 12.4 that the integral $\int_0^{\pi} \frac{1}{2}\sin^2\theta\, d\theta$ is equal to the area of a circle of radius $\frac{1}{2}$. So its value is $\pi(\frac{1}{2})^2 = \frac{\pi}{4}$. Letting $\alpha = \theta$ in the equality $\cos 2\alpha = 1 - 2\sin^2\alpha$ of Problem 1.26ii, and solving for $\sin^2\theta$, we get $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. Using this equality, we get

$$\int_0^{\pi} \tfrac{1}{2}\sin^2\theta\, d\theta = \int_0^{\pi} \tfrac{1}{4}(1 - \cos 2\theta)\, d\theta = \tfrac{1}{4}(\theta - \tfrac{1}{2}\sin 2\theta)\Big|_0^{\pi} = \tfrac{\pi}{4}.$$

12.34. Section 12.2 tells us that the graph of $r = f(\theta) = \frac{4}{1+\frac{2}{3}\cos\theta}$ is an ellipse with eccentricity $\varepsilon = \frac{2}{3}$ and $d = 4$. By part (ii) of Section 12.2 the semimajor and semiminor axes of this ellipse are $a = \frac{d}{1-\varepsilon^2} = \frac{4}{1-(\frac{2}{3})^2} = \frac{4}{\frac{5}{9}} = \frac{36}{5}$ and $b = \frac{d}{\sqrt{1-\varepsilon^2}} = \frac{4}{\sqrt{1-(\frac{2}{3})^2}} = \frac{4}{\sqrt{\frac{5}{9}}} = \frac{12}{\sqrt{5}}$. Recall from Section 5.7 that the area of an ellipse with semimajor and semiminor axes a and b is $ab\pi$. Since $\int_0^{\pi} \frac{1}{2}\big(\frac{4}{1+\frac{2}{3}\cos\theta}\big)^2\, d\theta$ is one-half the area of the ellipse it follows that $\int_0^{\pi} \frac{1}{2}\big(\frac{4}{1+\frac{2}{3}\cos\theta}\big)^2\, d\theta = \frac{1}{2}(\frac{36}{5})(\frac{12}{\sqrt{5}})\pi = \frac{216}{5\sqrt{5}}\pi$.

12.35. The derivative of $r = f(\theta) = \frac{d}{1+\varepsilon\cos\theta} = d(1+\varepsilon\cos\theta)^{-1}$ is

$$f'(\theta) = -d(1+\varepsilon\cos\theta)^{-2}(-\varepsilon\sin\theta) = \frac{d\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}.$$

Therefore

$$f(\theta)^2 + f'(\theta)^2 = \frac{d^2}{(1+\varepsilon\cos\theta)^2} + \frac{d^2\varepsilon^2\sin^2\theta}{(1+\varepsilon\cos\theta)^4} = \frac{d^2(1+\varepsilon\cos\theta)^2 + d^2\varepsilon^2\sin^2\theta}{(1+\varepsilon\cos\theta)^4} = \frac{d^2+2\varepsilon d^2\cos\theta+d^2\epsilon^2}{(1+\varepsilon\cos\theta)^4}.$$

Since $\sqrt{f(\theta)^2 + f'(\theta)^2} = \frac{d\sqrt{1+\varepsilon^2+2\varepsilon\cos\theta}}{(1+\varepsilon\cos\theta)^2}$, the integral $d\int_a^b \frac{\sqrt{1+\varepsilon^2+2\varepsilon\cos\theta}}{(1+\varepsilon\cos\theta)^2}\, d\theta$ expresses the length of the conic section $f(\theta) = \frac{d}{1+\varepsilon\cos\theta}$ between the rays $\theta = a$ to $\theta = b$.

In the parabolic case $\varepsilon = 1$ and $\sqrt{1+\varepsilon^2+2\varepsilon\cos\theta} = \sqrt{2+2\cos\theta} = \sqrt{2}(1+\cos\theta)^{\frac{1}{2}}$ and the integral is $\sqrt{2}d\int_a^b \frac{1}{(1+\cos\theta)^{\frac{3}{2}}}\, d\theta$.

