Solutions to Problems and Projects for Chapter 7

7.1. For $\sqrt{7-x}$ to make sense, we need $7-x \geq 0$ or $7 \geq x$. So the domain of $f(x)$ is $\{x \mid x \leq 7\}$. For $\sqrt{x+5}$ to make sense, $x+5 \geq 0$. So the domain of $g(x)$ is $\{x \mid x \geq -5\}$. For $h(x)$ to make sense, both $f(x)$ and $g(x)$ must make sense. So the domain of $h(x)$ is $\{x \mid -5 \leq x \leq 7\}$.

7.2. These expression make sense as long as the denominators are not zero. By the quadratic formula, $x^2-3x+2=0$ precisely when $x = \frac{3\pm\sqrt{(-3)^2-4\cdot1\cdot2}}{2} = \frac{3\pm1}{2}$. So $x=1$ or $x=2$. Since $x^3+x^2-2x = x(x^2+x-2)$, we need to check when $x^2+x-2=0$. Using the quadratic formula again, we see that this is so for $x = \frac{-1\pm\sqrt{1^2-4\cdot1\cdot(-2)}}{2} = \frac{-1\pm3}{2}$. So $x=-2$ or $x=1$.

7.3. For $\sqrt{3x-4}$ to be defined, we must have $3x-4 \geq 0$. So $x \geq \frac{4}{3}$ and hence the domain of $f(x)$ is $\{x \mid x \geq \frac{4}{3}\}$. For $\sqrt{2x-3}$ to make sense, $2x-3$ needs to be greater than or equal to 0. So $2x-3 \geq 0$ and hence the domain of $g(x)$ is $\{x \mid x \geq \frac{3}{2}\}$. Note that for $x \geq \frac{3}{2}$, both $f(x)$ and $g(x)$ are defined. For $k(x)$ to make sense, we also need $g(x) \neq 0$. So the domain of $k(x)$ is $\{x \mid x > \frac{3}{2}\}$.

7.4. In the case of $f(x)$, we need both $x+6 \geq 0$ and $3 \geq \sqrt{x+6}$. So $x \geq -6$ and $9 \geq x+6$. Hence $x \geq -6$ and $3 \geq x$. So the domain of $f(x)$ is $\{x \mid -6 \leq x \leq 3\}$. For $g(x)$ to make sense, we have to have $\frac{x-5}{x+3} \geq 0$. So either $x+3>0$ and $x-5 \geq 0$; or $x+3<0$ and $x-5 \leq 0$. So $x \geq 5$ or $x < -3$. Hence the domain of $g(x)$ is $\{x \mid x < -3 \text{ or } x \geq 5\}$.

7.5.

i. $\lim\limits_{x\to2} (x^2+1)(x^2+4x) = (5)(12) = 60$.

ii. $\lim\limits_{x\to1} \frac{x-2}{x^2+4x-3} = \frac{-1}{1+4-3} = -\frac{1}{2}$.

iii. $\lim\limits_{x\to4} \sqrt{x+\sqrt{x}} = \sqrt{4+2} = \sqrt{6}$.

iv. $\lim\limits_{x\to3} \frac{x^2-x+12}{x+3} = \frac{9-3+12}{3+3} = \frac{18}{6} = 3$.

7.6.

i. Notice that in the limit $\lim\limits_{x\to-3} \frac{x^2-x+12}{x+3}$, the numerator goes to 24 while the denominator goes to 0. So the ratio becomes larger and larger. It does not close in on a finite number, so that there is no limit. Because the numerator goes to 24 in each case, $\lim\limits_{x\to-3^-} \frac{x^2-x+12}{x+3} = -\infty$ and $\lim\limits_{x\to-3^+} \frac{x^2-x+12}{x+3} = +\infty$.

ii. This is a limit of "$\frac{0}{0}$" type. So we are looking for a cancellation. Because $x^2-x-12 = (x+3)(x-4)$, we see that $\lim\limits_{x\to-3} \frac{x^2-x-12}{x+3} = \lim\limits_{x\to-3} \frac{(x+3)(x-4)}{x+3} = \lim\limits_{x\to-3}(x-4) = -7$. The observation that $\lim\limits_{x\to-3} \frac{2x-1}{1} = -7$ tells us that L'Hospital's rule gives the same answer.

iii. This is another limit of "$\frac{0}{0}$" type. It is solved with a cancellation: $\lim\limits_{t\to-1} \frac{t^3-t}{t^2-1} = \lim\limits_{t\to-1} \frac{t(t^2-1)}{(t^2-1)} = \lim\limits_{t\to-1} t = -1$. Since $\lim\limits_{t\to-1} \frac{3t^2-1}{2t} = \frac{2}{-2} = -1$, L'Hospital's rule provides the same answer.

iv. This is a limit of "$\frac{0}{0}$" type. By L'Hospital's rule, $\lim\limits_{x\to1} \frac{x^3-1}{x^2-1} = \lim\limits_{x\to1} \frac{3x^2}{2x} = \frac{3}{2}$. This limit can also be determined via a cancellation. After checking that $x^3-1 = (x-1)(x^2+x+1)$, observe that $\lim\limits_{x\to1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim\limits_{x\to1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$.

v. By rationalizing, factoring, and canceling,

$$\frac{x^2-81}{\sqrt{x}-3} = \frac{x^2-81}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{x^2-81}{x-9}(\sqrt{x}+3) = \frac{(x-9)(x+9)}{x-9}(\sqrt{x}+3) = (x+9)(\sqrt{x}+3)\,.$$

So $\lim\limits_{x\to 9}\frac{x^2-81}{\sqrt{x}-3} = \lim\limits_{x\to 9}(x+9)(\sqrt{x}+3) = 108$. Because $\lim\limits_{x\to 9}\frac{2x}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim\limits_{x\to 9}4x^{\frac{3}{2}} = 4(3^3) = 108$, L'Hospital's rule gives the same result.

vi. By rationalizing, $\frac{x}{\sqrt{1+3x}-1} = \frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} = \frac{x(\sqrt{1+3x}+1)}{(1+3x)-1} = \frac{\sqrt{1+3x}+1}{3}$. So $\lim\limits_{x\to 0}\frac{x}{\sqrt{1+3x}-1} = \lim\limits_{x\to 0}\frac{\sqrt{1+3x}+1}{3} = \frac{2}{3}$. Since $\lim\limits_{x\to 0}\frac{1}{\frac{1}{2}(1+3x)^{-\frac{1}{2}}(3)} = \lim\limits_{x\to 0}\frac{2}{3}(1+3x)^{\frac{1}{2}} = \frac{2}{3}$, we get the same thing with L'Hospital's rule.

vii. By rationalizing, $\frac{4-\sqrt{s}}{s-16} = \frac{4-\sqrt{s}}{s-16}\cdot\frac{4+\sqrt{s}}{4+\sqrt{s}} = \frac{16-s}{(s-16)(4+\sqrt{s})} = \frac{-1}{4+\sqrt{s}}$. So $\lim\limits_{s\to 16}\frac{4-\sqrt{s}}{s-16} = \lim\limits_{s\to 16}\frac{-1}{4+\sqrt{s}} = -\frac{1}{8}$. What answer does L'Hospital's rule provide?

viii. The limit $\lim\limits_{x\to 2}\frac{|x-2|}{x-2}$ does not exist. To see this, check that $\lim\limits_{x\to 2^-}\frac{|x-2|}{x-2} = \lim\limits_{x\to 2^-}\frac{-(x-2)}{x-2} = -1$. This is so, because for $x<2$, we get $x-2<0$, so that $|x-2| = -(x-2)$. Show in a

similar way that $\lim\limits_{x\to 2^+}\frac{|x-2|}{x-2} = 1$. Since the limit from the left is not equal to the limit from the right, $\lim\limits_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

ix. This limit is a "$\frac{0}{0}$" limit that can be solved by a cancellation:

$$\lim_{h\to 0}\frac{(h-5)^2-25}{h} = \lim_{h\to 0}\frac{h^2-10h+25-25}{h} = \lim_{h\to 0}\frac{h(h-10)}{h} = \lim_{h\to 0}(h-10) = -10\,.$$

Replacing h by Δx we see that $\frac{(h-5)^2-25}{\Delta x} = \frac{(\Delta x-5)^2-25}{\Delta x}$. It follows that $\lim\limits_{h\to 0}\frac{(h-5)^2-25}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{(-5+\Delta x)^2-25}{\Delta x}$. For $f(x)=x^2$, we see that $f'(x) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$. It follows that the earlier limit is equal to the derivative $f'(x)=2x$ evaluated at $x=-5$. So it is $f'(-5)=-10$.

x. After letting $\pi = x$ and $h = \Delta x$, notice that this limit is the derivative of the function $f(x)=\sin x$ evaluated at $x=\pi$. Since $f'(x)=\cos x$ and $f'(\pi)=\cos\pi=-1$, the value of this limit is -1.

7.7. Because $\lim\limits_{x\to a} f(x) = 5$ and $\lim\limits_{x\to a} g(x) = 3$, we see that $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{5}{3}$. Similarly, $\lim\limits_{x\to a}\frac{2f(x)}{g(x)-f(x)} = \frac{2\cdot 5}{3-5} = \frac{10}{-2} = -5$. In the situation $\lim\limits_{x\to a}\frac{2f(x)}{5g(x)-3f(x)} = \frac{2\cdot 5}{3-5} = \frac{10}{-2} = -5$, notice that the numerator goes to 10 and the denominator goes to zero. So the limit does not exist.

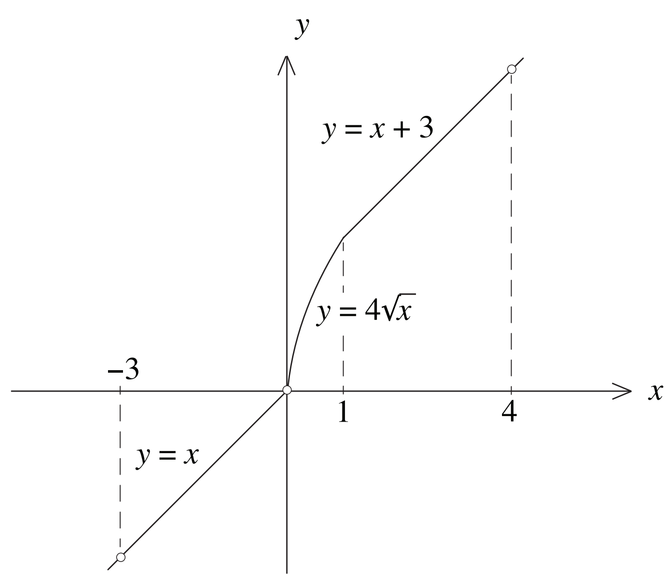
7.8. For (i), we need to check that the continuity criterion is satisfied for $c=5$. Because $f(5) = 1+\sqrt{5^2-9} = 1+\sqrt{16} = 5$, we know that $f(5)$ makes sense. In view of the fact that $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to 5}(1+\sqrt{x^2-9}) = 1+\sqrt{16} = 5 = f(5)$, the continuity criterion is satisfied. So $f(x)$ is continuous at $c=5$. In case (ii), we need to check the continuity criterion for $c=4$. Because $g(4) = \frac{5}{2\cdot 16-1} = \frac{5}{31}$, $g(x)$ is defined at $x=4$. Since $\lim\limits_{x\to 4} g(x) = \lim\limits_{x\to 4}\frac{x+1}{2x^2-1} = \frac{5}{31} = g(4)$, the criterion is met. So $g(x)$ is continuous at $c=4$.

7.9. For $\frac{x^4+17}{6x^2+x-1}$ to make sense we need only for $6x^2+x-1$ to be non-zero. By the quadratic formula, $6x^2+x-1=0$ when $x=\frac{-1\pm\sqrt{1+24}}{12}=\frac{-1\pm 5}{12}=-\frac{1}{2}$ or $\frac{1}{3}$. So the domain of $G(x)=\frac{x^4+17}{6x^2+x-1}$ is $\{x \mid x\neq -\frac{1}{2} \text{ and } x\neq \frac{1}{3}\}$. For $\frac{1}{\sqrt{x+1}}$ to make sense we need $x>-1$. So the domain of $H(x)=\frac{1}{\sqrt{x+1}}$ is $\{x \mid x>-1\}$. That $G(x)$ is continuous on its domain follows from Remark 7.2. That $H(x)$ is continuous on its domain follows from Example 7.8 and two applications of the continuity theorem. First to the composite of the functions $f(x)=\sqrt{x}$ and $g(x)=x+1$, and then to the quotient $\frac{f(x)}{g(x)}$ with $f(x)=1$ and $g(x)=\sqrt{x+1}$.

7.10. That the functions $cx+1$ and cx^2-11 are continuous for any constant c follows from Remark 7.1. (Notice that the constant c plays different roles in the remark and in the statement of the problem.) For the function $f(x)$ to be continuous, its graph must be in one connected piece. In view of what was already said, this will be so precisely if the graphs of $cx+1$ and cx^2-1 meet when $x=3$. So we need to have $3c+1=9c-1$. So $6c=2$ and $c=\frac{1}{3}$.

7.11. We know from Remark 7.2 that the function $\frac{x^2-2x-8}{x-4}$ is continuous, except when $x=4$ where it is not defined. If $\lim\limits_{x\to 4}\frac{x^2-2x-8}{x-4}=6$, then the definition $f(4)=6$ will close the gap in the graph. Because $\lim\limits_{x\to 4}\frac{x^2-2x-8}{x-4}=\lim\limits_{x\to 4}\frac{(x-4)(x+2)}{x-4}=\lim\limits_{x\to 4}(x+2)=6$, the gap is indeed closed. So $f(x)$ is continuous at $x=4$ and hence for all x. Notice that $f(x)$ and $g(x)=x+2$ are exactly the same function?

7.12. The first and third graphs are most easily sketched after observing that $g(x)=x$ when $x\neq -3$ and that $j(x)=\frac{(x-4)(x+3)}{x-4}=x+3$ when $x\neq 4$. The three holes in the graph occur for the values of x for which the corresponding functions are not defined. By replacing the function $g(x)$ by $g_1(x)=x$ for all $x\leq 0$, the function $j(x)$ by $j_1(x)=x+3$ for all $1\leq x$, and then defining the function $y=f(x)$ by $f(x)=g_1(x)=x$ for $x\leq 0$, $f(x)=h(x)$ for $0<x<1$, and



$f(x)=j_1(x)$ for $1\leq x$, the three holes are filled in, so that $y=f(x)$ is continuous for all x.

7.13. Let $f(x)=2x^3+x^2+2$. Because $f(-2)=2(-2)^3+(-2)^2+2=-16+4+2=-10<0$ and $f(-1)=2(-1)^3+(-1)^2+2=-2+1+2=1>0$, it follows from the intermediate value theorem that there exists some x in the interval $(-2,-1)$ such that $f(x) = 2x^3 + x^2 + 2 = 0$.

7.14. If m and M are the minimum and maximum values of f on $[-1,1]$, then $m \le 2 < 3 \le M$. So by the intermediate value theorem, there is, for any number v with $2 \le v \le 3$, a number u between -1 and 1 such that $f(u) = v$. Since e lies between 2 and 3, we get the x we need.

7.15. Enough detail has already been supplied.

7.16. Enough detail has been supplied.

7.17. **i.** $\lim\limits_{\Delta x\to 0} \frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\left(\frac{1}{x+\Delta x} - \frac{1}{x}\right) = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\frac{x-(x+\Delta x)}{(x+\Delta x)x} = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\frac{-\Delta x}{(x+\Delta x)x}$
$= \lim\limits_{\Delta x\to 0} \frac{-1}{(x+\Delta x)x} = \frac{-1}{x^2} = -x^{-2}$

ii. $\lim\limits_{\Delta x\to 0} \frac{(x+\Delta x)^3-x^3}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3)-x^3}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}$
$= \lim\limits_{\Delta x\to 0}(3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$

iii. $\lim\limits_{\Delta x\to 0} \frac{\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\left(\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}\right) = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\frac{x^2-(x+\Delta x)^2}{(x+\Delta x)^2x^2} = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\frac{x^2-x^2-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2}$
$= \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\frac{-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2} = \lim\limits_{\Delta x\to 0} \frac{-2x-\Delta x}{(x+\Delta x)^2x^2} = \frac{-2x}{x^4} = -2x^{-3}$

iv. $\lim\limits_{\Delta x\to 0} \frac{(x+\Delta x)-\frac{2}{x+\Delta x}-(x-\frac{2}{x})}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\left(\Delta x - (\frac{2}{x+\Delta x} - \frac{2}{x})\right) = \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\left(\Delta x - \frac{2x-2(x+\Delta x)}{(x+\Delta x)x}\right)$
$= \lim\limits_{\Delta x\to 0} \frac{1}{\Delta x}\left(\Delta x - \frac{-2\Delta x}{(x+\Delta x)x}\right) = \lim\limits_{\Delta x\to 0} \left(1+\frac{2}{(x+\Delta x)x}\right) = 1+\frac{2}{x^2} = 1+2x^{-2}$

v. $\lim\limits_{\Delta x\to 0} \frac{\sqrt{6-(x+\Delta x)}-\sqrt{6-x}}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{(\sqrt{6-(x+\Delta x)}-\sqrt{6-x})(\sqrt{6-(x+\Delta x)}+\sqrt{6-x})}{\Delta x(\sqrt{6-(x+\Delta x)}+\sqrt{6-x})}$
$= \lim\limits_{\Delta x\to 0} \frac{6-(x+\Delta x)-(6-x)}{\Delta x(\sqrt{6-(x+\Delta x)}+\sqrt{6-x})} = \lim\limits_{\Delta x\to 0} \frac{-\Delta x}{\Delta x(\sqrt{6-(x+\Delta x)}+\sqrt{6-x})} = \lim\limits_{\Delta x\to 0} \frac{-1}{\sqrt{6-(x+\Delta x)}+\sqrt{6-x}}$
$= \frac{-1}{2\sqrt{6-x}} = -\frac{1}{2}(x-6)^{-\frac{1}{2}}$

7.18. We saw in Section 7.4 that the definition of the derivative of a function $y = f(x)$ has the two formulations

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{and} \quad f'(x) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

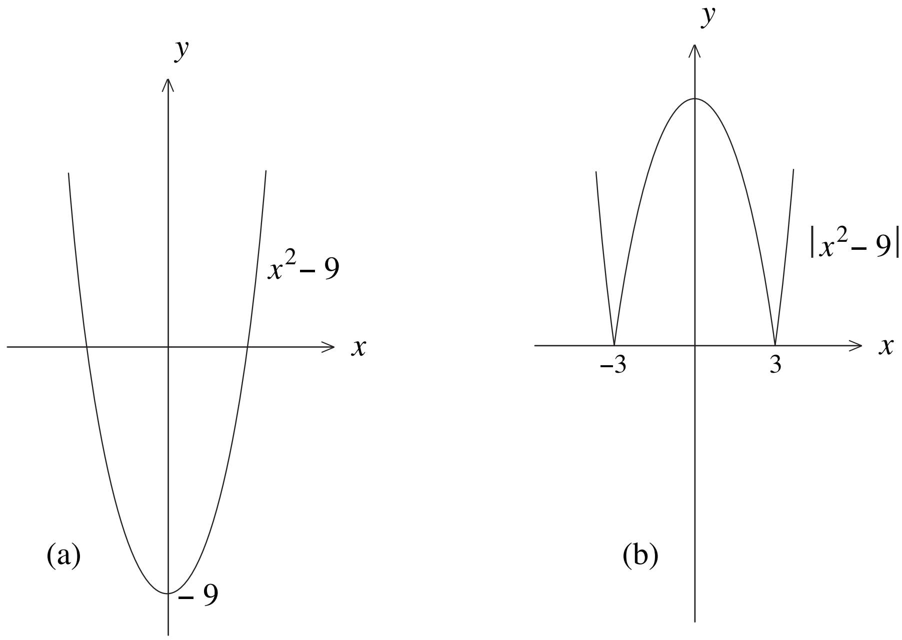
i. Take the function $f(x) = x^2$ and use the first formula to write down what $f'(5)$ means.

ii. Take the function $f(x) = \sqrt{x}$ and use the first formula to write down what $f'(1)$ means.

iii. Take the function $f(x) = x^9$ and use the second formula to write down what $f'(1)$ means.