12.36. A study of the solution of Problem 12.30 tells us that $\int_0^{\frac{\pi}{4}} \frac{1}{2}\big(\frac{4}{\cos\theta}\big)^2\, d\theta = \frac{1}{2}(4\cdot 4) = 8$. After changing notation from θ to x we get $\int_0^{\frac{\pi}{4}} \frac{3}{\cos^2 x}\, dx = \frac{3}{8}\int_0^{\frac{\pi}{4}} \frac{8}{\cos^2 x}\, dx = 3$.

By looking at the solution of Problem 12.33 and Figure 12.4 we see that $\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2\theta\, d\theta = \frac{1}{2}\pi(\frac{1}{2})^2 = \frac{\pi}{8}$. After changing the variable we see that $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{4}$.

After studying the solution of 12.34 we see that $\int_0^{2\pi} \frac{1}{2}\big(\frac{4}{1+\frac{2}{3}\cos\theta}\big)^2\, d\theta$ is the area of the entire ellipse with semimajor and semiminor axes $a = \frac{36}{5}$ and $b = \frac{12}{\sqrt{5}}$. So the value of this integral is $(\frac{36}{5})(\frac{12}{\sqrt{5}})\pi = \frac{432}{5\sqrt{5}}\pi$. It follows that $\int_0^{2\pi} \frac{2}{(1+\frac{2}{3}\cos x)^2}\, dx = \frac{1}{4}(\frac{432}{5\sqrt{5}}\pi) = \frac{108}{5\sqrt{5}}\pi$.

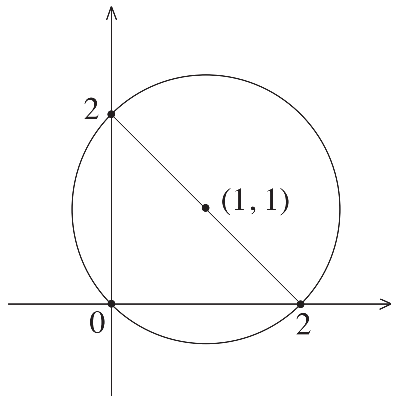
12.37. The circle $(x-1)^2 + (y-1)^2 = 2$ has center $(1,1)$ and radius $\sqrt{2}$. Its graph is sketched in the figure below. The Cartesian points $(2,0)$ and $(0,2)$ are on the circle and the segment joining them is on the line $y = -x+2$. It follows that $(1,1)$ is on this line as well so that the segment is a diameter of the circle.

Since $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$(x-1)^2 + (y-1)^2 = (r\cos\theta - 1)^2 + (r\sin\theta - 1)^2$$

$$= (r^2\cos^2\theta - 2r\cos\theta + 1) + (r^2\sin^2\theta - 2r\sin\theta + 1)$$
$$= r^2 - 2r(\cos\theta + \sin\theta) + 2$$

and hence $r^2 - 2r(\cos\theta + \sin\theta) = 0$. Assuming that $r \neq 0$, we get $r = 2(\cos\theta + \sin\theta)$. For $\theta = \frac{3\pi}{4}$, $r = 2(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) = 0$ so that the origin is on the graph of the polar



function $r = f(\theta) = 2(\cos\theta + \sin\theta)$. Hence its graph is the entire circle.

A look at the figure tells us that $\int_0^{\frac{\pi}{2}} \frac{1}{2}f(\theta)^2\,d\theta$ is the area consisting of half the circle plus the triangle with base and height equal to 2. So

$$\int_0^{\frac{\pi}{2}} \tfrac{1}{2}f(\theta)^2\,d\theta = \tfrac{1}{2}\pi(\sqrt{2})^2 + \tfrac{1}{2}(2\cdot 2) = \pi + 2.$$

It also follows from the figure that $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \frac{1}{2}(2\pi\sqrt{2}) = \sqrt{2}\pi$.

The two integrals can be solved directly. Since $f(\theta) = 2(\cos\theta + \sin\theta)$,

$$f(\theta)^2 = 4(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta) = 4(1 + 2\sin\theta\cos\theta).$$

Hence $\int_0^{\frac{\pi}{2}} \frac{1}{2}f(\theta)^2\,d\theta = \int_0^{\frac{\pi}{2}} 2(1 + 2\cos\theta\sin\theta)\,d\theta = (2\theta + 2\sin^2\theta)\Big|_0^{\frac{\pi}{2}} = \pi + 2$ as before. Since $f'(\theta) = 2(-\sin\theta + \cos\theta)$, we get

$$f'(\theta)^2 = 4(\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta) = 4(1 - 2\sin\theta\cos\theta).$$

So $f(\theta)^2 + f'(\theta)^2 = 8$ and $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = 2\sqrt{2}\theta\Big|_0^{\frac{\pi}{2}} = \sqrt{2}\pi$.

12.38. **i.** The area of the cardioid $r = f(\theta) = 1 + \cos\theta$ of Figure 12.38 is equal to

$$\int_0^{2\pi} \tfrac{1}{2}f(\theta)^2\,d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos\theta)^2\,d\theta.$$

With the equality $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ this integral is easily solved. Since

$(1+\cos\theta)^2 = 1+2\cos\theta+\cos^2\theta = 1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta) = \frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta$,

we get

$$\int_0^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta = \tfrac{1}{2}\big(\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\big)\Big|_0^{2\pi} = \tfrac{3}{2}\pi.$$

ii. The derivative of $f(\theta) = 1+\cos\theta$ is $f'(\theta) = -\sin\theta$. Therefore $f(\theta)^2 + f'(\theta)^2 = 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 + 2\cos\theta$ and hence

$$\int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_0^{\pi} \sqrt{2+2\cos\theta}\,d\theta = \sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\,d\theta.$$

As θ moves from 0 to π one-half of the cardioid is traced out, so that this integral represents one-half of the length of the cardioid.

iii. With $u = 1+\cos\theta$, we get $\frac{du}{d\theta} = -\sin\theta$ and $du = -\sin\theta\,d\theta$. From the fact that $\sin^2\theta + \cos^2\theta = 1$ and $\sin\theta \geq 0$ over $0 \leq \theta \leq \pi$ we get that $\sin\theta = (1-\cos^2\theta)^{\frac{1}{2}}$, and hence that

$$\sin\theta = (1-\cos^2\theta)^{\frac{1}{2}} = (1-(u-1)^2)^{\frac{1}{2}} = (1-u^2+2u-1)^{\frac{1}{2}} = (2u-u^2)^{\frac{1}{2}}.$$

So $\sqrt{1+\cos\theta}\,d\theta = -\frac{u^{\frac{1}{2}}}{(2u-u^2)^{\frac{1}{2}}}\,du = -\big(\frac{u}{2u-u^2}\big)^{\frac{1}{2}}du = -\big(\frac{1}{2-u}\big)^{\frac{1}{2}}du = \frac{-1}{\sqrt{2-u}}\,du$, and

$$\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\,d\theta = \sqrt{2}\int_2^0 \frac{-1}{\sqrt{2-u}}\,du = \sqrt{2}\int_0^2 \frac{1}{\sqrt{2-u}}\,du.$$

Observe that the integral $\displaystyle\int_0^2 \frac{1}{\sqrt{2-u}}\,du = \lim_{c\to 2^-}\int_0^c \frac{1}{\sqrt{2-u}}\,du$ is improper.

iv. Let's try $v = 2-u$ and $dv = -du$. So

$$\lim_{c\to 2^-}\int_0^c \frac{1}{\sqrt{2-u}}\,du = \lim_{c\to 2^-}\int_2^{2-c} \frac{-1}{\sqrt{v}}\,dv = \lim_{c\to 2^-}\int_{2-c}^2 v^{-\frac{1}{2}}\,dv = \lim_{c\to 2^-}\big(2v^{\frac{1}{2}}\big|_{2-c}^2\big)$$
$$= \lim_{c\to 2^-} 2(\sqrt{2}-\sqrt{2-c}) = 2\sqrt{2}.$$