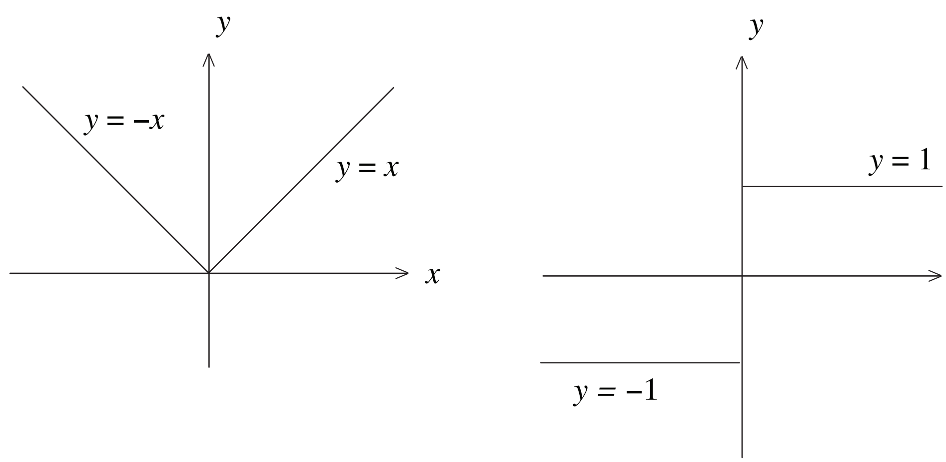
iv. Take the function $f(x) = \sqrt{1+x}$ and use the first formula to write down what $f'(0)$ means. Then change notation.

7.19. **i.** The negative part of the graph of $f(x) = x^2 - 9$ of Figure (a) is made positive by reflecting it as shown in Figure (b) to obtain the graph of $g(x) = |x^2-9|$. The derivative of $f(x) = x^2 - 9$ is $f'(x) = 2x$. This is also the derivative of $y = g(x)$ for $x < -3$ and $x > 3$. For $-3 < x < 3$ the function $y = g(x)$ is the same as $y = -(x^2-9) = -x^2+9$. So the derivative of $g(x) = |x^2 - 9|$ for $-3 < x < 3$ is $-2x$. The graph of $f'(x) = 2x$ is the



the line $y = 2x$. The graphs of $g'(x)$ is the indicated combination of the lines $y = 2x$ and $y = -2x$. The sharp corners at $x = -3$ and $x = 3$ tell us that $g(x)$ is not differentiable there.

ii. The graph of $f(x) = |x|$ coincides with that of $y = -x$ for $x < 0$ and with $y = x$ for $x > 0$. So $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$. Since the graph of $f(x) = |x|$



comes to a sharp point at $x = 0$, the function is not differentiable for $x = 0$.

7.20. **i.** Since $f'(x) = 3x^2$, the tangent has slope $f'(-2) = 12$ and equation $y + 8 = 12(x + 2)$.

ii. Since $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, the tangent has slope $g'(-3) = \frac{1}{9^{\frac{1}{3}}}$ and equation $(y - 3^{\frac{1}{3}}) = \frac{1}{9^{\frac{1}{3}}}(x+3)$

iii. Since $f'(x) = -x^{-2} = \frac{1}{x^2}$, the tangent has slope $f'(-\frac{1}{3}) = 9$ and equation $y+3 = 9(x+\frac{1}{3})$.

iv. Since $f'(x) = -2x^{-3} = -\frac{2}{x^3}$, the tangent has slope $f'(-2) = \frac{1}{4}$ and equation $y - \frac{1}{4} = \frac{1}{4}(x + 2)$.

7.21. **i.** $f'(x) = 0$

ii. For $g(x) = x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}, g'(x) = \frac{1}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

iii. $f'(x) = 14x - 5$

iv. For $y = h(x) = \frac{1}{x^4+x^2+1}$, $y' = \frac{2}{3}x^{-\frac{2}{3}} + 3\pi x^2$.

v. $g'(x) = -3x^{-2} + 3$

vi. $f'(x) = 6x^2 + 3 + 2x^{-3}$

vii. $g'(x) = 2x^{-\frac{1}{2}} - 5x^{-2}$

viii. $h'(x) = 18x^2 - \frac{7}{3}x^{-\frac{2}{3}}$

7.22. **i.** $G'(x) = 2x(2x-7) + (x^2+1)2 = 4x^2 - 14x + 2x^2 + 2 = 6x^2 - 14x + 2$

ii. $f'(x) = \frac{d}{dx}\frac{a-x^2}{1+x^2} = \frac{-2x(1+x^2)-(a-x^2)2x}{(1+x^2)^2} = \frac{-2x(1+x^2+a-x^2)}{(1+x^2)^2} = \frac{-2x(1+a)}{(1+x^2)^2}$

iii. For $s = g(t) = t^{\frac{1}{3}}(t+2) = t^{\frac{4}{3}} + 2t^{\frac{1}{3}}$, $\frac{ds}{dt} = \frac{4}{3}t^{\frac{1}{3}} + \frac{2}{3}t^{-\frac{2}{3}}$.

iv. For $y = h(x) = \frac{1}{x^4+x^2+1}$, $\frac{dy}{dx} = \frac{d}{dx}(x^4+x^2+1)^{-1} = -(x^4+x^2+1)^{-2}(4x^3+2x)$.

v. $\frac{dy}{dx} = \frac{d}{dx}(2x^3+4x^5)^6(7x^8+9x^{10})^{11}$

$= [6(2x^3+4x^5)^5(6x^2+20x^4)](7x^8+9x^{10})^{11} + (2x^3+4x^5)^6[11(7x^8+9x^{10})^{10}(56x^7+90x^9)]$

vi. $f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(2x) = x(1-x^2)^{-\frac{1}{2}}$

vii. Because $y = \frac{x}{\sqrt{9-4x}} = x(9-4x)^{-\frac{1}{2}}$, we get $\frac{dy}{dx} = (9-4x)^{-\frac{1}{2}} + x[-\frac{1}{2}(9-4x)^{-\frac{3}{2}}(-4)] = (9-4x)^{-\frac{1}{2}} + 2x(9-4x)^{-\frac{3}{2}} = \frac{(9-4x)+2x}{(9-4x)^{\frac{3}{2}}} = \frac{9-2x}{(9-4x)^{\frac{3}{2}}}$.

viii. $F'(x) = \frac{[5(x^2+4x+6)^4(2x+4)](x^3+4x^5)^{\frac{1}{2}} - (x^2+4x+6)^5[\frac{1}{2}(x^3+4x^5)^{-\frac{1}{2}}(3x^2+20x^4)]}{x^3+4x^5}$.

ix. Because $s(t) = \sqrt[4]{\frac{t^3+1}{t^3-1}} = \left(\frac{t^3+1}{t^3-1}\right)^{\frac{1}{4}}$, we find that

$s'(t) = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-\frac{3}{4}}\left[\frac{3t^2(t^3-1)-(t^3+1)3t^2}{(t^3-1)^2}\right] = \frac{1}{4}\left(\frac{t^3-1}{t^3+1}\right)^{\frac{3}{4}}\left[\frac{-6t^2}{(t^3-1)^2}\right] = -\frac{3}{2}\left(\frac{t^3-1}{t^3+1}\right)^{\frac{3}{4}}\frac{t^2}{(t^3-1)^2}$.

x. $\frac{d}{dx}f(g(h(x))) = f'(g(h(x)))\cdot(g(h(x)))' = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$.

7.23. **i.** Because $g'(x) = -3x^2$, the slope of the tangent to the graph at the point $(0,1)$ is $g'(0) = 0$. By the point-slope form of the equation of a line, the equation of the tangent line is $y - 1 = 0(x-0)$ or $y = 1$.

ii. Because $h'(x) = -1(2x-1)^{-2}(2) = \frac{-2}{(2x-1)^2}$, we see that $h'(-1) = \frac{-2}{(-3)^2} = -\frac{2}{9}$. By the point-slope form, the equation of the tangent line is $y + \frac{1}{3} = -\frac{2}{9}(x+1)$ or $y = -\frac{2}{9}x - \frac{5}{9}$.

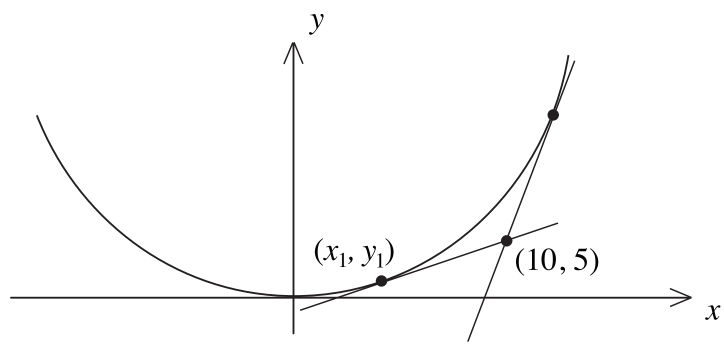
iii. Because $y' = \frac{1(x-3)-x(1)}{(x-3)^2} = \frac{-3}{(x-3)^2}$, we see that the slope of the tangent line is $\frac{-3}{(6-3)^2} = \frac{-3}{9} = -\frac{1}{3}$. By the point-slope form of the equation of a line, we get that the equation of the tangent is $y - 2 = -\frac{1}{3}(x-6)$ or $y = -\frac{1}{3}x + 4$.

7.24. Converting the equation $x - 2y = 1$ into slope-intercept form, we get $2y = x-1$ or $y = \frac{1}{2}x - \frac{1}{2}$. So $\frac{1}{2}$ is the slope of the line. Next, we need the point on the graph of $f(x) = x^2 - 1$ with the property that the tangent at that point has slope $\frac{1}{2}$. Because $f'(x) = 2x$, this occurs when $x = \frac{1}{4}$. So the point is $\left(\frac{1}{4}, f(\frac{1}{4})\right) = \left(\frac{1}{4}, -\frac{15}{16}\right)$. The equation we are looking for is that of the line through $\left(\frac{1}{4}, -\frac{15}{16}\right)$ with slope $\frac{1}{2}$. By the point-slope form of the equation of a line we get $y - \left(-\frac{15}{16}\right) = \frac{1}{2}(x - \frac{1}{4})$ or $y + \frac{15}{16} = \frac{1}{2}x - \frac{1}{8}$ or, finally, $y = \frac{1}{2}x - \frac{17}{16}$.

7.25. For the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ to have a horizontal tangent, we need to have $f'(x) = 6x^2 - 6x - 6 = 0$. By the quadratic formula, $6(x^2 - x - 1) = 0$ for $x = \frac{1 \pm \sqrt{5}}{2}$.

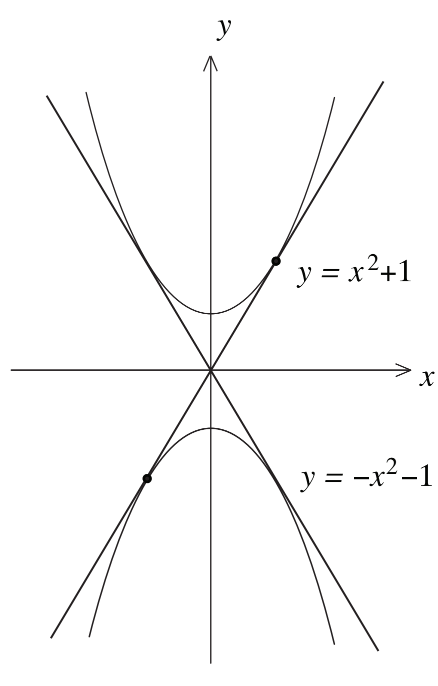
7.26. For $y = 6x^3 + 5x - 3$ to have a tangent line of slope 4, the derivative $y' = 18x^2 + 5$ must be equal to 4. But $4 = 18x^2 + 5$ implies that $x^2 = -\frac{1}{18}$ and this is impossible.

7.27. A reformulation of the question is this: For what point on the graph of $y = \frac{1}{10}x^2$ will the tangent line hit the point (10, 5)? Let this point be (x_1, y_1). Because the slope of the tangent line is $\frac{1}{5}x_1$ and the point (x_1, y_1) lies on it, we see that the equation of the tangent is $y - y_1 = \frac{1}{5}x_1(x - x_1)$. Since $(10, 5)$ must be on this line, $5 - y_1 = \frac{1}{5}x_1(10 - x_1)$. Since (x_1, y_1)



is also on the parabola, $y_1 = \frac{1}{10}x_1^2$. Therefore, $5 - \frac{1}{10}x_1^2 = \frac{1}{5}x_1(10 - x_1)$. Multiplying by 10 gives $50 - x_1^2 = 20x_1 - 2x_1^2$. So $x_1^2 - 20x_1 + 50 = 0$, and by the quadratic formula, $x_1 = 10 \pm 5\sqrt{2}$. It follows that there are two such points. Both are shown in the figure.

7.28. The goal is to find the points where the tangencies occur. See the figure below. Let $y = mx + b$ be one of the two lines and let (x_1, y_1) and (x_2, y_2) be the two points of tangency on the graphs



of $f(x) = x^2+1$ and $g(x) = -x^2-1$, respectively. Observe that $f'(x_1) = m = g'(x_2)$ and hence that $2x_1 = m = -2x_2$. So $x_2 = -x_1$. Therefore, $y_2 = -x_2^2-1 = -(-x_1)^2-1 = -x_1^2-1 = -y_1$. Because $y_1 = mx_1+b$ and $y_2 = mx_2+b$, we get $y_1 = mx_1+b$ and $-y_1 = -mx_1+b$, and hence that $2b = 0$ and $b = 0$. Because $m = 2x_1$, and (x_1, y_1) lies on the graphs of both $y = mx$ and $y = x^2+1$, we get $x_1^2+1 = y_1 = mx_1 = 2x_1^2$. So $x_1^2 = 1$ and hence $x_1 = \pm 1$. When $x_1 = 1$, we get $y_1 = 1^2+1 = 2, x_2 = -1$ and $y_2 = -(-1)^2-1 = -2$. So the two points are $(1, 2)$ and $(-1, -2)$, the points in the figure. When $x_1 = -1$, we get $y_1 = (-1)^2+1 = 2, x_2 = 1$, and $y_2 = -1^2-1 = -2$. So the other two points are $(-1, 2)$ and $(1, -2)$.

7.29. The facts to remember are these: If $f'(x) > 0$ for all x in an interval I, then $f(x)$ is increasing over I; and if $f'(x) < 0$ for all x in I, then $f(x)$ is decreasing over I. If $f'(x) = 0$, then the graph of f has a horizontal tangent. Going from left to right: We see that the function whose derivative has graph a is increasing, then suddenly decreasing, then suddenly increasing, and then suddenly decreasing again. This is the pattern of graph (ii). The function whose derivative has graph b is increasing, then has a horizontal tangent, then is decreasing, has another horizontal tangent, then increases until it has another horizontal tangent, and it is decreasing thereafter. This is the pattern of graph (iv). Similar considerations match graph c with graph (iii) and graph d with graph (i).

7.30. **i.** It's best to separate the two cases $y_0 > 0$ and $y_0 < 0$. We'll do the second. The first is similar. For $y < 0$, $y = -(r^2-x^2)^{\frac{1}{2}}$. So $y' = -\frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x) = \frac{x}{(r^2-x^2)^{\frac{1}{2}}}$. So the slope of the circle at (x_0, y_0) is $\frac{x_0}{(r^2-x_0^2)^{\frac{1}{2}}} = -\frac{x_0}{y_0}$.

ii. Let (x_0, y_0) be a point of tangency. Applying the answer above, we see that $-\frac{x_0}{y_0} = -\frac{1}{3}$. So $y_0 = 3x_0$. Since (x_0, y_0) satisfies $x_0^2+y_0^2 = 1$, we get $x_0^2+(3x_0)^2 = 1$ and hence $x_0 = \pm\frac{1}{\sqrt{10}}$. Since $(x_0, 3x_0)$ is on the tangent line, $3(\frac{\pm 1}{\sqrt{10}}) = -\frac{1}{3}(\frac{\pm 1}{\sqrt{10}}) + b$ and it follows that $b = 3(\frac{\pm 1}{\sqrt{10}}) + \frac{1}{3}(\frac{\pm 1}{\sqrt{10}}) = \frac{10}{3}\frac{\pm 1}{\sqrt{10}} = \pm\frac{\sqrt{10}}{3}$.

7.31. **i.** Because $y = \sin(x^{-1})$, we get $y' = \cos(x^{-1}) \cdot (-x^{-2}) = -\frac{\cos x^{-1}}{x^2}$.

ii. Because $\sin^2(\cos 4x) = [\sin(\cos 4x)]^2$, we get
$y' = 2[\sin(\cos(4x))] \cdot \cos(\cos 4x) \cdot (-\sin 4x) \cdot 4 = -8(\sin(\cos 4x))(\cos(\cos 4x))(\sin 4x)$.

iii. $y' = \frac{[(2\sin x)(\cos x)]\cos x - (\sin^2 x)(-\sin x)}{\cos^2 x} = \frac{2\sin x\cos^2 x + \sin^3 x}{\cos^2 x}$

iv. Because $y = x\sin(x^{-1})$, we get

$$y' = \sin(x^{-1}) + x \cdot \cos(x^{-1}) \cdot (-x^{-2}) = \sin(x^{-1}) - x^{-1}\cos(x^{-1}).$$

vi. $y' = \sec^2(3x) \cdot 3 = 3\sec^2(3x)$

vii. $y' = -5(\cos\sqrt{x^2+1})^{-6}(-\sin\sqrt{x^2+1})(\frac{1}{2})(x^2+1)^{-\frac{1}{2}}(2x) = \frac{5x\sin\sqrt{x^2+1}}{(x^2+1)^{\frac{1}{2}}\cos^6(\sqrt{x^2+1})}$

viii. $y' = 6(1+\sec^3 x)^5(3\sec^2 x)(\sec x\tan x) = 18\tan x(\sec^3 x)(1+\sec^3 x)^5$

ix. $y' = \sec^2(x^2) \cdot (2x) + 2\tan x\sec^2 x = 2x\sec^2(x^2) + 2\tan x\sec^2 x$

x. $y' = \frac{1}{2}(1+2\tan x)^{-\frac{1}{2}}(2\sec^2 x) = \frac{\sec^2 x}{\sqrt{1+2\tan x}}$

7.32. **i.** $6x\cos(3x^2+1)$

ii. $2\sin(\sqrt{t})\cdot\cos(\sqrt{t})\cdot\frac{1}{2}t^{-\frac{1}{2}}+2\cos(\sqrt{t})\cdot(-\sin(\sqrt{t}))\cdot\frac{1}{2}t^{-\frac{1}{2}}=0$

iii. $\sec^2\sqrt{u^2+27u}\cdot\frac{1}{2}(u^2+27u)^{-\frac{1}{2}}\cdot(2u+27)$

iv. $\frac{d}{dx}\sec^2 x=2(\sec x)\cdot(\sec x)(\tan x)=2\sec^2 x\tan x$

7.33. **i.** $(\cos\alpha(t))\alpha'(t)$

ii. $(-\sin\beta(t))\beta'(t)$

7.34. Note first that $\lim\limits_{\theta\to\frac{\pi}{3}}\frac{\cos\theta-0.5}{\theta-\frac{\pi}{3}}=\lim\limits_{\theta\to\frac{\pi}{3}}\frac{\cos\theta-\cos\frac{\pi}{3}}{\theta-\frac{\pi}{3}}$. Since this is the derivative of the function $\cos\theta$ evaluated at $\theta=\frac{\pi}{3}$, this limit is equal to $-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$.