Combining the conclusions of (ii), (iii), and (iv), we get that one-half the length of the cardioid is

$$\int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\,d\theta = \sqrt{2}(2\sqrt{2}) = 4.$$

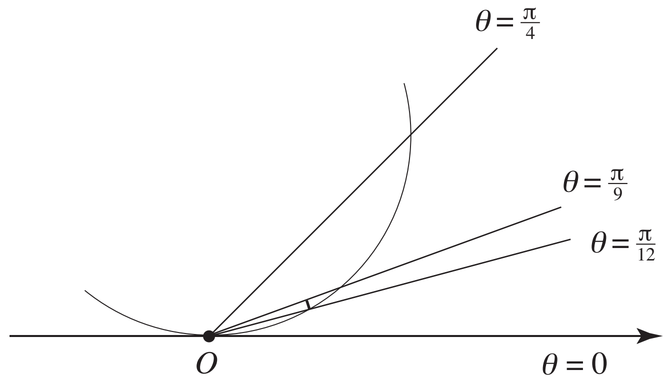
v. So the length of the cardioid $r = 1+\cos\theta$ is 8.

12.39. With regard to Figure 12.41 the simple fact is that for some graphs (or parts of graphs) the length $f(\theta_i)\,d\theta$ of the circular arc does not approximate the length of the graph between the rays determined by θ_i and θ_{i+1} with sufficient accuracy.

Consider the function $r = f(\theta) = \sin\theta$. Its graph is the circle sketched in Figure 12.4. Notice that as θ moves from 0 through small positive angles, the point (r,θ) on the graph recedes quickly from the origin. This part of the graph illustrates the problem of "sufficient accuracy." Consider the rays $\theta = \frac{\pi}{12}$ and $\theta = \frac{\pi}{9}$. The ray $\theta = \frac{\pi}{12}$ cuts the graph at the point $(\sin\frac{\pi}{12}, \frac{\pi}{12}) \approx (0.26, 0.26)$. By the length formula for a circular arc, the circular arc centered at O from this point to the ray $\theta = \frac{\pi}{9}$ has length

$$f(\tfrac{\pi}{12})(\tfrac{\pi}{9}-\tfrac{\pi}{12}) = (\sin\tfrac{\pi}{12})(\tfrac{\pi}{9}-\tfrac{\pi}{12}) = (\sin\tfrac{\pi}{12})(\tfrac{\pi}{36}) \approx 0.0226.$$

Let's compare this against the length of the graph of the function between these two



rays. This length is

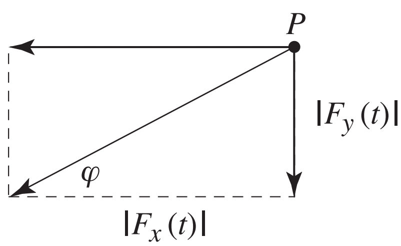
$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sqrt{\sin^2\theta + \cos^2\theta}\,d\theta = \theta\,\Big|_{\frac{\pi}{12}}^{\frac{\pi}{9}} = (\tfrac{\pi}{9}-\tfrac{\pi}{12}) = \tfrac{\pi}{36} \approx 0.0873,$$

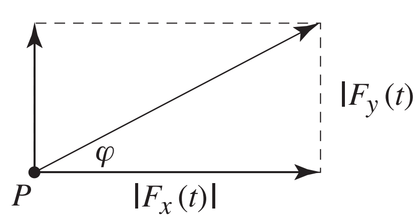
or about 4 times the length of the circular arc.