7.35. Because $y'=\sec^2 x$, the slope of the tangent line is $\sec^2\frac{\pi}{3}=4$. So its equation is $y-\sqrt{3}=4(x-\frac{\pi}{3})$ or $y=4x-\frac{4\pi}{3}+\sqrt{3}$.

7.36. **i.** Since $x(t)^2=t^2=y(t)$, every point on the path of the point lies on the parabola $y=x^2$. This is a standard parabola rising from the origin. Since $x(t)=t\geq 0$, the point starts at the origin and moves up along the right side of the parabola with speed $\sqrt{x'(t)^2+y'(t)^2}=\sqrt{1+(2t)^2}=\sqrt{1+4t^2}$. It's initial speed is 1 and it moves faster and faster with increasing t.

ii. Squaring both $x(t)$ and $y(t)$ we get $x(t)^2=t^2$ and $y(t)^2=1-t^2=1-x(t)^2$. So $x(t)^2+y(t)^2=1$ and it follows that the point moves on the circle $x^2+y^2=1$. With $t=0$, $x(0)=0$ and $y(0)=1$. So the point starts at $(0,1)$ on the circle and moves in the direction of the point $(1,0)$, arriving there when $t=1$. Since $x'(t)=1$ and $y'(t)=\frac{1}{2}(1-t^2)^{-\frac{1}{2}}(-2t)$, the speed of the point at any time t with $0\leq t\leq 1$ is $\sqrt{x'(t)^2+y'(t)^2}=\sqrt{1+\frac{t^2}{1-t^2}}=\sqrt{\frac{1}{1-t^2}}$. So the speed is 1 and time $t=0$. The point increases its speed and slams into the point $(1,0)$ with infinite speed.

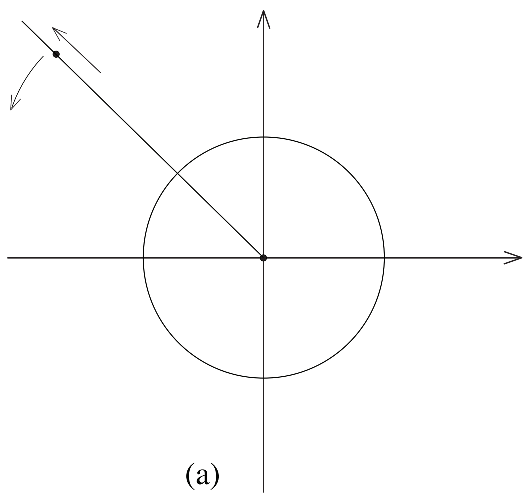
iii. Since $y(t)=\cos x(t)$, for any $t\geq 0$, the point moves on the curve $y=\cos x$ with $x\geq 0$. Since $x'(t)=1$ its x-coordinate moves with constant speed. Since $\sqrt{x'(t)^2+y'(t)^2}=\sqrt{1+\sin^2 t}$. It follows that the speed of the point varies between 1 and $\sqrt{2}$.

7.37. **i.** Since $x(t)^2+y(t)^2=1$, the point moves on the circle $x^2+y^2=1$. When $t=0$ the point is at $(1,0)$. Since $y(t)=\sin t$ increases from 0 to 1 as t flows from $t=0$ to $t=\frac{\pi}{2}$, the point moves from $(1,0)$ to $(0,1)$ during this time. The point continues its counterclockwise motion around the circle. The speed of the point is constant because $\sqrt{x'(t)^2+y'(t)^2}=\sqrt{\sin^2 t+\cos^2 t}=1$.

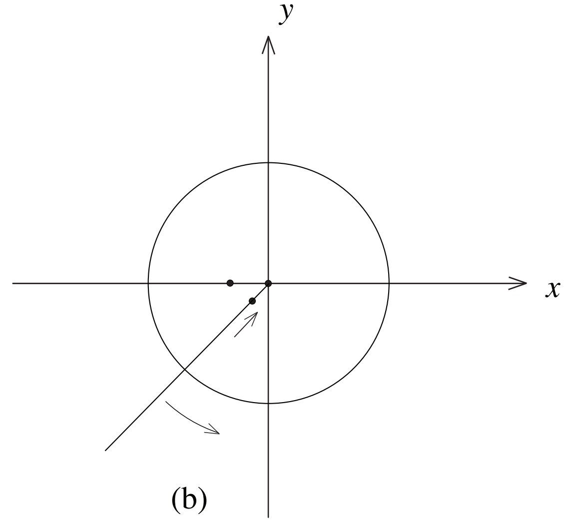
ii. This motion is a combination of the circular motion described in (i) and an outward motion. Think of it this way. For any time t consider the point (x,y) given by $x=\cos t$ and $y=\sin t$ as well as the ray from the origin through this point. To locate the position $(x(t),y(t))$ of the moving point at time t observe that the distance between $(x(t),y(t))$ and the origin $(0,0)$ is $\sqrt{(x(t)-0)^2+(y(t)-0)^2}=\sqrt{(t\cos t)^2+(t\sin t)^2}=\sqrt{t^2}=t$. It follows form (i) that the ray rotates counterclockwise at a constant rotational speed of 1. At the same time, the point moves outward on the ray, so that at any time t it is a distance t from the origin. So the speed of the point on the ray is also constant and equal to 1. The point's path is the composite of the two motions: an outward spiral that opens in a counterclockwise way. See figure (a). The speed of the point along this spiral at any time $t \geq 0$ is

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{1+t^2}.$$

So at time $t = 0$, the point is at $(0, 0)$ and has an initial speed of 1.



(a)



(b)

iii. The point's motion is also a combination of two components. The ray behaves exactly as before: it rotates counterclockwise at a constant rotational speed of 1. At the start $t = \pi$, the point is positioned at $(\frac{1}{\pi}(-1), 0)$ and the ray goes through it (and the origin). This time the distance from the point to the origin at any time $t \geq \pi$ is $\sqrt{(x(t)-0)^2 + (y(t)-0)^2} = \sqrt{(\frac{1}{t}\cos t)^2 + (\frac{1}{t}\sin t)^2} = \frac{1}{t}$. So as the ray rotates, the point moves closer and closer to the origin. See figure (b). So it moves toward the origin in a spiral that grows ever smaller. The speed of the point along the rotating ray at any time $t \geq \pi$ is $\left|\frac{d}{dt}\frac{1}{t}\right| = \left|-\frac{1}{t^2}\right| = \frac{1}{t^2}$. The speed of the point along its spiral is

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-\tfrac{1}{t^2}\cos t - \tfrac{1}{t}\sin t)^2 + (-\tfrac{1}{t^2}\sin t + \tfrac{1}{t}\cos t)^2} = \sqrt{\tfrac{1}{t^4} + \tfrac{1}{t^2}}.$$

As our discussion already suggests, the point's speed approaches 0 with increasing time.

7.38. Because $y = (x^2 + 2x + 3)^2$, we get $\frac{dy}{dx} = 2(x^2+2x+3)(2x+2)$. On the other hand, $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 2x + 2$, so that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u(2x+2) = 2(x^2+2x+3)(2x+2)$$

as before. Finally, $\frac{dy}{dx}\Big|_{x=1} = 2(1+2+3)(2+2) = (2)(6)(4) = 48$.

7.39. Since the equation of the tangent line also involves an x and a y, let's change the notation for the coordinates of the point P to (x_0, y_0). Since $f'(x) = 10x - 6x^2$, the slope of the tangent to the curve at this point is $10x_0 - 6x_0^2$. Since $P = (x_0, y_0)$ is on the tangent line, the equation of the tangent line is $y - y_0 = (10x_0 - 6x_0^2)(x - x_0)$.

7.40. Let (x, y) be a possible point of intersection. Since x and y satisfy both equations, $x^2 = 3x - 4$ so that $x^2 - 3x + 4 = 0$. By the quadratic formula, $x = \frac{3 \pm \sqrt{3^2 - 4 \cdot 4}}{2}$. Since $\sqrt{-7}$ does not exist as a real number there can be no such x and hence there is no point of intersection. Since the point $(0, -4)$ is on the line, it follows that the line lies entirely below the parabola. Any line parallel to the given line has an equation of the form $y = 3x + b$. By moving it up, it will touch the parabola at a point where the tangent of the parabola has slope 3. This happens for $2x = 3$ and hence $x = \frac{3}{2}$. The corresponding y coordinate is $y = (\frac{3}{2})^2 = \frac{9}{4}$. So $(\frac{3}{2}, \frac{9}{4})$ is the point where the line touches the parabola.

7.41. The distance between any point (x, y) on the line $y = \frac{1}{2}x + 5$ and the point $(-4, 3)$ is

$$\begin{aligned}\sqrt{(x-(-4))^2 + (y-3)^2} &= \sqrt{(x+4)^2 + ((-\tfrac{1}{2}x + 5) - 3)^2} = \sqrt{(x+4)^2 + (-\tfrac{1}{2}x + 2)^2} \\ &= \sqrt{(x^2 + 8x + 16) + (\tfrac{1}{4}x^2 - 2x + 4)} = \sqrt{\tfrac{5}{4}x^2 + 6x + 20}.\end{aligned}$$

This expresses the distance between the points (x, y) and $(-4, 3)$ as a function $d(x) = \sqrt{\frac{5}{4}x^2 + 6x + 20}$ of the x-coordinate of (x, y). The distance between $(-4, 3)$ and the line is determined by that point (x, y) on the line for which $d(x)$ is a minimum. So we need to find the x for which $d(x) = \sqrt{\frac{5}{4}x^2 + 6x + 20} = (\frac{5}{4}x^2 + 6x + 20)^{\frac{1}{2}}$ attains its minimum value. This task involves the derivative

$$d'(x) = \tfrac{1}{2}(\tfrac{5}{4}x^2 + 6x + 20)^{-\frac{1}{2}}(\tfrac{5}{2}x + 6) = \frac{2(\frac{5}{2}x+6)}{(\frac{5}{4}x^2+6x+20)^{\frac{1}{2}}} = \frac{5x+12}{(\frac{5}{4}x^2+6x+20)^{\frac{1}{2}}}.$$

Since $(-4, 3)$ is not on the line, $(\frac{5}{4}x^2 + 6x + 20)^{\frac{1}{2}}$ is always positive. Notice that $5x + 12 = 0$ when $x = -\frac{12}{5}$, and that this term is negative for $x = -\frac{12}{5}$ and positive for $x > -\frac{12}{5}$. This information about $d'(x)$ tells us that $d(x)$ reaches its minimum value at $x = -\frac{12}{5}$. It follows that the distance between $(-4, 3)$ and the line $y = -\frac{1}{2}x + 5$ is

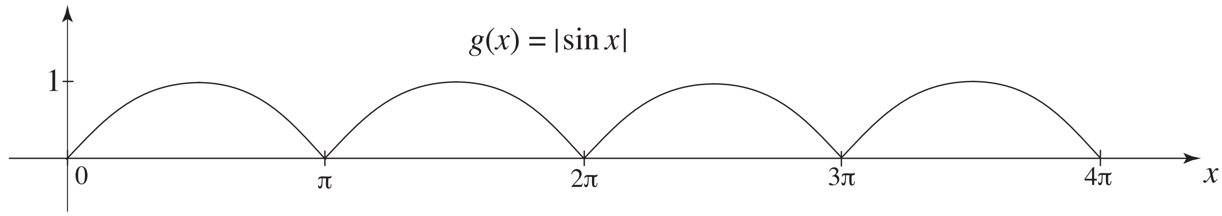
$$d(-\tfrac{12}{5}) = \sqrt{\tfrac{5}{4}(-\tfrac{12}{5})^2 + 6(-\tfrac{12}{5}) + 20} = \sqrt{\tfrac{36}{5} - \tfrac{72}{5} + 20} = \sqrt{\tfrac{64}{5}} = \tfrac{8}{\sqrt{5}}.$$

The x that minimizes the function $d(x)$ is also the x that minimizes the function $d(x)^2 = \frac{5}{4}x^2 + 6x + 20$. In terms of the calculus involved this function is more easily dealt with than $d(x)$. (This explains the last part of the hint.)

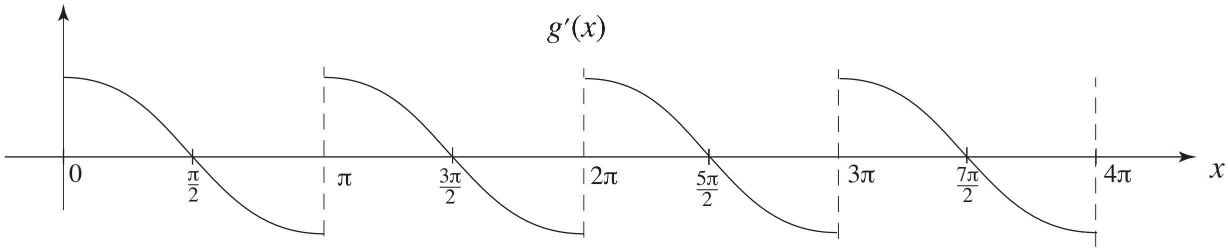
7.42. The term $\left(\frac{f}{g}\right)'(3)$ is the derivative of the quotient of $\frac{f(x)}{g(x)}$ evaluated at $x = 3$. Since $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, it follows that $\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{g(3)^2} = \frac{(-6)(2)-(4)(5)}{4} = -8$. By the chain rule, $(f(g(x)))' = f'(g(x)) \cdot g'(x)$. Evaluating this at $x = 3$, we get $f'(g(3)) \cdot g'(3) = f'(2) \cdot 5 = (-3)(5) = -15$.

7.43. This problem was already considered. See Problem 7.19i .

7.44. For the graphs of $f(x) = \sin x$ and $f'(x) = \cos x$ refer to Figures 4.23 and 4.24 and extend/restrict the pattern to the interval $[0, 4\pi]$. The graph of $g(x) = |\sin x|$ is sketched below. Its shape is explained by the fact that the absolute value makes things positive. The perhaps instinctive response to say that $g'(x) = |\cos x|$ is wrong! Since $g(x) = |\sin x| = \sin x$ over the interval $(0, \pi)$, it follows that $g'(x) = \cos x$ over $(0, \pi)$. The fact that the graph of $g(x) = |\sin x|$ repeats itself over the intervals $(\pi, 2\pi), (2\pi, 3\pi)$, and $(3\pi, 4\pi)$ means that the



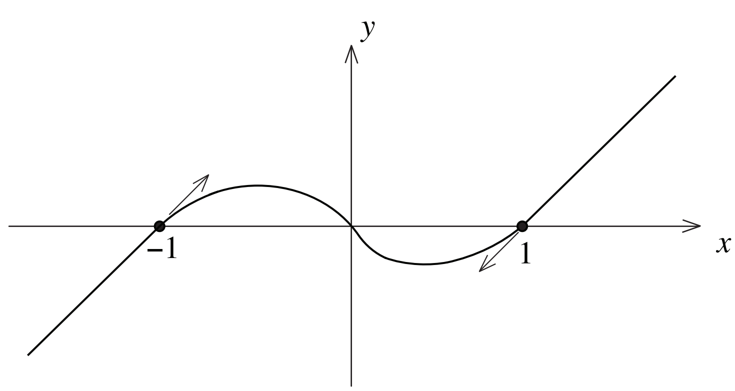
same is true for $g'(x)$ as the graph below illustrates. Notice that $g(x) = |\sin x|$ is not differ-



entiable at $0, \pi, 2\pi, 3\pi$, and 4π.

7.45. As the question is phrased, the answer is that there are no such c and d because the $d\sqrt{x}+c$ is differentiable only for $x > 0$. So we'll add the condition $x > 0$ to the assumptions. The discussion in Section 7.5 about the rules of differentiation tells us that $y = cx^2 + 12$ is differentiable for all x no matter what the constant c is. By the same discussion, $y = d\sqrt{x}+c$ is differentiable for all $x > 0$ no matter what d is (but not for $x \leq 0$). The remaining question is this: For which c and d do the graphs of $y = cx^2 + 12$ and $y = d\sqrt{x} + c$ fit together in such a way that the graph of $f(x)$ is smooth at $x = 1$? The first thing we need is that the two graphs are connected when $x = 1$ because the condition of differentiability implies that of continuity. So we need to have $\lim\limits_{x\to 1^+}(d\sqrt{x}+c) = c \cdot 1^2 + 12$. But this means that $d + c = c + 12$ and hence that $d = 12$. To ensure that the two pieces connect smoothly for $x = 1$, we'll take the derivatives $y' = 2cx$ of $y = cx^2 + 12$ and $y' = \frac{1}{2}dx^{-\frac{1}{2}}$ of $y = d\sqrt{x}+c$ and set them equal to each other with $x = 1$. This gives us $2c = \frac{1}{2}d = 6$ and hence $c = 3$. With $c = 3$ and $d = 12$, the function $f(x)$ is differentiable for all x with $x > 0$.

7.46. The graphs of the two functions $y = x + 1$ for $x \leq -1$ and $y = x - 1$ for $1 \leq x$ are sketched in the figure below. Any function $f(x)$ with $-1 \leq x \leq 1$ that completes these two functions to one that is differentiable for all x needs to satisfy: Its graph must connect smoothly to the graph of $y = x + 1$ at $(-1, 0)$ and to the graph of $y = x - 1$ at $(1, 0)$ and it must be



differentiable over the interval $-1 < x < 1$.

i. There are infinitely many curves that can be drawn between the points $(-1, 0)$ and $(1, 0)$ that have slope 1 at these two endpoints and that are smooth with nonvertical tangents over $-1 \le x \le 1$. There is no limit on the number and variation of wiggles that such a curve can have. One such curve is drawn into the figure above.

ii. A function of the form $f(x) = ax^3 + bx^2 + cx + d$ with $a, b, c,$ and d constants, has derivative $f'(x) = 3ax^2 + 2bx + c$. Other than the observation that polynomial functions are differentiable, we need for $f(x)$ to satisfy the four equations:

$$f(-1) = -a + b - c + d = 0,\ f(1) = a + b + c + d = 0,$$
$$f'(-1) = 3a - 2b + c = 1,\ \text{ and } f'(1) = 3a + 2b + c = 1.$$

What's left is to solve these equations for $a, b, c,$ and d. Subtracting the third equation from the fourth, tells us that $4b = 0$ and hence that $b = 0$. So $3a + c = 1$. Subtracting the first equation from the second, gives us $2a + 2c = 0$, so that $c = -a$. It follows that $2a = 1$, and hence that $a = \frac{1}{2}$. So $c = -\frac{1}{2}$. Inserting the values for $a, b,$ and c into the second equation, we get $\frac{1}{2} + 0 - \frac{1}{2} + d = 0$, so that $d = 0$. So $f(x) = \frac{1}{2}x^3 - \frac{1}{2}x$.

iii. The graph of $y = 0$ lies on the x-axis. So instead of having a slope of 1 at $(1, 0)$, the graph of $f(x)$ needs to have slope 0 at $(1, 0)$. All the other requirements are the same as before. In terms of a function of the form $f(x) = ax^3 + bx^2 + cx + d$ with derivative $f'(x) = 3ax^2 + 2bx + c$, only the last equation is different. We now need to solve:

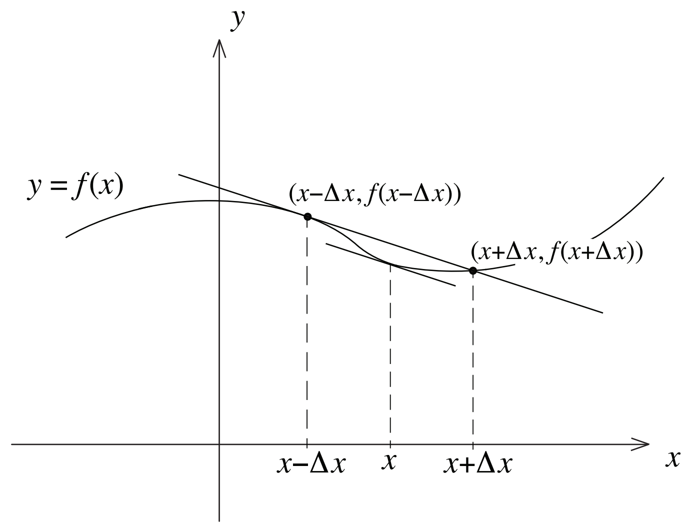
$$f(-1) = -a + b - c + d = 0,\ f(1) = a + b + c + d = 0,$$
$$f'(-1) = 3a - 2b + c = 1,\ \text{ and }\ f'(1) = 3a + 2b + c = 0.$$

Subtracting the fourth equation from the third, we get $-4b = 1$, so that $b = -\frac{1}{4}$. So $3a + c = \frac{1}{2}$. As in the previous example, $2a + 2c = 0$ so that $a + c = 0$. By another subtraction, $2a = \frac{1}{2}$ and $a = \frac{1}{4}$. So $c = -\frac{1}{4}$ and using the second equation, $\frac{1}{4} - \frac{1}{4} - \frac{1}{4} + d = 0$. Since $d = \frac{1}{4}$, the function $f(x)$ is given by $f(x) = \frac{1}{4}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{4}$.

7.47. **i.** Note first that any polynomial function is continuous for all real numbers. So for the two functions $f(x) = x^2 + x$ for $x < 0$ and $g(x) = ax^2 + bx + c$ where $x \ge 0$ and $a, b,$ and c constants to splice together to a continuous functions for all real numbers, we only need have $\lim_{x \to 0^-} (x^2 + x) = g(0) = c$ and therefore that $c = 0$.

ii. The rules of differentiation referred to in Section 7.5 imply that any polynomial functions is differentiable for all real numbers. So we only need to see to it that the two functions splice smoothly at $(0, 0)$. The derivative of $y = x^2 + x$ is $\frac{dy}{dx} = 2x + 1$ and the derivative of $y = ax^2 + bx + c$ is $\frac{dy}{dx} = 2ax + b$. For the two functions to splice smoothly at $x = 0$, we need—in addition to $c = 0$—only that $2a \cdot 0 + b = 1$, so $b = 1$. Therefore any function of the form $g(x) = ax^2 + x$ will satisfy the required differentiability. Since a can be any constant, there are infinitely many such functions.

7.48. A typical graph of a differentiable function $y = f(x)$ is sketched in the figure below. For a given x and Δx, the interval $[x - \Delta x, x + \Delta x]$ is in the domain of the function. We have assumed that $\Delta x > 0$, but this is not essential. (By interchanging some minuses and pluses the case $\Delta x < 0$ can be handled in the same way.) Consider the two points $(x-\Delta x, f(x-\Delta x))$ and $(x+\Delta x, f(x+\Delta x))$ on the graph of the function and the line that they determine. The slope of this line is equal to $\frac{f(x+\Delta x)-f(x-\Delta x)}{x+\Delta x-(x-\Delta x)} = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$. Refer to the figure and let Δx shrink to zero. It seems intuitively clear that in the process the line that the two points



determine should close in on the tangent line to the graph at $(x, f(x))$, so that

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} = f'(x).$$

This equality can be verified analytically as well. Since $f'(x) = \lim\limits_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ and $f'(x) = \lim\limits_{\Delta x\to 0} \frac{f(x-\Delta x)-f(x)}{-\Delta x}$, it follows that

$$\tfrac{1}{2}f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{2\Delta x} \text{ and } \tfrac{1}{2}f'(x) = \lim_{\Delta x\to 0} \frac{f(x-\Delta x)-f(x)}{-2\Delta x} = \lim_{\Delta x\to 0} \frac{-f(x-\Delta x)+f(x)}{2\Delta x},$$

so that $f'(x) = \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = \lim\limits_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)-f(x-\Delta x)+f(x)}{2\Delta x} = \lim\limits_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$. Therefore $f'(x) = \lim\limits_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$ as asserted earlier.

7.49. This is done by rationalizing as follows:

$$\lim_{x\to a} \frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a} \frac{(f(x)-f(a))(\sqrt{x}+\sqrt{a})}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} = \lim_{x\to a}(\sqrt{x}+\sqrt{a})\Big(\frac{f(x)-f(a)}{x-a}\Big) = 2\sqrt{a}f'(a).$$

7.50. **i.** $\lim\limits_{x\to -3} \frac{x^2-x-12}{x+3} = \lim\limits_{x\to -3} \frac{2x-1}{1} = -7$.

ii. $\lim\limits_{x\to 1} \frac{x^3-1}{x^2-1} = \lim\limits_{x\to 1} \frac{3x^2}{2x} = \frac{3}{2}$.

iii. $\lim\limits_{x\to 9} \frac{x^2-81}{\sqrt{x}-3} = \lim\limits_{x\to 9} \frac{x^2-81}{x^{\frac{1}{2}}-3} = \lim\limits_{x\to 9} \frac{2x}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim\limits_{x\to 9} 4x\cdot x^{\frac{1}{2}} = (4)(9)(3) = 108$.

iv. $\lim\limits_{s\to 4} \frac{s^3-7s^2+17s-20}{s^2-5s+4} = \lim\limits_{s\to 4} \frac{3s^2-14s+17}{2s-5} = \frac{9}{3} = 3$.

7.51. From the example, $f(x) = x^5 + x^4 - x - 1$ and $g(x) = x^3 + x^2 - x - 1$. Notice that 1 is a root of both of these polynomials so that $x - 1$ divides both of them. (See segment 4E of the Problems and Projects section of Chapter 4.) Dividing $x - 1$ into $x^5 + x^4 - x - 1$, we get the result $x^4 + 2x^3 + 2x^2 + 2x + 1$ and dividing $x - 1$ into $x^3 + x^2 - x - 1$, we get $x^2 + 2x + 1$.

After canceling the term $x-1$ we see that $\frac{x^5+x^4-x-1}{x^3+x^2-x-1} = \frac{2x^4+2x^3+2x^2+2x+1}{x^2+2x+1}$ and hence that

$$\lim_{x\to 1} \frac{x^5+x^4-x-1}{x^3+x^2-x-1} = \lim_{x\to 1} \frac{x^4+2x^3+2x^2+2x+1}{x^2+2x+1} = \frac{8}{4} = 2.$$

7.52. By the Mean Value Theorem we know that there is a number c between 0 and 9 such that $f'(c) = \frac{f(9)-f(0)}{9-0} = \frac{12}{9} = \frac{4}{3}$. Because $f'(x) = 1 + \frac{1}{2\sqrt{x}}$, we need to solve $\frac{4}{3} = 1 + \frac{1}{2\sqrt{x}}$ for x. Doing so, we get $\frac{1}{3} = \frac{1}{2\sqrt{x}}$, so $2\sqrt{x} = 3$, and hence $x = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

7.53. **i.** Note that $f(3) = 17$ and $f(7) = 9$. Combining the fact that a differentiable function is continuous with the intermediate value theorem (in Section 7.3) tells us that for every number v with $m \le v \le M$, where m and M are the minimum and maximum values of f on $[3,7]$, there is a number u in the interval $[3,7]$ such that $f(u) = v$. Since $m \le 9$ and $17 \le M$, $m \le 4\pi \le M$, there is a d in $[3,7]$ such that $f(d) = 4\pi$,

ii. The mean value theorem (of Section 7.6) with $a = 3$ and $b = 7$ informs us that there is a c between 3 and 7 such that $f(7) - f(3) = f'(c)(7-3)$. Since $9 - 17 = -8$ we get $4f'(c) = -8$ and hence$f'(c) = -2$.

7.54. Treating y as a function of x and using some of the standard rules for differentiating, we get

i. $g'(x) = y^3 + x(3y^2y') = y^3 + 3xy^2y'$

ii. $h'(x) = \frac{3}{2}y^{-\frac{1}{2}}y' + y + xy'$

iii. $k'(x) = \frac{4y^2-4x(2yy')}{y^4} = 4y^{-2} - 8xy^{-3}y'$

iv. $g'(x) = 4(4x + y^{-\frac{3}{2}})^3(4 - \frac{3}{2}y^{-\frac{5}{2}}y')$

v. $g'(x) = 2(2x^2 + 3y^{\frac{1}{2}})(4x + \frac{3}{2}y^{-\frac{1}{2}}y')$.

7.55. **i.** Because $f'(x) = 3x^2 - 3$, the critical numbers are those x for which $3x^2 - 3 = 0$ for x. So $x = \pm 1$.

ii. $F'(x) = \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2 + x^{\frac{4}{5}}(2(x-4)) = \frac{\frac{4}{5}(x-4)^2+x(2x-8)}{x^{\frac{1}{5}}} = \frac{\frac{4}{5}(x-4)^2+2x(x-4)}{x^{\frac{1}{5}}} = \frac{(x-4)[\frac{4}{5}(x-4)+2x]}{x^{\frac{1}{5}}}$
$= \frac{(x-4)\frac{1}{5}(4x-16+10x)}{x^{\frac{1}{5}}} = \frac{(x-4)(14x-16)}{5x^{\frac{1}{5}}}$.

It follows that the critical numbers are $0, 4$, and $\frac{16}{14} = \frac{8}{7}$.

iii. Note that

$$T'(x) = 2x(2x-1)^{\frac{2}{3}} + x^2\frac{2}{3}(2x-1)^{-\frac{1}{3}}(2) = 2x(2x-1)^{\frac{2}{3}} + \frac{4x^2}{3(2x-1)^{\frac{1}{3}}} = \frac{6x(2x-1)+4x^2}{3(2x-1)^{\frac{1}{3}}}$$
$$= \frac{16x^2-6x}{3(2x-1)^{\frac{1}{3}}} = \frac{16x(x-\frac{6}{16})}{3(2x-1)^{\frac{1}{3}}}.$$

So the critical numbers are $\frac{1}{2}, 0$, and $\frac{6}{16} = \frac{3}{8}$.

7.56. **i.** Because $f'(x) = 3x^2 - 4x + 1$, the critical numbers are $\frac{4\pm\sqrt{16-4\cdot 3}}{6} = \frac{4\pm 2}{6}$ and hence $x = \frac{1}{3}$ and $x = 1$. Take $0, \frac{1}{2}$, and 2 as test points. Since $f'(0) = 1, f'(\frac{1}{2}) = \frac{3}{4} - 1 = -\frac{1}{4}$, and $f'(2) = 5$, we find that f is increasing over the intervals $(-\infty, \frac{1}{3})$ and $(1, \infty)$ and decreasing over $(\frac{1}{3}, 1)$. It follows that f has a local maximum value at $\frac{1}{3}$ and a local minimum value at 1.

ii. Check that $f'(x) = 4x^3 - 12x^2 - 16x = 4x(x^2 - 3x - 4) = 4x(x-4)(x+1)$. So the critical numbers are $-1, 0$ and 4. Take $-2, -\frac{1}{2}, 1$, and 5 to be the test points. Check that $f'(-2) = (-8)(-6)(-1) = -48$; $f'(-\frac{1}{2}) = -2(-\frac{9}{2})(\frac{1}{2}) = \frac{9}{2}$, $f'(1) = 4(-3)(2) = -24$, and $f'(5) = 20(1)6 = 120$. It follows that f is increasing over $(-1, 0)$ and $(4, \infty)$, and decreasing over $(-\infty, -1)$ and $(0,\ 4)$. So f has local minima at -1 and 4 and a local maximum at 0.

iii. Observe first that $f(x)$ is defined only when $1 \geq x^2$ or for $-1 \leq x \leq 1$. Note that

$$f'(x) = (1-x^2)^{\frac{1}{2}} + x\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = (1-x^2)^{\frac{1}{2}} - \tfrac{x^2}{(1-x^2)^{\frac{1}{2}}} = \tfrac{1-x^2-x^2}{(1-x^2)^{\frac{1}{2}}} = \tfrac{1-2x^2}{(1-x^2)^{\frac{1}{2}}}.$$

It follows that the critical numbers are ± 1 and $\pm\frac{1}{\sqrt{2}}$, so they are in increasing order: -1, $\frac{-1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, and 1. Because $\frac{1}{\sqrt{2}} \approx 0.71$ and $-1 \leq x \leq 1$, we take $-0.8, 0$, and 0.8 as test points. Check that $f'(-0.8) < 0$, $f'(0) > 0$, and $f'(0.8) < 0$. So f is decreasing over $(-1, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, 1)$ and increasing over $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Notice that f has a local minimum at $-\frac{1}{\sqrt{2}}$ and a local maximum at $\frac{1}{\sqrt{2}}$.

iv. For $f(x)$ to be defined we need $x \geq x^2$. Observe that $x < 0$ is not possible and that $1 \geq x$ if $x \geq 0$. So the domain of f consists of the interval $0 \leq x \leq 1$. Check that

$$\begin{aligned} f'(x) &= (x-x^2)^{\frac{1}{2}} + x\tfrac{1}{2}(x-x^2)^{-\frac{1}{2}}(1-2x) = (x-x^2)^{\frac{1}{2}} + \tfrac{x(1-2x)}{2(x-x^2)^{\frac{1}{2}}} = \tfrac{2(x-x^2)+x(1-2x)}{2(x-x^2)^{\frac{1}{2}}} \\ &= \tfrac{-4x^2+3x}{2(x-x^2)^{\frac{1}{2}}} = \tfrac{-4x(x-\frac{3}{4})}{2(x-x^2)^{\frac{1}{2}}}. \end{aligned}$$

So the critical numbers are $0, \frac{3}{4}$, and 1. Because $0 \leq x \leq 1$, we only need the test points $\frac{1}{2}$ and $\frac{4}{5}$. Check that $f'(\frac{1}{2}) > 0$ and $f'(\frac{4}{5}) < 0$. Therefore f is increasing over $(0, \frac{3}{4})$ and decreasing over $(\frac{3}{4}, 1)$. Hence f has a local maximum at $\frac{3}{4}$.

7.57. Let $f(x) = x + \frac{1}{x}$. Check that $f'(x) = 1 - \frac{1}{x^2}$. When $x > 1$, $\frac{1}{x^2} < 1$, and hence $f'(x) = 1 - \frac{1}{x^2} > 0$. So f is increasing for $x > 1$. Because $f'(1) = 0$, the graph of f has a horizontal tangent at the point $(1,\ 2)$. It follows that f is increasing over $[1, \infty)$. The verification of the inequality follows from the definition of increasing function.

7.58. Consider the function $f(x) = (1+x)^n - (1+nx)$ for $x \geq 0$. Differentiating, we get $f'(x) = n(1+x)^{n-1} - n = n[(1+x)^{n-1} - 1]$. So $f'(x) > 0$ whenever $x > 0$. Therefore $f(x)$ is an increasing function for $x > 0$. Since $f(0) = 0$ it follows that $f(x) > 0$ for $x > 0$.

7.59. **i.** Since $f'(x) = 1 - 2\cos x$, the critical points are those x with $1 - 2\cos x = 0$ or $\cos x = \frac{1}{2}$. A look at Figure 4.24 tells us that $x = \frac{\pi}{3}$. Take $\frac{\pi}{4}$ and $\frac{\pi}{2}$ as test points. Since $f'(\frac{\pi}{4}) = 1 - 2\frac{\sqrt{2}}{2} = 1 - \sqrt{2} < 0$ and $f'(\frac{\pi}{2}) = 1 > 0$, we know that $f(x)$ is decreasing over $(0, \frac{\pi}{3})$ and increasing over $(\frac{\pi}{3}, \pi)$. There is a local minimum at $\frac{\pi}{3}$.

ii. Check that $f'(x) = \sin x + x\cos x - \sin x = x\cos x$. Because $-\pi \leq x \leq \pi$, the critical numbers are $-\frac{\pi}{2}, 0$, and $\frac{\pi}{2}$. Take $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}$, and $\frac{3\pi}{4}$ as test points. By Figure 4.24, $f'(-\frac{3\pi}{4}) > 0$, $f'(-\frac{\pi}{4}) < 0$, $f'(\frac{\pi}{4}) > 0$, and $f'(\frac{3\pi}{4}) < 0$. So the function $f(x)$ is increasing on $(-\pi, -\frac{\pi}{2})$, decreasing on $(-\frac{\pi}{2}, 0)$, increasing on $(0, \frac{\pi}{2})$, and decreasing on $(\frac{\pi}{2}, \pi)$. There are local maxima at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and there is a local minimum at 0.

iii. Refer to Figure 4.26 and notice that f is not defined for $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. Check that $f'(x) = 2\sec^2 x - 2\tan x \sec^2 x = 2\sec^2 x(1 - \tan x) = \frac{2(1-\tan x)}{\cos^2 x}$. So the critical points occur when $\tan x = 1$ and $\cos x = 0$. Since $\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = 1$, the graphs in Figures 4.24 and 4.26 tell us that the only critical number is $\frac{\pi}{4}$. (While $\cos\frac{\pi}{2} = 0$, $\frac{\pi}{2}$ is not a critical number because $f(x)$ is not defined at $\frac{\pi}{2}$.) Take 0 and $\frac{\pi}{3}$ as test points. Since $f'(0) = 1 > 0$ and $f'(\frac{\pi}{3}) = \frac{2(1-\tan\frac{\pi}{3})}{\cos^2\frac{\pi}{3}} = \frac{2(1-\sqrt{3})}{(\frac{1}{2})^2} < 0$, f is increasing over $(-\frac{\pi}{2}, \frac{\pi}{3})$ and decreasing over $(\frac{\pi}{3}, \frac{\pi}{2})$. There is a local maximum at $\frac{\pi}{3}$.

iv. The derivative is $g'(x) = \cos x - \sin x$. By the discussion in Section 4.6, $x = \frac{\pi}{4}$ is the only x with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ such that $\cos x = \sin x$. So $x = \frac{\pi}{4}$ is the only critical number. Take 0 and $\frac{\pi}{3}$ as test points to see that $g(x)$ is increasing on $(-\frac{\pi}{2}, \frac{\pi}{4})$ and decreasing on $(\frac{\pi}{4}, \frac{\pi}{2})$. So f has a local maximum at $\frac{\pi}{4}$.