A comparison of the correct and incorrect formulas for one-half the length of the circle of Figure 12.4 shows how these differences accumulate. The incorrect formula tells us that $\int_0^{\frac{\pi}{2}} f(\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta = -\cos\theta\Big|_0^{\frac{\pi}{2}} = -(0-1) = 1$, whereas the correct result is $\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta + \cos^2\theta}\,d\theta = \theta\,\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} \approx 1.57$.

12.40. **i.** Putting the equations for $x(t)$ and $F_x(t)$ together tells us that $F_x(t) = -mx(t)$ where m is the mass of the point. So when for a given time t one of $x(t)$ or $F_x(t)$ is positive, the other is negative. Since $F_y(t) = -my(t)$ there is a similar connection between $y(t)$ and $F_y(t)$. It follows that $F_x(t)$ always points in the direction of the y-axis and that $F_y(t)$ points in the direction of the y-axis.

ii. The slope of the slanting segment in Figure 12.42a connecting P and O is $\frac{y(t)-0}{x(t)-0} = \frac{y(t)}{x(t)}$. Turn to Figure 12.42b and assume that P is in either the first of third quadrant. Since the parallelogram law holds, the resultant runs in the direction of the diagonal of the rectangle in each case. The slope of the diagonal is $\tan\varphi = \frac{|F_y(t)|}{|F_x(t)|}$ in each case. Since $F_x(t) = -mx(t)$ and $F_y(t) = -my(t)$ and $x(t)$ and $y(t)$





have the same sign, $\tan\varphi = \frac{y(t)}{x(t)}$ is also equal to the slope of the slanting line of Figure 12.42b. It follows that the resultant of the forces $F_x(t)$ and $F_y(t)$ lies on the line joining P and O. A similar argument works when P is in the second or fourth quadrants.

iii. A combination of (i) and (ii) shows that the resultant of $F_x(t)$ and $F_y(t)$ is a centripetal force on P acting in the direction of the origin O.

iv. This is confirmed by the computation

$$F(t) = \sqrt{F_x(t)^2 + F_y(t)^2} = m\sqrt{a^2\cos^2 t + b^2\sin^2 t} = m\sqrt{x(t)^2 + y(t)^2} = mr(t).$$

12.41. If the angle $\theta(t)$ of Figure 12.23 increases at a constant rate, then the formula $r(t)^2\theta'(t) = 2\kappa$ of Section 12.8 informs us that $r(t)^2$ is constant and hence that $r(t)$ is constant as well. This means that the orbit is a circle. Since $r = f(\theta)$ is constant and $g(\theta) = f(\theta)^{-1}$, it follows that $g(\theta)$ is constant and hence that $g'(\theta)$ and $g''(\theta)$ are both zero. In turn, $F(t) = 4m\kappa^2 g(\theta(t))^3$ is constant.

12.42. Since $a(r\sin\theta) + b(r\cos\theta) = c$, the Cartesian version of the equation $r = f(\theta)$ is the line $ay + bx = c$. Since $c \neq 0$, the origin is not on the line. Note that $g(\theta) = \frac{1}{c}(a\sin\theta + b\cos\theta)$. So $g'(\theta) = \frac{1}{c}(a\cos\theta - b\sin\theta)$ and $g''(\theta) = \frac{1}{c}(-a\sin\theta - b\sin\theta)$. It follows that $g''(\theta) = -g(\theta)$. An application of the centripetal force equation tells us that $F(t) = 0$.

12.43. The assumption that the magnitude of the centripetal force satisfies the inverse cube law $F(t) = K\frac{m}{r(t)^3} = Kmg(\theta(t))^3$ for a constant K combined with the centripetal force equation $F(t) = 4m\kappa^2 g(\theta(t))^2\big[g(\theta(t)) + g''(\theta(t))\big]$ implies that $4\kappa^2[g(\theta(t)) + g''(\theta(t))] = Kg(\theta(t))$ and hence $g''(\theta(t)) + \big[1 - \frac{K}{4\kappa^2}\big]g(\theta(t)) = 0$. It follows that the function $g(\theta)$ satisfies

$$g''(\theta) + \big[1 - \tfrac{K}{4\kappa^2}\big]g(\theta) = 0.$$

So $y = g(\theta)$ is a solution of a second-order differential equation of the form studied in Section 11.6. With respect to the constants $A, B,$ and C introduced there, observe that $A = 1, B = 0,$ and $C = 1 - \frac{K}{4\kappa^2}$. Therefore $B^2 - 4AC = -4\big(1 - \frac{K}{4\kappa^2}\big) = \frac{K}{\kappa^2} - 4 = \frac{K - 4\kappa^2}{\kappa^2}$.