7.60. **i.** Since $f'(x) = 2(x+1)$, $x = -1$ is the only critical number. The values of $f(x)$ at -1 and at the endpoints $-2, 5$ are $f(-2) = 2,\ f(-1) = 1$, and $f(5) = 37$. So the maximum value of f is $f(5) = 37$ and the minimum value is $f(-1) = 1$.

ii. The derivative is $f'(x) = 3x^2 - 12 = 3(x^2-4)$. So the critical numbers are ± 2. Evaluating f at the critical numbers and also at -3 and 5, we get $f(-3) = 10,\ f(-2) = 17,\ f(2) = -15$, and $f(5) = 66$. So the maximum value is $f(5) = 66$ and the minimum value is $f(2) = -15$.

iii. The derivative is $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1)$. So the critical numbers are $-1, 0$, and 1. Evaluating f at the required points, we get $f(-2) = -57,\ f(-1) = 1,\ f(0) = -1,\ f(1) = -3$, and $f(2) = 55$. So the maximum value is $f(2) = 55$ and the minimum value is $f(-2) = -57$.

iv. Check that the derivative is $f'(x) = \frac{-x}{\sqrt{9-x^2}}$. So the only critical number in $[-1, 2]$ is 0. Evaluating the function at $x = -1, 0$ and 2, we get $f(-1) = \sqrt{8},\ f(0) = 3$, and $f(2) = \sqrt{5}$. So the maximum value is $f(0) = 3$ and the minimum value is $f(2) = \sqrt{5}$.

7.61. The derivative of $f(x) = x^2 - x + 6$ is $f'(x) = 2x - 1 = 2(x - \frac{1}{2})$. So $f'(x) = 0$ only when $x = \frac{1}{2}$. Notice that $f'(x) < 0$ for $x < \frac{1}{2}$ and $f'(x) > 0$ for $x > \frac{1}{2}$. It follows that the minimum value of $f(x) = x^2 - x + 6$ occurs when $x = \frac{1}{2}$. This minimum value is $f(\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} + 6 = 5\frac{3}{4}$. By completing the square for $x^2 - x + 6$, we get $x^2 - x + (\frac{1}{2})^2 - (\frac{1}{2})^2 + 6 = (x - \frac{1}{2})^2 + 5\frac{3}{4}$. A look at this last expression confirms that it attains its smallest value when $x = \frac{1}{2}$ and that this smallest value is $5\frac{3}{4}$.

7.62. Because $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, the upper right corner of the rectangle is the point $(x, \frac{b}{a}\sqrt{a^2-x^2})$ with $x > 0$. The area of the rectangle is equal to $A(x) = (2x)\left(2\frac{b}{a}\sqrt{a^2-x^2}\right) = 4\frac{b}{a}x(a^2-x^2)^{\frac{1}{2}}$. We are looking for the value of x for which the function $A(x)$ attains its maximum value. Differentiating $A(x)$, we get $A'(x) = 4\frac{b}{a}\left[(a^2-x^2)^{\frac{1}{2}} + x \cdot \frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x)\right]$. By taking common denominators, $A'(x) = \frac{4b}{a}\left[\frac{a^2-x^2-x^2}{(a^2-x^2)^{\frac{1}{2}}}\right] = \frac{4b(a^2-2x^2)}{a(a^2-x^2)^{\frac{1}{2}}}$. The value $x = a$ can be ignored because $A(x) = 0$ in this case. Notice that $A'(x) = 0$ when $x = \frac{a}{\sqrt{2}}$. When $x < \frac{a}{\sqrt{2}}$, then $x^2 < \frac{a^2}{2}$. So $2x^2 < a^2$ and hence $A'(x) > 0$. When $x > \frac{a}{\sqrt{2}}$, then $x^2 > \frac{a^2}{2}$. So $2x^2 > a^2$, and this time $A'(x) < 0$. It follows that $A(x)$ is increasing to the left of $x = \frac{a}{\sqrt{2}}$ and decreasing to the right. Therefore $x = \frac{a}{\sqrt{2}}$ gives us the maximum we are looking for. This maximal rectangle has base $2 \cdot \frac{a}{\sqrt{2}} = \sqrt{2}a$ and height $2\frac{b}{a}\sqrt{a^2 - \frac{a^2}{2}} = 2\frac{b}{a}\sqrt{\frac{a^2}{2}} = \frac{2}{\sqrt{2}}\,b = \sqrt{2}\,b$. Its area is $2ab$.

7.63. The y-coordinate of the point in the first quadrant where the circle and the rectangle meet $y = \sqrt{r^2 - x^2}$.

i. The volume of a cylinder is equal to the area of its circular base times its height, so that the volume is $V(x) = (\pi x^2)(2y) = 2\pi x^2\sqrt{r^2 - x^2}$. The domain of this function is $[0, r]$.

ii. The derivative of $V(x)$ is

$$\begin{aligned} V'(x) &= 2\pi\big(2x(r^2 - x^2)^{\frac{1}{2}} + x^2\tfrac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x)\big) \\ &= 2\pi\big(2x(r^2 - x^2)^{\frac{1}{2}} - \tfrac{x^3}{(r^2-x^2)^{\frac{1}{2}}}\big) = 2\pi\big(\tfrac{2x(r^2-x^2)-x^3}{(r^2-x^2)^{\frac{1}{2}}}\big) \\ &= 2\pi x\big(\tfrac{2r^2-3x^2}{(r^2-x^2)^{\frac{1}{2}}}\big). \end{aligned}$$

Since neither $x = 0$ nor $x = r$ provides a maximum (because $V(x) = 0$ in either case), the only remaining possibility occurs when $3x^2 = 2r^2$, or $x = \sqrt{\frac{2}{3}}r$. So $x = \sqrt{\frac{2}{3}}r$ gives us the maximum volume.

iii. Because $V\big(\sqrt{\frac{2}{3}}r\big) = 2\pi\frac{2}{3}r^2\sqrt{r^2 - \frac{2}{3}r^2} = \frac{4}{3}\pi r^2\sqrt{\frac{1}{3}r^2} = \frac{4}{3\sqrt{3}}\pi r^3$, this is the maximum volume that an inscribed cylinder has. So the ratio of the volumes is $\frac{\frac{4}{3}\pi r^3}{\frac{4}{3\sqrt{3}}\pi r^3} = \sqrt{3}$.

7.64. Let d be the length of the segment. The large right triangle has base $x + a$ and hypotenuse d and the lower right triangle has base x and hypotenuse $\sqrt{x^2 + b^2}$. By similar triangles, $\frac{d}{a+x} = \frac{\sqrt{x^2+b^2}}{x}$. So

$$d = (a + x)\tfrac{1}{x}(x\sqrt{1 + b^2x^{-2}}) = (a + x)\sqrt{1 + \tfrac{b^2}{x^2}}$$

and we have expressed $d = d(x)$ as a function of x with $x > 0$. Let $D(x) = d(x)^2 = (a + x)^2\big(1 + \frac{b^2}{x^2}\big)$. Differentiating, we get

$$\begin{aligned} D'(x) &= 2(a + x)\big(1 + \tfrac{b^2}{x^2}\big) + (a + x)^2\big(-\tfrac{2b^2}{x^3}\big) \\ &= 2(a + x)\big(\tfrac{x^3+xb^2-(a+x)b^2}{x^3}\big) = \tfrac{2(x+a)(x^3-ab^2)}{x^3}. \end{aligned}$$

So $D'(x) = 0$ only when $x = (ab^2)^{\frac{1}{3}}$. The fact that $D'(x)$ is not defined at $x = 0$ can be ignored. (Why?) That $x = (ab^2)^{\frac{1}{3}}$ gives us the minimal $D(x)$ can be confirmed by noticing that $D'(x) < 0$ when $x < (ab^2)^{\frac{1}{3}}$ and that $D'(x) > 0$ when $x > (ab^2)^{\frac{1}{3}}$. It remains to notice that the x that provides the minimal $D(x)$ also provides the minimal $d(x)$. For $x = (ab^2)^{\frac{1}{3}}$, we get

$$d(x) = (a + (ab^2)^{\frac{1}{3}})\sqrt{1 + \tfrac{b^2}{(ab^2)^{\frac{2}{3}}}} = (a + (ab^2)^{\frac{1}{3}})\sqrt{1 + \tfrac{b^{(2-\frac{4}{3})}}{a^{\frac{2}{3}}}} = (a + (ab^2)^{\frac{1}{3}})\sqrt{1 + (\tfrac{b}{a})^{\frac{2}{3}}}.$$

This is the shortest that d can be.

7.65. Refer to Figure 7.56. With x and y the dimensions of the base and h the height of the box, of and note that $xyh = a^3$. So $y = \frac{a^3}{hx} = \frac{a^3}{h}x^{-1}$. Refer to Figure 8.34 and note that the surface area of the box is $2hx + 2hy + 2yx$. Substituting $y = \frac{a^3}{h}x^{-1}$, gives the surface area as the function

$$S(x) = 2hx + 2h\tfrac{a^3}{h}x^{-1} + 2\tfrac{a^3}{h}x^{-1}x = 2hx + 2a^3x^{-1} + 2\tfrac{a^3}{h}$$

of x. The question now is this: For which x does $S(x)$ attain its minimum value?
The answer requires the analysis of the derivative $S'(x) = 2h - 2a^3x^{-2} = 2\frac{hx^2-a^3}{x^2}$. Since $hx^2 = a^3$ implies that $x = \sqrt{\frac{a^3}{h}}$, it follows that $\sqrt{\frac{a^3}{h}}$ is a critical number of $S(x)$. (It is the only critical number because $S(x)$ is not defined at 0.) So the focus is on $x = \sqrt{\frac{a^3}{h}}$. Does this provide the minimum we are looking for? That it does can be seen from the following observation. If the substitution $x = \sqrt{\frac{a^3}{h}}$ makes $hx^2 - a^3$ equal to 0, then substituting any x smaller than $\sqrt{\frac{a^3}{h}}$ must make $hx^2 - a^3$ negative, and substituting any x larger than $\sqrt{\frac{a^3}{h}}$ must make $hx^2 - a^3$ positive. It follows that $S'(x) = 2\frac{hx^2-a^3}{x^2}$ is negative for $x < \sqrt{\frac{a^3}{h}}$ and positive for $x > \sqrt{\frac{a^3}{h}}$. So $S(x)$ is increasing on the left of $x = \sqrt{\frac{a^3}{h}}$ and decreasing on its right. So as asserted, $S(x)$ has its minimum value when $x = \sqrt{\frac{a^3}{h}}$. Substituting this value of x into $y = \frac{a^3}{h}x^{-1}$ gives $y = \frac{a^3}{h}(\frac{a^3}{h})^{-\frac{1}{2}} = \frac{a^3}{h}\frac{a^{-\frac{3}{2}}}{h^{-\frac{1}{2}}} = \frac{a^{\frac{3}{2}}}{h^{\frac{1}{2}}} = \sqrt{\frac{a^3}{h}}$. So the base of the box with minimal surface area is a square. What is the height of this box?

7.66. **i.** Since $f(x) = e^{x^{\frac{1}{2}}}$, we get $f'(x) = e^{x^{\frac{1}{2}}} \cdot \frac{d}{dx}x^{\frac{1}{2}} = e^{x^{\frac{1}{2}}} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

ii. $g'(x) = e^{-5x}(-5)\cos 3x + e^{-5x}(-\sin 3x)\cdot 3 = e^{-5x}(-5\cos 3x - 3\sin 3x)$

iii. $\frac{dy}{dx} = e^{x+e^x}\frac{d}{dx}(x+e^x) = e^{x+e^x}(1+e^x)$

iv. $f'(x) = 2xe^x + x^2e^x = (2x+x^2)e^x$

v. $\frac{dy}{dx} = e^{x^2} + xe^{x^2}\cdot 2x = (1+2x^2)e^{x^2}$

vi. $\frac{dy}{dx} = e^{\frac{1}{1-x^2}}\frac{d}{dx}(1-x^2)^{-1} = e^{\frac{1}{1-x^2}}(-1)(1-x^2)^{-2}(-2x) = \frac{2xe^{\frac{1}{1-x^2}}}{(1-x^2)^2}$

vii. $\frac{dy}{dx} = \sec^2(e^{3x-2})\frac{d}{dx}e^{3x-2} = \sec^2(e^{3x-2})\cdot e^{3x-2}\cdot 3 = 3e^{3x-2}\sec^2(e^{3x-2})$

viii. $\frac{dy}{dx} = \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2} = \frac{e^{2x}+e^{-2x}-2-(e^{2x}+e^{-2x}+2)}{(e^x-e^{-x})^2} = \frac{-4}{(e^x-e^{-x})^2}$

7.67. **i.** The slope of the tangent is $\frac{dy}{dx}$ evaluated at $x = 1$. Because $\frac{dy}{dx} = 2xe^{-x} + x^2(-e^{-x}) = (2x-x^2)e^{-x}$, this value is $2e^{-1} - e^{-1} = e^{-1} = \frac{1}{e}$. By the point-slope form of the equation of a line, we get that the tangent has equation $y - \frac{1}{e} = \frac{1}{e}(x-1)$ or $y = \frac{1}{e}x$.

ii. Observe that $y' = 2e^{2x} - 3e^{-3x}$ and $y'' = 4e^{2x} + 9e^{-3x}$. Therefore by substituting, we get $y'' + y' - 6y = 4e^{2x} + 9e^{-3x} + 2e^{2x} - 3e^{-3x} - 6e^{2x} - 6e^{-3x} = 0$.

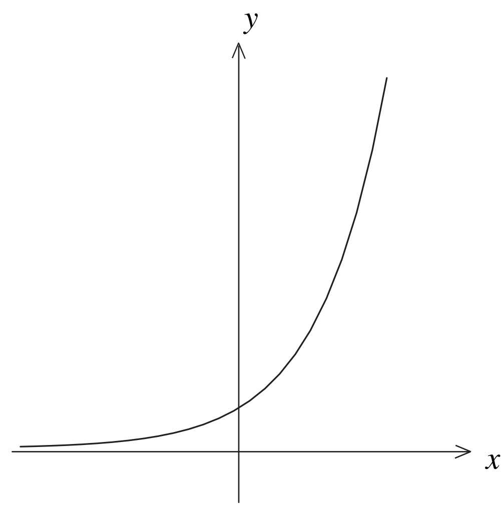
iii. The solution is a matter of recognizing a pattern in the flow of consecutive derivatives of $f(x)$:

$$\begin{aligned}
f'(x) &= e^{-x} + x(-e^{-x}) = -(x-1)e^{-x} \\
f''(x) &= -e^{-x} + (x-1)e^{-x} = (x-2)e^{-x} \\
f'''(x) &= e^{-x} - (x-2)e^{-x} = -(x-3)e^{-x} \\
f^{(4)}(x) &= -e^{-x} + (x-3)e^{-x} = (x-4)e^{-x} \\
f^{(5)}(x) &= e^{-x} - (x-4)e^{-x} = -(x-5)e^{-x} \\
f^{(6)}(x) &= -e^{-x} + (x-5)e^{-x} = (x-6)e^{-x}
\end{aligned}$$

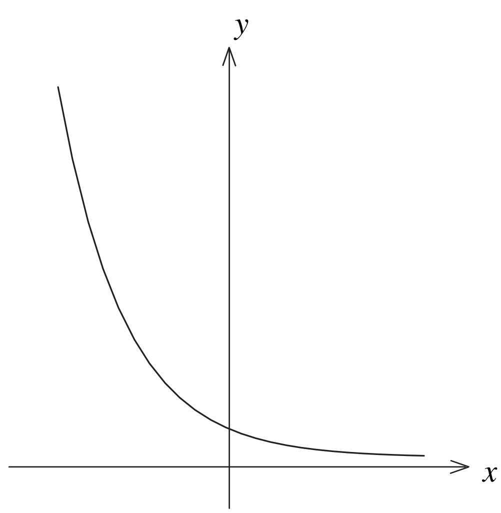
Two patterns have emerged, one for the odd derivatives, the other for the even derivatives. Following it, we see that the one hundredth derivative of $f(x)$ is $(x - 100)e^{-x}$. Provide a more definitive solution by using the principle of mathematical induction (developed in segment 3E of Section 3.8).

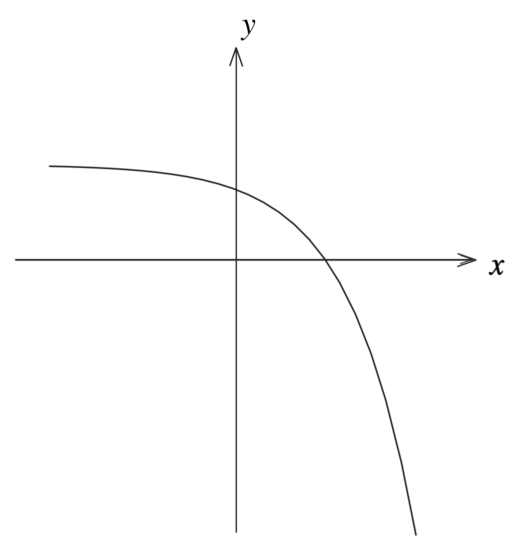
iv. Let $f(x) = e^x + x$. Because $f(0) = 1$ and $f(-1) = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$, it follows by the intermediate value theorem and the continuity of the function f that $f(x) = 0$ for some x with $-1 < x < 0$.

v. Starting with the graph of e^x,



we get the graphs of e^{-x} and $3 - e^x$, see below, after thinking a little.





vi. $g'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$. Observe that $g'(x) = 0$ precisely when $x = 1$. Notice that $g'(x) < 0$ when $0 < x < 1$ and that $g'(x) > 0$ when $x > 1$. So $g(x) = \frac{e^x}{x}$ is decreasing to the left of $x = 1$ and increasing to the right of $x = 1$. So g has its absolute minimum at $x = 1$. The absolute minimum value is $g(1) = e$.

7.68. Clarity is added by plotting the graphs of the inverse functions $2^x < e^x < 10^x$ (see Figures 7.39 and 7.40 for instance) and then to reflect these about the line $y = x$.

7.69. **i.** $\log_2 x + 3\log_2(x+1) + \frac{1}{4}\log_2(x-1) = \log_2 x + \log_2(x+1)^3 + \log_2(x-1)^{\frac{1}{4}}$
$= \log_2 x(x+1)^3(x-1)^{\frac{1}{4}}$.

ii. $\frac{1}{3}\ln x - 4\ln(2x+3) = \ln x^{\frac{1}{3}} - \ln(2x+3)^4 = \ln \frac{x^{\frac{1}{3}}}{(2x+3)^4}$.

7.70. **i.** $2^{\log_2 x} = 2^3$, so $x = 2^3 = 8$.

ii. $\ln 2^{x^2-5} = \ln 3$, so $(x^2-5)\ln 2 = \ln 3$. Hence $x^2 - 5 = \frac{\ln 3}{\ln 2}$. Therefore $x = \pm\sqrt{5 + \frac{\ln 3}{\ln 2}}$.

iii. $\ln 5^{x^2-1} = \ln 2$, so $x^2 - 1 = \frac{\ln 2}{\ln 5}$, and $x = \pm\sqrt{1 + \frac{\ln 2}{\ln 5}}$.

iv. $\ln 4^{x^2+1} = \ln 3$, so $x^2 + 1 = \frac{\ln 3}{\ln 4}$ and $x^2 = \frac{\ln 3}{\ln 4} - 1$. Because $\ln x$ is an increasing function, $\ln 4 > \ln 3$. Hence $\frac{\ln 3}{\ln 4} < 1$. It follows that $x^2 < 0$, impossible. So the equation has no solution. (This is also evident from its first formulation.)

v. From $\log_9(4x^2 - 11) = 7$, we get $9^{\log_9(4x^2-11)} = 9^7$ and hence $4x^2 - 11 = 9^7$. So $4x^2 =$ 4,782,969 + 11 = 4,782,980, and therefore $x^2 =$ 1,195,745. So $x \approx \pm\, 1093.5$.

vi. Since $\log_5(\log_5 x) = 6$, we get $5^{\log_5(\log_5 x)} = 5^6$ and hence $\log_5 x = 5^6$. Therefore, $5^{\log_5 x} = (5)^{5^6}$ and hence $x = (5)^{5^6} = 5^{15625}$.

vii. From basic properties of $\ln x$, we get $\ln[(x+6)(x-3)] = \ln[5 \cdot 7]$. By applying the exponential function e^x to both sides, we get $(x+6)(x-3) = 35$. So $x^2 + 3x - 18 = 35$ and hence $x^2 + 3x - 53 = 0$. Thus, by the quadratic formula, $x = \frac{-3\pm\sqrt{9+212}}{2} = \frac{-3\pm\sqrt{221}}{2}$.

viii. From the given equation $\ln \frac{x-2}{x+1} - \ln \frac{x-3}{x+1} = 1$. So $\ln\left(\frac{x-2}{x+1}\Big/\frac{x-3}{x+1}\right) = 1$ and hence $\ln\left[\frac{x-2}{x+1} \cdot \frac{x+1}{x-3}\right] = 1$. It follows that $\ln \frac{x-2}{x-3} = 1$. Therefore, $\frac{x-2}{x-3} = e^{\ln \frac{x-2}{x-3}} = e$ and hence $x - 2 = e(x-3)$. So $(e-1)x = 3e - 2$ and $x = \frac{3e-2}{e-1}$.

ix. Observe first that $3x - 2 > 0$ since $\ln(3x-2)$ needs to be defined. So $x > \frac{2}{3}$. Because e^x is an increasing function, $3x - 2 = e^{\ln(3x-2)} \le e^0 = 1$. Therefore $3x \le 3$ and hence $x \le 1$. It follows that $\frac{2}{3} < x \le 1$.

x. From $4^x - 2^{x+3} + 12 = 0$ we get $(2^2)^x - 2^x \cdot 2^3 + 12 = 0$, so $(2^x)^2 - 8 \cdot 2^x + 12 = 0$. Let $y = 2^x$. Since $y^2 - 8y + 12 = (y-2)(y-6) = 0$, we get $y = 2^x = 2$ or $y = 2^x = 6$. Taking $\log_2$ of both sides, $x = \log_2 2^x = \log_2 2$ or $\log_2 6$. So $x = 1$ or $x = \log_2 6$.

7.71. **i.** We need $1 - x > 0$. So $1 > x$.

ii. We need both $t \ge 0$ and $t^2 - 1 > 0$. So $t \ge 0$ and $t^2 > 1$. Hence $t > 1$.

7.72. **i.** For $\ln x$ to make sense we need $x > 0$. Because $\cos\theta$ makes sense for any θ, the domain of $f(x)$ is $\{x \mid x > 0\}$. Note that $f'(x) = -\sin(\ln x) \cdot \frac{1}{x}$. The considerations above tell us that the domain of $f'(x)$ is also $\{x \mid x > 0\}$.

ii. For $f(x)$ to make sense, we need $2 - x - x^2 > 0$ or $x^2 + x - 2 < 0$. Note that $x^2 + x - 2 = (x+2)(x-1) = 0$ when $x = -2$ or 1. Consider the x-axis and take $-3, 0$, and 2 as test points to see that $x^2 + x - 2 < 0$ precisely when $-2 < x < 1$. So the domain of $f(x)$ is $\{x \mid -2 < x < 1\}$. Check that $f'(x) = \frac{-1-2x}{2-x-x^2} = \frac{2x+1}{x^2+x-2}$. Since the domain of $f'(x)$ can be no larger than the domain of $f(x)$ (this follows from the definition $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ of $f'(x)$), it follows that the domain of $f'(x)$ is also $\{x \mid -2 < x < 1\}$.

iii. For $f(x)$ to make sense we need both $x \geq 0$ and $x - 1 \geq 0$, so we need $x \geq 1$. For $\ln(\sqrt{x} - \sqrt{x-1})$ to make sense, we must have $\sqrt{x} > \sqrt{x-1}$. Since $x > x - 1$, this is so for $x \geq 1$. So the domain of $f(x)$ is $\{x \mid x \geq 1\}$. Because $f(x) = \ln(x^{\frac{1}{2}} - (x-1)^{\frac{1}{2}})$,

$$f'(x) = \frac{\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}(x-1)^{-\frac{1}{2}}}{x^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}} = \frac{\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-1}}}{\sqrt{x} - \sqrt{x-1}}.$$

So only $x = 1$ has to be excluded from the domain of $f(x)$ is $\{x \mid x \geq 1\}$ and it follows that the domain of $f'(x)$ is $\{x \mid x > 1\}$.

iv. We need $x^4 + 3x^2 > 0$ for $f(x)$ to make sense. For $x^4 + 3x^2 = x^2(x^2+3) > 0$ all we need to have is $x \neq 0$. So the domain of $f(x)$ is $\{x \mid x \neq 0\}$. By one of the laws of logarithms, $f(x) = \log_{11}(x^4 + 3x^2) = \frac{\ln(x^4+3x^2)}{\ln 11}$. So

$$f'(x) = \frac{1}{\ln 11}\left(\frac{4x^3+6x}{x^4+3x^2}\right).$$

It follows that $\{x \mid x \neq 0\}$ is also the domain of $f'(x)$.

v. For $f(x)$ to be defined we need $x + 3x^2 > 0$. Because $x + 3x^2 = x(1+3x)$, this is so for $x > 0$. If $x < 0$, we need to have $1 + 3x < 0$, to get $x(1+3x) > 0$. But $1 + 3x < 0$ means that $3x < -1$ and hence $x < -\frac{1}{3}$. So the domain of $f(x)$ is

$$\left\{x \mid x < -\tfrac{1}{3} \text{ or } 0 < x\right\}.$$

Because $f(x) = \ln(x + 3x^2)^{\frac{1}{2}}$, we get $f'(x) = \frac{\frac{1}{2}(x+3x^2)^{-\frac{1}{2}}(1+6x)}{(x+3x^2)^{\frac{1}{2}}}$. Because $x + 3x^2 > 0$ for all x in the domain of $f(x)$, the domain of $f'(x)$ is the same as that of $f(x)$.

7.73. **i.** $y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$ and $y'' = \frac{1}{x}$.

ii. Because $\frac{d}{dx}\log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$ for any base a, $y' = \frac{1}{\ln 10} \cdot \frac{1}{x} = \frac{1}{\ln 10} \cdot x^{-1}$ and $y'' = \frac{-1}{\ln 10}x^{-2}$.

iii. $y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x = (\cos x)^{-1}$ and $y'' = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = (\sec x)(\tan x)$.

7.74. **i.** Because $g(x) = (\ln x)^{\frac{1}{2}}$, we get $g'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$.

ii. $f'(t) = \frac{1}{\ln 7} \cdot \frac{1}{t^4 - t^2 + 1}(4t^3 - 2t) = \frac{4t^3 - 2t}{(\ln 7)(t^4 - t^2 + 1)}$.

iii. $f'(x) = e^x \cdot \ln x + e^x \cdot \frac{1}{x}$.

iv. $h'(t) = 3t^2 - (\ln 3)3^t$.

iv. Let $g(x) = x^{\sin x}$. Since $\frac{d}{dx}\ln x^{\sin x} = \frac{d}{dx}(\sin x \cdot \ln x) = \cos x \cdot \ln x + \frac{\sin x}{x}$, we get by logarithmic differentiation, that $g'(x) = (\frac{d}{dx}\ln x^{\sin x})g(x) = (\cos x \cdot \ln x + \frac{\sin x}{x})x^{\sin x}$.

7.75. $f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$. If $\ln x > -1$, that is if $x = e^{\ln x} > e^{-1} = \frac{1}{e}$, then $f'(x) > 0$ and if $\ln x < -1$, that is if $x = e^{\ln x} < e^{-1} = \frac{1}{e}$, then $f'(x) < 0$. So $f(x)$ is decreasing to the left of $x = \frac{1}{e}$ and increasing to the right. It follows that $f(x)$ has its absolute minimum value when $x = \frac{1}{e}$. This value is $f(\frac{1}{e}) = \frac{1}{e} \cdot \ln\frac{1}{e} = \frac{1}{e}\ln(e^{-1}) = -\frac{1}{e}$.

7.76. The line determined by $(1,1)$ and $(x_0, \frac{1}{x_0})$ has slope $\frac{1-\frac{1}{x_0}}{1-x_0} = \frac{\frac{x_0-1}{x_0}}{1-x_0} = -\frac{1}{x_0}$. So $y - 1 = -\frac{1}{x_0}(x-1)$

is an equation of the line. A little algebra converts it to $y = -\frac{1}{x_0}x + (1 + \frac{1}{x_0})$. To find the area under this line and over the interval $[1, x_0]$, notice first that the line intersects the x-axis when $-\frac{1}{x_0}x + (1 + \frac{1}{x_0}) = 0$, hence when $\frac{1}{x_0}x = 1 + \frac{1}{x_0}$ and therefore at $x = x_0 + 1$. A look at Figure 7.57a, informs us that the area under the line and over $[1, x_0]$ is the difference

$$\tfrac{1}{2}[(1 + x_0) - 1](1) - \tfrac{1}{2}[(1 + x_0) - x_0]\tfrac{1}{x_0} = \tfrac{1}{2}(x_0 - \tfrac{1}{x_0})$$

between two triangles. Since $x_0 > 1$, $\ln x_0$ is the area under $y = \frac{1}{x}$ over interval $[1, x_0]$. A look at Figure 7.57a confirms that $\ln x_0 < \frac{1}{2}(x_0 - \frac{1}{x_0})$ and also, when $x_0 \approx 1$, that $\ln x_0 \approx \frac{1}{2}(x_0 - \frac{1}{x_0})$. Taking $x_0 = 2$, we get $\ln 2 < \frac{1}{2}(2 - \frac{1}{2}) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} = 0.75$. A calculator tells us that $\ln 2 \approx 0.693$.

If $\ln x_0 \approx \frac{1}{2}(x_0 - \frac{1}{x_0})$ for x_0 much greater than 1, then $\ln x_0 \approx \frac{1}{2}x_0$. But this is not the case for large x_0 as Figure 7.57b illustrates.

7.77. The verifications combine the inverse relationship between the log and exponential function and the rules for exponents from Section 7.10.

i. Set $\log_a x_1 = u_1$ and $\log_a x_2 = u_2$. By the inverse relationship that connects the log and exponential functions, $a^{u_1} = x_1$ and $a^{u_2} = x_2$. Since $x_1x_2 = a^{u_1}a^{u_2} = a^{u_1+u_2}$, it follows that

$$\log_a(x_1x_2) = u_1 + u_2 = \log_a x_1 + \log_a x_2.$$

ii. Again let $\log_a x_1 = u_1$ and $\log_a x_2 = u_2$. As in part (i), $a^{u_1} = x_1$ and $a^{u_2} = x_2$. Since $\frac{1}{x_2} = a^{-u_2}$, we have $\frac{x_1}{x_2} = a^{u_1}a^{-u_2} = a^{u_1-u_2}$. Therefore

$$\log_a \tfrac{x_1}{x_2} = u_1 - u_2 = \log_a x_1 - \log_a x_2.$$

iii. Let $\log_a x_1 = u_1$. So $a^{u_1} = x_1$ and hence $a^{u_1x_3} = (a^{u_1})^{x_3} = x_1^{x_3}$. So by the inverse relationship, $\log_a(x_1^{x_3}) = u_1x_3 = x_3u_1 = x_3 \log_a x_1$.

7.78. Set $y = \log_a x_1 = \log_b x_2$. So $a^y = x_1$ and $b^y = x_2$. Therefore $(ab)^y = x_1x_2$, and hence $\log_{ab} x_1x_2 = y$.

7.79. The x can be any real number, but it is fixed for the entire discussion. The answer to "Why?" is given by the inverse relationship between the exponential function "e" and the log function "ln". So we need to show that $\ln\big(\lim\limits_{n\to\infty}\big(1 + \frac{x}{n}\big)^n\big) = x$. With $h = \frac{x}{n}$,

$$\ln\Big(\lim_{n\to\infty}\big(1 + \tfrac{x}{n}\big)^n\Big) = \ln\Big(\lim_{h\to 0}\big(1 + h\big)^{\frac{x}{h}}\Big)$$

so it remains to show that $\ln \lim\limits_{h\to 0}(1 + h)^{\frac{x}{h}} = x$.

We will proceed somewhat differently and more simply than the outline proposes. The use of the theorem proved in Problem 7.15 is key. It says the following about the composite $y = f(g(x))$ of two functions. If $\lim\limits_{x\to c} g(x) = b$ and if f is continuous at b, then

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(b).$$

Note first that since x is fixed, the function $f(z) = z^x$ is continuous for all $z > 0$. (In Example 7.44 the function $f(x) = x^r$ for $x > 0$ was shown to be differentiable—and hence continuous—for any real number r.) Next let $g(h) = (1 + h)^{\frac{1}{h}}$ and observe (as a consequence of the limit definition of e developed in Section 7.10) that $\lim\limits_{h\to 0} g(h) = \lim\limits_{h\to 0}(1 + h)^{\frac{1}{h}} = e$. So by the theorem,

$$\lim_{h\to 0}(1+h)^{\frac{x}{h}} = \lim_{h\to 0}\left[(1+h)^{\frac{1}{h}}\right]^x = \left[\lim_{h\to 0}(1+h)^{\frac{1}{h}}\right]^x = e^x.$$

Therefore

$$\ln \lim_{h\to 0}(1+h)^{\frac{x}{h}} = \ln e^x = x$$

and we are done.

The approach outlined in the text makes similar use of the theorem. The continuity of the natural log allows $\lim\limits_{h\to 0}$ to be moved past $\ln$ in the expression $\ln(1+h)^{\frac{x}{h}}$ with the result that $\lim\limits_{h\to 0}\ln(1+h)^{\frac{x}{h}} = \ln\big(\lim\limits_{h\to 0}(1+h)^{\frac{x}{h}}\big)$. This leaves $\lim\limits_{h\to 0}\ln(1+h)^{\frac{x}{h}}$. By a basic property of logs, $\lim\limits_{h\to 0}\ln(1+h)^{\frac{x}{h}} = \lim\limits_{h\to 0}\frac{x}{h}\cdot\ln(1+h) = x\cdot\lim\limits_{h\to 0}\frac{\ln(1+h)}{h}$. It remains to observe that $\lim\limits_{h\to 0}\frac{\ln(1+h)}{h} = \lim\limits_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}$ and that this is nothing but the derivative $\frac{d}{dx}\ln x = \frac{1}{x}$ evaluated at $x=1$. Since this is equal to 1, we have again verified that $\ln\big(\lim\limits_{n\to\infty}\big(1+\frac{x}{n}\big)^n\big) = x$. Notice that it is the differentiability of the natural log at 1 that is needed, not just the continuity.

7.80. Recall that $\cosh x = \frac{1}{2}(e^x + e^{-x})$. A look at the graph of $y = \cosh x$ in Figure 7.46 tells us that $\cosh 0 = \frac{1}{2}(1+1) = 1$ is the smallest value of the function $y = \cosh x$. So $y = \operatorname{sech} x = \frac{1}{\cosh x}$ has its largest value $\operatorname{sech}(0) = 1$ at $x = 0$. Since $\lim\limits_{x\to\pm\infty}\cosh x = \infty$ we know that $\lim\limits_{x\to\pm\infty}\operatorname{sech} x = 0$. Since $y = \cosh x$ is always positive, the same is the case for $y = \operatorname{sech} x$. The combination of these observations has the consequence that the graph of $y = \operatorname{sech} x$ has the form that Figure 7.48 already depicts. The graphing calculator

https://www.desmos.com/calculator

and Example 7.48 inform us that the points of inflection of the graph occur for $x \approx \pm 0.881$.