Case 1. If $K > 4\kappa^2$, then $B^2 - 4AC > 0$ and Case 1 of Section 11.6 applies. So $g(\theta)$ is given by

$$g(\theta) = D_1 e^{\frac{\sqrt{K-4\kappa^2}}{2\kappa}\theta} \ + \ D_2 e^{-\frac{\sqrt{K-4\kappa^2}}{2\kappa}\theta}$$

for some real constants D_1 and D_2.

Case 2. If $K = 4\kappa^2$, then $B^2 - 4AC = 0$ so that Case 2 of Section 11.6 applies. Since $A = 1$, and $B = C = 0$, the only root of the characteristic polynomial x^2 is zero, and hence $g(\theta)$ is given by

$$g(\theta) = D_1 \ + \ D_2\theta$$

for some real constants D_1 and D_2.

Case 3. If $K < 4\kappa^2$, then Case 3 of Section 11.6 applies, so that

$$g(\theta) = D_1 \sin\left(\tfrac{\sqrt{4\kappa^2 - K}}{2\kappa}\,\theta\right) \;+\; D_2 \cos\left(\tfrac{\sqrt{4\kappa^2 - K}}{2\kappa}\,\theta\right)$$

where D_1 and D_2 are real constants.

12.44. **i.** Since $r = f(\theta) = \frac{1}{a\theta + c}$, we know that $g(\theta) = a\theta + c$. So $g'(\theta) = a, g''(\theta) = 0$, and

$$F(t) = 4m\kappa^2 g(\theta(t))^2\big[g(\theta(t))\big] = 4m\kappa^2 g(\theta(t))^3 = 4\kappa^2 \tfrac{m}{r(t))^3}.$$

ii. For $r = f(\theta) = \frac{1}{a\cos(b\theta + c)}$, we get $g(\theta) = a\cos(b\theta + c)$. So $g'(\theta) = -ab\sin(b\theta + c)$ and $g''(\theta) = -ab^2\cos(b\theta + c) = -b^2 g(\theta)$. Hence $g(\theta) + g''(\theta) = g(\theta) - b^2 g(\theta) = (1 - b^2)g(\theta)$ and therefore

$$F(t) = 4m\kappa^2 g(\theta(t))^2\big[g(\theta(t)) + g''(\theta(t))\big] = 4m\kappa^2(1 - b^2)g(\theta(t))^3 = 4\kappa^2(1 - b^2)\tfrac{m}{r(t)^3}.$$

iii. Given $r = f(\theta) = \frac{1}{a\cosh(b\theta + c)}$, we know that $g(\theta) = a\cosh(b\theta + c)$. Hence $g'(\theta) = ab\sinh(b\theta + c)$ and $g''(\theta) = ab^2\cosh(b\theta + c) = b^2 g(\theta)$. Therefore $g(\theta) + g''(\theta) = g(\theta) + b^2 g(\theta) = (1 + b^2)g(\theta)$ and

$$F(t) = 4m\kappa^2 g(\theta(t))^2\big[g(\theta(t)) + g''(\theta(t))\big] = 4m\kappa^2(1 + b^2)g(\theta(t))^3 = 4\kappa^2(1 + b^2)\tfrac{m}{r(t)^3}.$$

Let's look at the conclusions of Problems 12.43 and 12.44 side by side. Problem 12.43 describes *all orbits* of point-masses driven by a centripetal force that satisfies an inverse cube law. It classifies all the polar functions $r = f(\theta)$ that have such orbits as graphs by showing that they belong to one of three basic types. Problem 12.44 goes on to provide *three examples* of functions with graphs that describe orbits of point-masses that are pushed by a centripetal force satisfying an inverse cube law. The logical implication is that these three examples should all appear in the classification that Problem 12.43 provides. Example (i) appears as Case 2 of the conclusion of Problem 12.43. But what about examples (ii) and (iii)? It turns out they belong to Cases 3 and 1 of this classification, respectively. In the situation of example (ii) this is a consequence of the addition formula for the cosine (Problem 1.25). For example (iii) it follows from the addition formula for the hyperbolic cosine (Example 7.47).

12.45. It was shown in Part 3 of Section 12.10 that the mass M_i of the typical spherical shell within the sphere of radius R (as depicted in Figure 12.29) is approximately

$$M_i \approx (4\pi c_i^2 \Delta x_i)\rho(c_i) = 4\pi c_i^2 \rho(c_i)\Delta x_i.$$

Since the sum of the masses of the n shells is the mass M of the sphere, we see that

$$M = \sum_{i=0}^{n-1} M_i \approx \sum_{i=0}^{n-1} 4\pi c_i^2 \rho(c_i)\,\Delta x_i.$$

By repeating this computation again and again for partitions $\mathcal{P}$ of smaller and smaller norm $\|\mathcal{P}\|$ the approximations involved get tighter and tighter so that in the limit,

$$M = \lim_{\|\mathcal{P}\|\to 0} \sum_{i=0}^{n-1} 4\pi c_i^2 \rho(c_i)\,\Delta x_i.$$

Since this limit is nothing but $\displaystyle\int_0^R 4\pi x^2 \rho(x)\,dx$ we have verified that the mass M of the sphere is given by this integral.

The last two problems show that the inverse square law—that correctly expresses the magnitude of the gravitational force between point-masses and homogeneous spheres—is the exception rather than the rule. For instance, it fails to describe the magnitude of the force with which a homogeneous disc or cylinder attracts a point-mass.

12.46. The solution will make use of the short-hand approach to integration (rather than the longer and more detailed path using partitions, norms, and limits). Since the disc D is homogeneous with radius R and mass M it has constant density equal to $\rho = \frac{M}{\pi R^2}$ (given by mass over area). Let's focus on the typical thin circular ring of radius x and thickness dx shown in Figure 12.43. Cut the ring and straighten it out to form a thin rectangle of height dx and length equal to the circumference $2\pi x$ of the ring. The area of this rectangle is $(2\pi x)dx$ so that it has mass $2\pi\rho x\,dx$. By applying the conclusion of Part 1 of Section 12.10, the gravitational force with which this ring attracts the point-mass m acts in the direction of its center O with magnitude $\frac{Gm(2\pi\rho x\,dx)c}{(c^2+x^2)^{\frac{3}{2}}}$. Since this is so for each of these circular rings as x varies from 0 to R, it follows that the resultant of these forces (that is to say the force with which the entire disc D attracts the point-mass m) acts in the direction of O with magnitude equal to the sum of the magnitudes $\frac{Gm(2\pi\rho x\,dx)c}{(c^2+x^2)^{\frac{3}{2}}}$ (as x varies from 0 to R). By arguing as we have so many times before we know that this sum is given by the integral

$$F = \int_0^R \frac{Gm(2\pi\rho x\,dx)c}{(c^2+x^2)^{\frac{3}{2}}}\,dx = Gmc\int_0^R \frac{2\pi(\frac{M}{\pi R^2})\,x}{(x^2+c^2)^{\frac{3}{2}}}\,dx = G\frac{mMc}{R^2}\int_0^R \frac{2x}{(x^2+c^2)^{\frac{3}{2}}}\,dx.$$