The graph of $y = \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$ is next. We see from the graph of $y = \tanh x$ in

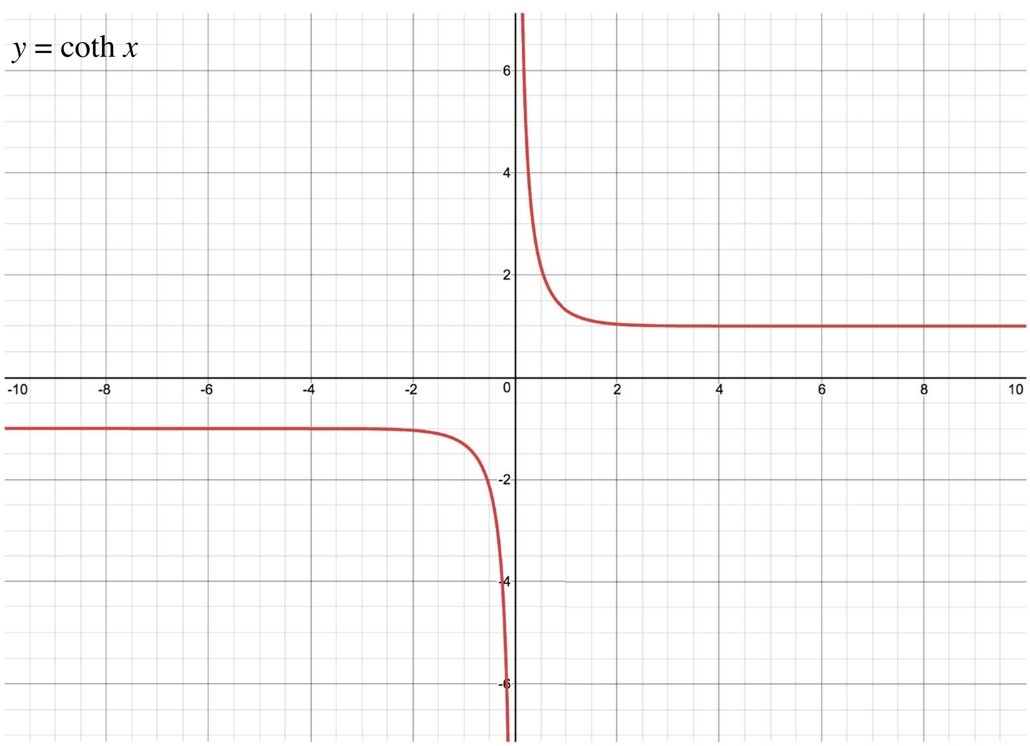
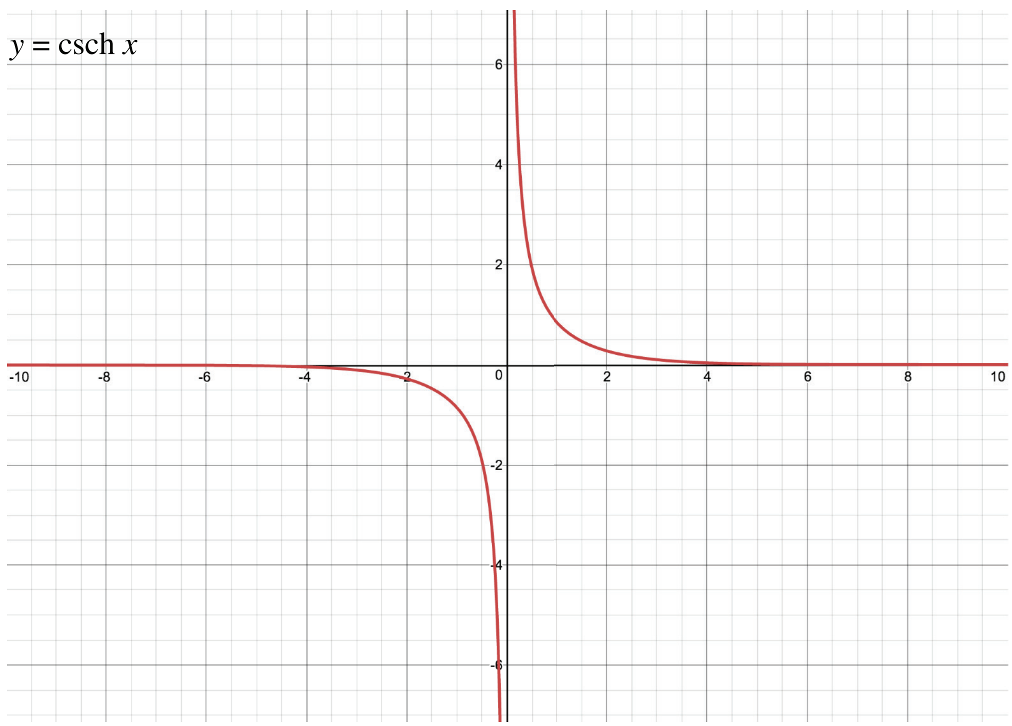


Figure 7.47 that $\lim_{x\to 0^+} \coth x = +\infty$ and $\lim_{x\to 0^-} \coth x = -\infty$. Since $-1 < \tanh x < 0$ for all negative x and $\lim_{x\to -\infty} \tanh x = -1$, we know that $\coth x < -1$ for all negative x and $\lim_{x\to -\infty} \coth x = -1$. Since $0 < \tanh x < 1$ for all positive x and $\lim_{x\to -\infty} \tanh x = 1$ we find in the same way, that $\coth x > 1$ for all positive x and $\lim_{x\to -\infty} \coth x = 1$. Combining this information, we get the graph of $y = \coth x$ sketched above. It is also drawn with https://www.desmos.com/calculator.

Finally to the graph of $y = \operatorname{csch} x = \frac{1}{\sinh x}$. From the graph of $y = \sinh x$, $\lim_{x\to 0^+} \operatorname{csch} x = +\infty$ and $\lim_{x\to 0^-} \operatorname{csch} x = -\infty$. We also see that $\lim_{x\to +\infty} \operatorname{csch} x = 0$ and $\lim_{x\to -\infty} \operatorname{csch} x = 0$. Since $y = \sinh x$ is increasing over the interval $(0, +\infty)$, $y = \operatorname{csch} x$ is decreasing over this interval. An analogous thing is true for $(-\infty, 0)$. So $y = \operatorname{csch} x$ has the shape shown in the figure below. The graphing calculator https://www.desmos.com/calculator provides the specifics.



7.81. Since $\sinh \frac{x}{2} = \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}$,

$$(\sinh \tfrac{x}{2})^2 = \Big(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}\Big)^2 = \frac{(e^{\frac{x}{2}})^2 - 2(e^{\frac{x}{2}})(e^{-\frac{x}{2}}) + (e^{-\frac{x}{2}})^2}{4} = \frac{e^x - 2 + e^{-x}}{4} = \frac{1}{2}\Big(\frac{e^x + e^{-x}}{2} - 1\Big)$$

and therefore $\sinh^2 \frac{x}{2} = \frac{1}{2}(\cosh x - 1)$. The equality $\cosh^2 \frac{x}{2} = \frac{1}{2}(\cosh x + 1)$ is verified in the same way.

To verify the last identity, we'll take a detour that will illustrate more of the similarities between the properties of the hyperbolic and trigonometric functions. We'll start by verifying the sum formulas

$$\sinh(x + y) = \sinh x \cdot \cosh y + \cosh x \cdot \sinh y \quad \text{and}$$
$$\cosh(x + y) = \cosh x \cdot \cosh y + \sinh x \cdot \sinh y$$

of Example 7.47. Both involve nothing but straightforward multiplication of exponentials. In the first case,

$$\begin{aligned}\left(\tfrac{e^x-e^{-x}}{2}\right)\left(\tfrac{e^y+e^{-y}}{2}\right)+\left(\tfrac{e^x+e^{-x}}{2}\right)\left(\tfrac{e^y-e^{-y}}{2}\right) &= \tfrac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}}{4}+\tfrac{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}}{4}\\ &= \tfrac{2e^{x+y}-2e^{-x-y}}{4}=\tfrac{e^{x+y}-e^{-x-y}}{2}.\end{aligned}$$

The sum formula for cosh follows in the same way. By dividing the sum formula for sinh by the sum formula for cosh, we get

$$\begin{aligned}\tanh(x+y) &= \tfrac{\sinh(x+y)}{\cosh(x+y)}=\tfrac{\sinh x\cdot\cosh y+\cosh x\cdot\sinh y}{\cosh x\cdot\cosh y+\sinh x\cdot\sinh y}=\tfrac{\sinh x\cdot\cosh y+\cosh x\cdot\sinh y}{(\cosh x\cdot\cosh y)(1+\frac{\sinh x\cdot\sinh y}{\cosh x\cdot\cosh y})}\\ &= \tfrac{\sinh x\cdot\cosh y}{(\cosh x\cdot\cosh y)(1+\frac{\sinh x\cdot\sinh y}{\cosh x\cdot\cosh y})}+\tfrac{\cosh x\cdot\sinh y}{(\cosh x\cdot\cosh y)(1+\frac{\sinh x\cdot\sinh y}{\cosh x\cdot\cosh y})}\\ &= \tfrac{\sinh x}{(\cosh x)(1+\frac{\sinh x\cdot\sinh y}{\cosh x\cdot\cosh y})}+\tfrac{\sinh y}{(\cosh y)(1+\frac{\sinh x\cdot\sinh y}{\cosh x\cdot\cosh y})}=\tfrac{\tanh x+\tanh y}{1+(\tanh x)(\tanh y)}.\end{aligned}$$

Applying this addition formula for tanh with $\frac{x}{2}$ in place of both x and y provides the last of the analogues that the solution of Problem 7.81 calls for.

7.82. Refer to Example 7.48 for the fact that $\frac{d^2}{dx^2}\operatorname{sech} x = (\operatorname{sech} x)(2\tanh^2 x - 1)$. The calculator http://web2.0calc.com shows that $(\operatorname{sech} x)(2\tanh^2 x - 1) \approx -0.000374$ for $x = 0.881$ and $(\operatorname{sech} x)(2\tanh^2 x - 1) \approx 0.000626$ for $x = 0.882$. So the graph of $y = \operatorname{sech} x$ is concave down for $x = 0.881$ and concave up for $x = 0.882$ and it has a point of inflection for some x with $0.881 < x < 0.882$. By another such calculation, the graph of $y = \operatorname{sech} x$ is concave up for $x = -0.882$ and concave down for $x = -0.881$ and it has a point of inflection for some x with $-0.882 < x < -0.881$. Check that this information is consistent with Figure 7.48.

7.83. At $x = 1, 2, 3, 4, 5, 10, 15, 20$ the values of $f(x) = x^2$ and $g(x) = 2^x$ are $1, 4, 9, 16, 25, 100,$ $225, 400$ and $2, 4, 8, 16, 32$, 1024, 32,768, 1,048,576, respectively. So from 1 to 5 the values of the two functions are close, but thereafter, the values of $g(x) = 2^x$ far outpace those of $f(x) = x^2$. Their graphs were already sketched in Figures 7.13 and 7.40.

7.84. Consider the function $f(x) = 2x^2 - x^4 = x^2(2 - x^2)$. By the discussion about symmetry in Section 7.13 the graph of the function is symmetric about the y-axis.

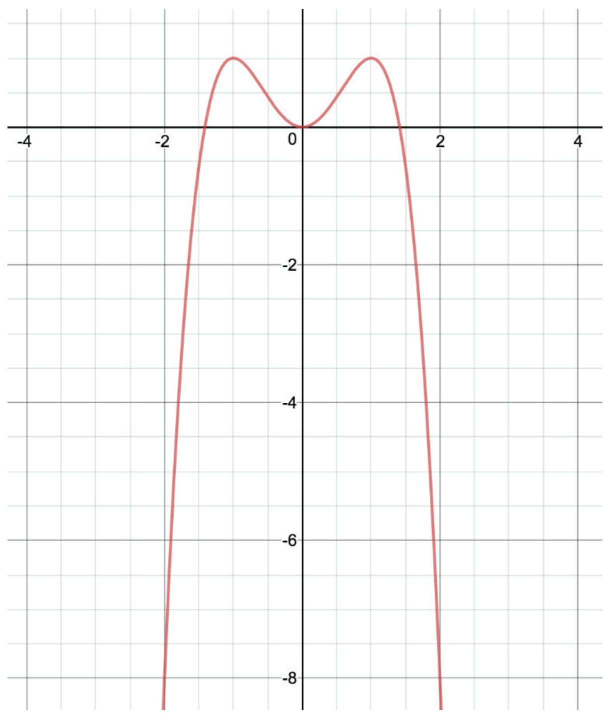
i. Notice that $f(x) \geq 0$ when $x^2 \leq 2$ and that this is case precisely for $-\sqrt{2} \leq x \leq \sqrt{2}$.

ii. Since $f'(x) = 4x - 4x^3 = 4x(1 - x^2)$, the critical numbers are $-1, 0,$ and 1. Choose $-2, -\frac{1}{2}, \frac{1}{2}$, and 2 as test points. Since $f'(\frac{1}{2}) = 2(\frac{3}{4}) > 0$ and $f'(2) = 8(-3) < 0$, $f(x)$ is increasing over the interval $(0, 1)$ and decreasing over $(1, +\infty)$. The symmetry of the graph tells us that $f(x)$ is increasing over over $(-\infty, -1)$ and decreasing over $(-1, 0)$.

iii. For a large negative x both factors of the product $f'(x) = 4x(1 - x^2)$ are positive and large. As x moves toward $x = -1$ the factors continue to be positive, but both get smaller and at $x = -1$ one of the factors is zero so that $f'(-1) = 0$. At $x = 1$ one of the factors is zero, so that $f'(1) = 0$. For $x > 1$ and increasing one factor is positive the other negative so that the product $f'(x)$ is increasing negatively. It follows that for $x > -1$, the tangent lines have positive slope, but that their steepness decreases with increasing x until at $x = 1$ the tangent is horizontal. In the same way, the tangent is horizontal at $x = 1$ and becomes steeper negatively as $x \geq 1$ increases.

iv. From $f'(x) = 4x - 4x^3$, it follows that $f''(x) = 4 - 12x^2 = -4(3x^2 - 1)$. So $f''(x) = 0$ for $x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.58$. The relevant intervals are $(-\infty, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and $(\frac{1}{\sqrt{3}}, \infty)$. Using $x = -1, 0$, and 1 as test points, we conclude that the graph of $f(x) = 2x^2 - x^4$ over these intervals is concave down, up, and down again, respectively.

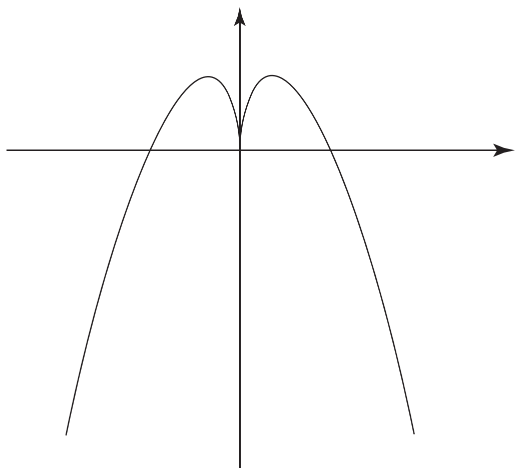
v. The graph of $f(x) = 2x^2 - x^4$ is now easy to sketch. Use the information above and plot a few points. The graphing calculator https://www.desmos.com/calculator provides the figure below. In reference to the discussion in Section 7.13 about dominant terms, notice that x^2 dominates the graph for x with $|x| < 1$ (the smaller $|x|$, the greater the dominance) and that $-x^4$ dominates the graph for x with $|x| > 1$ (the larger the $|x|$, the greater the dominance).

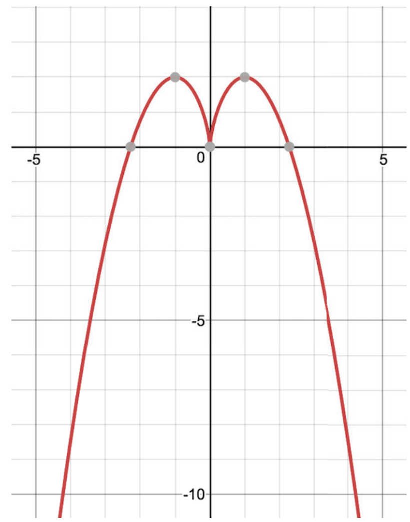


7.85. Solving $f(x) = 3x^{\frac{2}{3}} - x^2 = 0$ for x, we get $x^2 = 3x^{\frac{2}{3}}$, hence $x^{\frac{4}{3}} = 3$, and therefore that $x = \pm 3^{\frac{3}{4}} \approx \pm 2.28$. So the graph crosses the x-axis at the points $(\pm 3^{\frac{3}{4}}, 0) \approx (\pm 2.28, 0)$. The derivative $f'(x) = 2x^{-\frac{1}{3}} - 2x = -2x^{-\frac{1}{3}}(x^{\frac{4}{3}} - 1) = \frac{-2(x^{\frac{4}{3}}-1)}{x^{\frac{1}{3}}}$ is zero for $x = \pm 1$ and is undefined for $x = 0$. Since $f(\pm 1) = 3 - 1 = 2$, the graph has horizontal tangents at the points $(\pm 1, 2)$.

Since $f(-x) = f(x)$ for any x, the graph is symmetric with respect to the y-axis. So it is enough to analyze the graph for $x < 0$. For $x < -1$, the term $x^{\frac{1}{3}}$ is negative and $x^{\frac{4}{3}} = (x^{\frac{1}{3}})^4 > 1$, so that $f'(x) = \frac{-2(x^{\frac{4}{3}}-1)}{x^{\frac{1}{3}}}$ is positive. For $-1 < x < 0$, $x^{\frac{1}{3}}$ is still negative and $x^{\frac{4}{3}} = (x^{\frac{1}{3}})^4 < 1$, so that $f'(x) = \frac{-2(x^{\frac{4}{3}}-1)}{x^{\frac{1}{3}}}$ is negative. The closer x moves to zero, the larger negatively $f'(x)$ becomes. It follows that for $x < 0$ the graph increases to the left of $x = -1$, has a horizontal tangent at $x = -1$ and then decreases more and more steeply to a vertical tangent at $x = 0$.

The second derivative is $f''(x) = -\frac{2}{3}x^{-\frac{4}{3}} - 2 = -2\big(\frac{1}{3x^{\frac{4}{3}}} + 1\big)$. So the second derivative is negative except at $x = 0$ where it is not defined. So the graph is concave down over the intervals $(-\infty, 0)$ and $(0, +\infty)$. The information collected above confirms that the graph of $f(x) = 3x^{\frac{2}{3}} - x^2$ provided by Figure 7.58c (the graph below on the left) is correct. The graph from https://www.desmos.com/calculator (below on the right) is more precise confirmation.





7.86. We'll develop basic relevant information about the nine functions.

(1) For $g(x) = x^3 - 3x$, we have $g'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$. So the graph of $g(x)$ is increasing over $(-\infty, -1)$ and $(1, \infty)$ and decreasing over $(-1, 1)$. It has horizontal tangents at the points $(-1, 2)$ and $(1, -2)$. The slope of its tangent line at $(0, 0)$ is -3. The only graph that satisfies all these properties is (b).

(2) For $g(x) = x^3$, we see that $g'(x) = 3x^2$ and $g''(x) = 6x$. It follows that the graph is increasing throughout and that it concave up over $(-\infty, 0)$ and concave down over $(0, \infty, -1)$. The slope of its tangent at $(0, 0)$ is zero. The only graph that satisfies all these properties is (a).

(3) For $g(x) = x^3 - 1$ the derivatives are $g'(x) = 3x^2$ and $g''(x) = 6x$. So the graph of $g(x)$ is increasing, goes through $(0, -1)$, and its tangent has slope zero there. Graph (g) is the only possibility.

(4) The function $g(x) = x^3 + x$ has $g'(x) = 3x^2 + 1$ and $g''(x) = 6x$. Again the graph is increasing throughout and its tangent at $(0, 0)$ has slope 1. The information corresponds to graph (c).

(5) The function $g(x) = x^3 + 4x$ has $g'(x) = 3x^2 + 4$ and $g''(x) = 6x$. Once more the graph is increasing throughout and its tangent at $(0, 0)$ has slope 4. This matches graph (h).

(6) The function $g(x) = x^3 + x^2$ has $g'(x) = 3x^2 + 2x = 3x(x + \frac{2}{3})$ and $g''(x) = 6x + 2$. The graph has horizontal tangents at the points $(0, 0)$ and $\big(-\frac{2}{3}, \frac{4}{27}\big)$. There is a point of inflection for $x = -\frac{1}{3}$ with the graph concave down to the left of $x = -\frac{1}{3}$ and concave up to the right. Note that (e) is the only possibility.

(7) The function $g(x) = x^3 - x$ has $g'(x) = 3x^2 - 1 = 3(x^2 - \frac{1}{3})$ and $g''(x) = 6x$. The graph has horizontal tangents for $x = \pm\frac{1}{\sqrt{3}}$ and its tangent has slope -1 at $(0,0)$. This fits graph (i).

(8) For $g(x) = x^3 - 2x$, we have $g'(x) = 3x^2 - 2 = 3(x^2 - \frac{2}{3})$. So the graph has horizontal tangents at $x = \pm\sqrt{\frac{2}{3}}$ and the tangent at $(0,0)$ has slope -2. Graph (f) matches this.