The integral $\displaystyle\int_0^R \frac{2x}{(x^2+c^2)^{\frac{3}{2}}}\,dx$ is easy to evaluate. With $u = x^2+c^2$ and $du = 2x\,dx$,

$$\int_0^R \frac{2x}{(x^2+c^2)^{\frac{3}{2}}}\,dx = \int_{c^2}^{R^2+c^2} u^{-\frac{3}{2}}\,du = -2u^{-\frac{1}{2}}\Big|_{c^2}^{R^2+c^2} = -2\left(\frac{1}{\sqrt{R^2+c^2}} - \frac{1}{\sqrt{c^2}}\right) = 2\left(\frac{1}{c} - \frac{1}{\sqrt{R^2+c^2}}\right).$$

We have shown that the gravitational force of the disc on the point-mass is directed to the center O of the disc with a magnitude $F = G\frac{2mM}{R^2}\left(1 - \frac{c}{\sqrt{R^2+c^2}}\right)$.

12.47. Since the mass of the cylinder is homogeneously distributed its density ρ is the constant obtained by dividing its mass M by its volume $\pi R^2 h$. So $\rho = \frac{M}{\pi R^2 h}$. Let's focus on one of the typical thin discs of thickness dx that the cylinder has been sliced into. See Figure 12.44. The volume of the disc is $\pi R^2 dx$ so that its mass is $\pi R^2 dx \cdot \rho$. The point-mass m lies on the central axis of the cylinder at a distance $c-x$ from the disc's center. By the conclusion of the previous problem, the gravitational force with which the disc attracts the point-mass acts in the direction of the center of the disc with a magnitude of

$$
\begin{aligned}
G\tfrac{2m(\pi R^2 dx\cdot\rho)}{R^2}\big(1-\tfrac{c-x}{\sqrt{R^2+(c-x)^2}}\big) &= G\cdot 2m\pi(\tfrac{M}{\pi R^2 h})dx\big(1-\tfrac{c-x}{(R^2+(c-x)^2)^{\frac{1}{2}}}\big)\\
&= G\tfrac{2mM}{R^2h}\big(1-\tfrac{c-x}{(R^2+(c-x)^2)^{\frac{1}{2}}}\big)\,dx.
\end{aligned}
$$

Summing things up over all the discs from $x = 0$ to $x = h$, we find that the force with which the cylinder attracts the point mass is directed along the central axis of the cylinder with a magnitude of

$$F = \int_0^h G\tfrac{2mM}{R^2h}\big(1-\tfrac{c-x}{(R^2+(c-x)^2)^{\frac{1}{2}}}\big)\,dx = G\tfrac{2mM}{R^2h}\int_0^h \big(1+\tfrac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\big)\,dx.$$

Since $\displaystyle\int_0^h \big(1+\tfrac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\big)\,dx = h+\int_0^h \tfrac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\,dx$, it remains to solve the integral $\displaystyle\int_0^h \tfrac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\,dx$. To this end, we'll start with the substitution $u = x-c$ and $du = dx$ to get $\displaystyle\int_0^h \tfrac{x-c}{(R^2+(c-x)^2)^{\frac{1}{2}}}\,dx = \int_{-c}^{h-c} \tfrac{u}{(R^2+u^2)^{\frac{1}{2}}}\,du$. Next, we'll let $v = R^2+u^2$ and $dv = 2u\,du$ so that

$$\int_{-c}^{h-c} \tfrac{u}{(R^2+u^2)^{\frac{1}{2}}}\,du = \int_{R^2+c^2}^{R^2+(h-c)^2} \tfrac{1}{2}v^{-\frac{1}{2}}\,dv = v^{\frac{1}{2}}\Big|_{R^2+c^2}^{R^2+(h-c)^2} = (R^2+(h-c)^2)^{\frac{1}{2}}-(R^2+c^2)^{\frac{1}{2}}.$$

Putting things together, we finally see that $F = G\frac{2mM}{R^2h}\big(h-(R^2+c^2)^{\frac{1}{2}}+(R^2+(c-h)^2)^{\frac{1}{2}}\big)$.