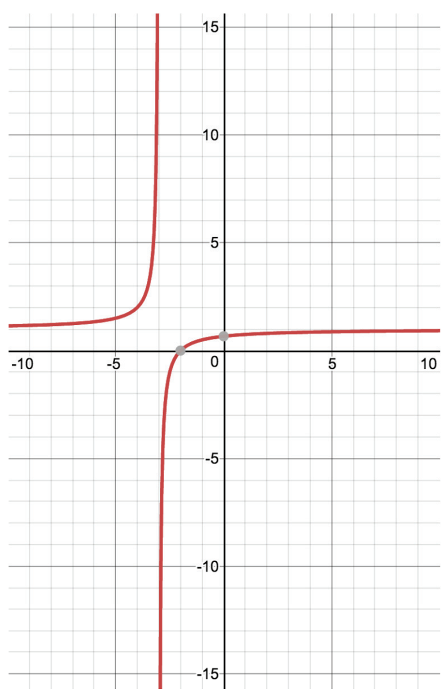
(9) The function $g(x) = x^3 - x^2$ has $g'(x) = 3x^2 - 2x = 3x(x - \frac{2}{3})$ and $g''(x) = 6x - 2$. The graph has horizontal tangents at the points $(0,0)$ and $\left(\frac{2}{3}, -\frac{4}{27}\right)$. There is a point of inflection for $x = \frac{1}{3}$ with the graph concave down to the left of $x = \frac{1}{3}$ and concave up to the right. Graph (d) agrees with this information.

7.87. **i.** If the line $x = 3$ were to be a vertical asymptote of the graph of f, then $\lim_{x\to 3^-} f(x) = \pm\infty$ or $\lim_{x\to 3^+} f(x) = \pm\infty$ (or both). But this is not the case because $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} g(x) = \frac{5}{6}$ and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} g(x) = \frac{5}{6}$.

ii. For any $x < -3$, we have $x + 3 < 0$ and $x + 2 < 0$, so that $\frac{x+2}{x+3} > 0$. On the other hand, for any $x > -3$, we have $x + 3 > 0$ and $x + 2 < 0$, so that $\frac{x+2}{x+3} < 0$. That $\lim_{x\to -3^-} \frac{x+2}{x+3} = \pm\infty$ and $\lim_{x\to -3^+} \frac{x+2}{x+3} = \pm\infty$ is clear. Using what was just observed, we now know more precisely that

$$\lim_{x\to -3^-} \frac{x+2}{x+3} = +\infty \quad \text{and} \quad \lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty.$$

So $x = -3$ is a vertical asymptote of $f(x) = \frac{x^2-x-6}{x^2-9}$. Because $f(x)$ is defined for all x except $x = \pm 3$ it is the only vertical asymptote. The fact that $\lim_{x\to\pm\infty} \frac{x+2}{x+3} = \lim_{x\to\pm\infty} \frac{1+\frac{2}{x}}{1+\frac{3}{x}} = 1$ tells us that $y = 1$ is a horizontal asymptote for the graph.



iii. By the quotient rule, $g'(x) = \frac{(x+3)-(x+2)}{(x+3)^2} = \frac{1}{(x+3)^2}$ and by the chain rule $g''(x) = \frac{-2}{(x+3)^3}$. (Note the incorrect double minus in the text's formulation of $g''(x)$.) Because $\frac{1}{(x+3)^2} > 0$ for all x, the graph is increasing both to the left and right of its vertical asymptote $x = -3$. For $x < -3$, $x + 3 < 0$, so that $g''(x) > 0$. For $x > -3$, $x + 3 > 0$, so that $g''(x) < 0$. So the graph is concave up for $x < -3$ and concave down for $x > -3$. Since $g(x)$ is not defined at $x = -3$ there is no point of inflection. The above graph of the function was drawn with https://www.desmos.com/calculator.

7.88. Setting $f(x) = \frac{1}{2}x^2 - 1 = 0$, we get $\frac{1}{2}x^2 = 1$ and hence $x^2 = 2$. So $x = \pm\sqrt{2}$ are the roots of $\frac{1}{2}x^2 - 1$. Let's see what Newton's method gives us. Note that $f'(x) = x$. Starting with $c_1 = 2$, we get

$$\begin{aligned}
c_2 &= 2 - \tfrac{f(2)}{f'(2)} = 2 - \tfrac{\frac{1}{2}\cdot 2^2 - 1}{2} = 2 - \tfrac{1}{2} = \tfrac{3}{2} \\
c_3 &= \tfrac{3}{2} - \tfrac{f(\frac{3}{2})}{f'(\frac{3}{2})} = \tfrac{3}{2} - \tfrac{\frac{1}{2}(\frac{3}{2})^2 - 1}{\frac{3}{2}} = \tfrac{3}{2} - \tfrac{2}{3}\cdot\tfrac{1}{2}\cdot\tfrac{9}{4} + \tfrac{2}{3} = 1.4167 \\
c_4 &= 1.4167 - \tfrac{f(1.4167)}{f'(1.4167)} = 1.4167 - 0.0025 = 1.4142.
\end{aligned}$$

Checking with a calculator that $\sqrt{2} = 1.414213562\ldots$ we see that Newton's method has already closed in on the root $\sqrt{2}$ to within the required four decimal place accuracy. This should mean that $c_5 = 1.4142$ rounded to four decimal places. Let's check. Because $c_4 = 1.4142$,

$$c_5 = 1.4142 - \tfrac{f(1.4142)}{f'(1.4142)} = 1.414213562\ldots.$$

So c_5 turns out to be an approximation of $\sqrt{2}$ that is accurate not only up to four, but in fact, up to nine decimal places.

7.89. We get $f'(x) = 3x^2 + 2x - 7$. Starting with $c_1 = 3$ gives us

$$\begin{aligned}
c_2 &= 3 - \tfrac{f(3)}{f'(3)} = 3 - \tfrac{8}{26} = 2.6923 \\
c_3 &= 2.6923 - \tfrac{f(2.6923)}{f'(2.6923)} = 2.6923 - \tfrac{0.9175}{20.1300} = 2.6467 \\
c_4 &= 2.6467 - \tfrac{f(2.6467)}{f'(2.6467)} = 2.6467 - \tfrac{0.0183}{19.3085} = 2.6458 \\
c_5 &= 2.6458 - \tfrac{f(2.6458)}{f'(2.6458)} = 2.6458 - \tfrac{0.0009}{19.2924} = 2.64575335.
\end{aligned}$$

This agrees with c_5 when rounded off. So the process is finished.

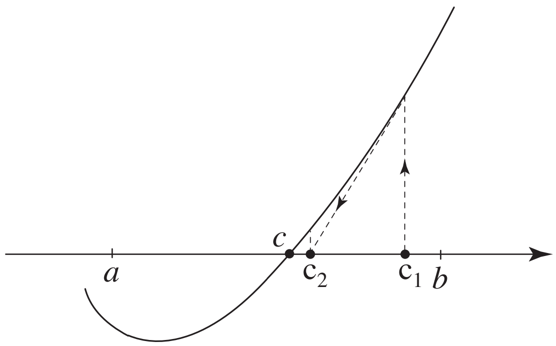
Refer to segment 4E of Section 4.7. From the fact that $f(-1) = -1 + 1 + 7 - 7 = 0$, it follows that $x + 1$ divides $x^3 + x^2 - 7x - 7$. Doing the division $x + 1\,\overline{)x^3 + x^2 - 7x - 7}$ we get that $x^3 + x^2 - 7x - 7 = (x + 1)(x^2 - 7)$. So the roots are -1 and $x = \pm\sqrt{7}$. So c_6 can only be an approximation of $\sqrt{7}$. Because $\sqrt{7} \approx 2.645751311$, this is indeed so.

7.90. We get $f'(x) = 3x^2 + 2x + 7$. Starting with $c_1 = 3$ gives us

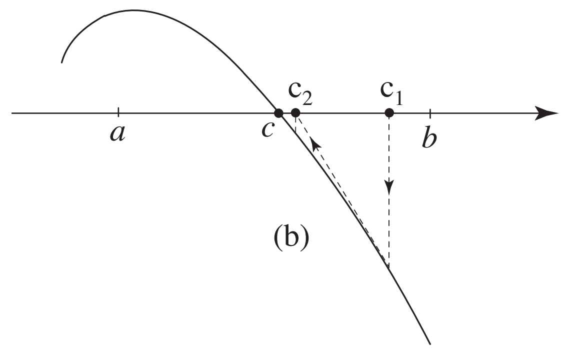
$$\begin{aligned}
c_2 &= 3 - \tfrac{f(3)}{f'(3)} = 3 - \tfrac{64}{40} = 1.4000 \\
c_3 &= 1.4000 - \tfrac{f(1.4000)}{f'(1.4000)} = 1.4000 - \tfrac{21.504}{15.6800} = 0.0286 \\
c_4 &= 0.0286 - \tfrac{f(0.0286)}{f'(0.0286)} = 0.0286 - \tfrac{7.2010}{7.0597} = -0.9914 \\
c_5 &= -0.9914 - \tfrac{f(-0.9914)}{f'(-0.9914)} = -0.9914 - \tfrac{0.0687}{7.9658} = -1.00002437.
\end{aligned}$$

The convergence is to the root -1 of x^3+x^2+7x+7. By dividing x^3+x^2+7x+7 by $x+1$, we get $x^3+x^2+7x+7=(x+1)(x^2+7)$. It follows that $x=-1$ is the only root of $f(x)$.

7.91. **i.** The generic graphs below illustrate the convergence of Newton's method in each case.

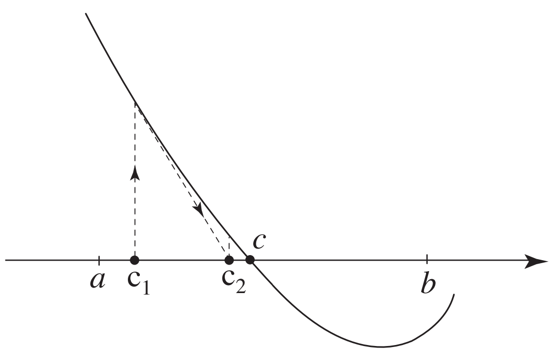


(a)

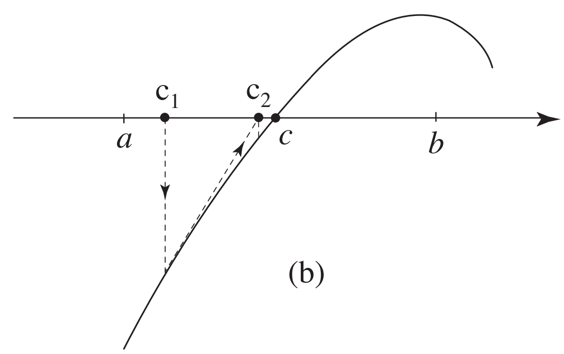


(b)

ii. Two more generic graphs illustrate the convergence of Newton's method.

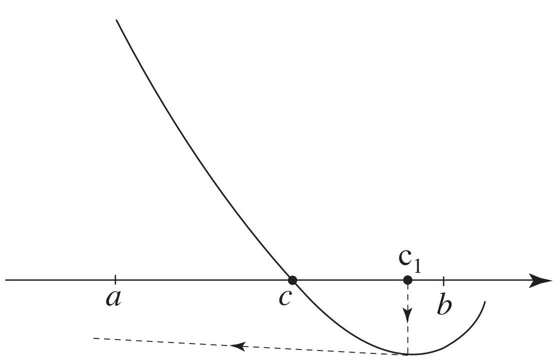


(a)

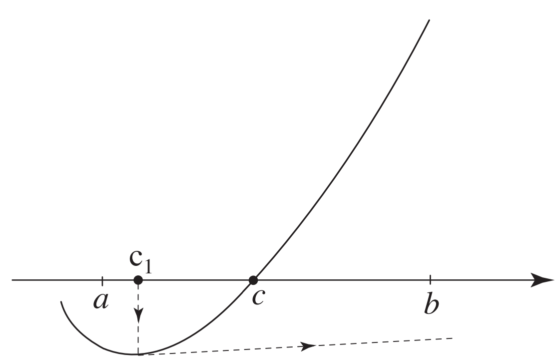


(b)

iii. In the two situations below, the point that the second iteration produces ends up far outside the interval $[a,b]$. Depending on the behavior of the graph "out there" anything



(a)



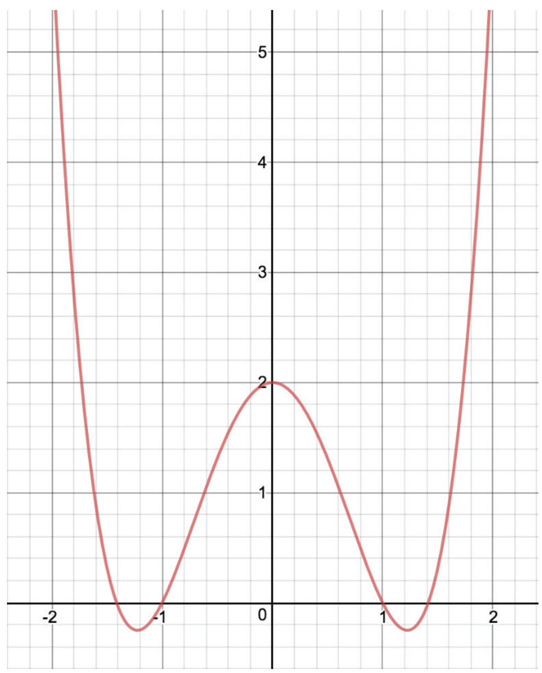
(b)

can happen. For instance, the process may converge to a solution d of $f(x) = 0$ that satisfies one of the conditions (i) and (ii) above (if there is such a d).

7.92. The function $f(x) = x^4 - 3x^2 + 2$ has derivative $f'(x) = 4x^3 - 6x = 4x(x^2 - \frac{3}{2})$ and second derivative $f''(x) = 12x^2 - 6 = 12(x^2 - \frac{1}{2})$.

Since the solutions of $f'(x) = 0$ are $x = -\sqrt{\frac{3}{2}}, 0$, and $\sqrt{\frac{3}{2}}$, the graph has horizontal tangents at the three points listed. Taking test points in each of the four intervals $(-\infty, -\sqrt{\frac{3}{2}}), (-\sqrt{\frac{3}{2}}, 0), (0, \sqrt{\frac{3}{2}})$, and $(\sqrt{\frac{3}{2}}, \infty)$, say $-2, -1, 1$, and 2, we see that $f'(-2) < 0$, $f'(-1) > 0, f'(1) < 0$, and $f'(2) > 0$, so that the graph of $f(x)$ is decreasing over the first of these intervals, increasing over the second, decreasing over the third, and increasing over the last.

The points of inflection occur for $x = \pm\sqrt{\frac{1}{2}}$. Picking as test points $-1, 0$, and 1 in the intervals $(-\infty, -\sqrt{\frac{1}{2}}), (-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$, and $(\sqrt{\frac{1}{2}}, \infty)$, respectively, we see that $f''(-1) > 0, f''(0) < 0$, and $f''(1) > 0$. Therefore the graph of $f(x)$ is concave up over the first and third of these



intervals and concave down over the second. The graph of $f(x) = x^4 - 3x^2 + 2$ sketched above was drawn with https://www.desmos.com/calculator.

7.93. For the guess $c_1 = 3$ first case of Problem 7.91i applies to show that Newton's method converges to the root $\sqrt{2}$ of $f(x)$.

For $c_1 = \sqrt{\frac{3}{2}}$, there is a horizontal tangent. This horizontal tangent intersects the graph at $(-\sqrt{\frac{3}{2}}, -\frac{1}{4})$ so that Newton's method goes nowhere.

For $c_1 = 1.1$, we get $c_2 = 1.1 - \frac{f(1.1)}{f'(1.1)} = 1.1 - \frac{-0.1659}{-1.2760} \approx 0.9700$. Since the graph of $f(x)$ is increasing and concave up over the interval $[c_2, 1]$, the first case of Problem 7.91ii tells us that Newton's method will converge to the root 1 of $f(x)$. The same observation applies to $c_1 = 0.9$.

Finally, $c_1 = 0.1$. The slope of the tangent to the graph of $f(x)$ at $(0.1, 1.9701)$ is $f'(0.1) = 4(0.1)^3 - 6(0.1) = -0.5960$. So this tangent line has equation $y - 1.9701 = -0.5960(x - 0.1)$ or $y = -0.5960x + 0.0596 + 1.9701 = -0.5960x + 2.0297$. Setting $y = 0$, we get $0.5960x = 2.0297$ and hence $x = c_2 \approx 3.4055$. We are now in the same situation as the case $c_1 = 3$.

The site http://keisan.casio.com/exec/system/1244946907 carries out Newton's method for any differentiable function $f(x)$.

We'll close the set of solutions for Chapter 7 by returning to Example 7.14 and the question of the convergence of the sawtooth pattern depicted in Figure 7.24 to the origin (both from the left and the right). The answer that this is so was provided to me by my Notre Dame colleague Laurence Taylor. His argument follows. It is much more subtle than one might have expected.

Let $y = x^2$ and pick a point (s, s^2) in the first quadrant. Pick a positive slope m. There are two generic problems to solve.

1. Pick any point (t_0, t_0^2), $t_0 > 0$ and find the intersection of the line of slope m with the x-axis. This is the line $y = mx + (t_0^2 - mt_0)$, so that the x-coordinate of the point is

$$u_0 = t_0\Big(1 - \frac{t_0}{m}\Big).$$

As long as $m > t_0$, we have

$$0 < u_0 < t_0. \tag{1}$$

Let's suppose hereafter that $m > s$.

2. Start with any point $(w_0, 0)$, $w_0 > 0$ and run the line of slope $-m$ through this point up to the point (v_0, v_0^2) on $y = x^2$ in the first quadrant. Since this line is $y = -mx + mw_0$, we get $v_0^2 = -mv_0 + mw_0$ or $v_0^2 + mv_0 - mw_0 = 0$. Therefore

$$v_0 = \frac{-m \pm \sqrt{m^2 + 4mw_0}}{2}.$$

If we were to take the minus sign we would be in the second quadrant so

$$v_0 = \frac{-m + \sqrt{m^2 + 4mw_0}}{2}.$$

Note that

$$0 < v_0 < w_0, \tag{2}$$

since $m^2 < m^2 + 4mw_0 < (m + 2w_0)^2$. Note also that

$$v_0^2 = m(w_0 - v_0) \tag{3}$$

which also proves $w_0 > v_0$.

Now define the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ inductively as follows.

- $a_0 = s$, $b_0 = u_0$ computed with $t_0 = s$.
- For $n > 0$, $a_{n+1} = u_0$ computed with $t_0 = b_n$.
- For $n > 0$, $b_n = v_0$ computed with $w_0 = a_n$.

From (1) we see that $a_{n+1} < b_n$, and from (2) it follows that $b_n < a_n$. Since $a_0 = s$ is greater than a_n, $n > 0$ and b_n for all n, $m > a_n$ and $m > b_n$ for all n. It further follows that $a_n > 0$ and $b_n > 0$ for all n. The sequence $(a_n, 0)$ are the points of the sawtooth on the x-axis and the sequence (b_n, b_n^2) are the points on the parabola. Note that

$$0 < \cdots b_n < a_n < b_{n-1} < \cdots b_0 < a_0 = s < m.$$

Hence $0 \leqslant \lim a_n = \lim b_n$. From equation (3) above

$$\lim b_n^2 = m \lim(a_n - b_n).$$

Hence $\lim b_n^2 = 0$ and since $\lim b_n$ exists, $\lim b_n = 0$. Finally $\lim a_n = \lim b_n = 0$.