Solutions to Problems and Projects for Chapter 11

11.1. Since $y = \tan x + \sec x$, we know that $\frac{dy}{dx} = \sec^2 x + (\sec x)(\tan x)$ by facts from Section 7.7. The restriction $-\frac{\pi}{2} < x < \frac{\pi}{2}$ insures the continuity and differentiability of the functions $y = \tan x$ and $y = \sec x$ (see Figures 4.26 and 4.27).

$$\begin{aligned} 2\tfrac{dy}{dx} - y^2 &= 2\sec^2 x + 2(\sec x)(\tan x) - \tan^2 x - 2(\tan x)(\sec x) - \sec^2 x \\ &= \sec^2 x - \tan^2 x. \end{aligned}$$

The trig identity $\tan^2 + 1 = \sec^2 x$ informs us that $\sec^2 x - \tan^2 x = 1$.

11.2. Let $y = \sin x$. Since $\frac{dy}{dx} = \cos x$ and $\sin^2 x + \cos^2 x = 1$, we see that $(\frac{dy}{dx})^2 = 1 - \sin^2 x = 1 - y^2$. In the same way, for $y = \cos x$, $(\frac{dy}{dx})^2 = 1 - \cos^2 x = 1 - y^2$.

The solutions of the differential equations that follow involve a constant of integration C that is often modified in the course of the solution (by multiplication by a constant for instance). We will usually continue to denote such a constant by C even after such modifications.

11.3. After separating variables the equation $\frac{dy}{dx} = xy$ becomes $\frac{dy}{y} = x\,dx$. So $\displaystyle\int \frac{dy}{y} = \int x\,dx$ and hence $\ln y = \frac{1}{2}x^2 + C$. Therefore $y = e^{\ln y} = e^{\frac{1}{2}x^2 + C} = e^{\frac{1}{2}x^2}e^C$. So the general solution has the form $y = f(x) = Ae^{\frac{1}{2}x^2}$ for a positive constant A. The condition $f(0) = 4$ provides the particular solution $f(x) = 4e^{\frac{1}{2}x^2}$.

11.4. By separating the variables the equation $\frac{dy}{dx} = \frac{y}{x}$ becomes $\frac{dy}{y} = \frac{dx}{x}$. So $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x}$ and hence $\ln y = \ln x + C$. So $y = e^{\ln y} = e^{\ln x + C} = e^{\ln x}e^C = e^C x$. So the general solution has the form $f(x) = Ax$ for a positive constant A. The condition $f(1) = 2$ gives rise to the particular solution $f(x) = 2x$. Since all solutions satisfy $f(0) = 0$, the condition $f(0) = 1$ cannot arise.

11.5. The equation $y' = y \sin x - \sin x$ can be rewritten as $\frac{dy}{dx} = (\sin x)(y - 1)$ and hence as $\frac{dy}{y-1} = \sin x\,dx$. So $\displaystyle\int \frac{dy}{y-1} = \int \sin x\,dx$ and therefore $\ln(y - 1) = -\cos x + C$. It follows that $y - 1 = e^{\ln(y-1)} = e^{-\cos x + C} = e^C e^{-\cos x}$. It follows that the general solution is $y = f(x) = Ae^{-\cos x} + 1$ for a positive constant A. With the initial condition $f(\pi) = 3$, we get $3 = Ae^{-\cos\pi} + 1 = Ae^1 + 1$. So $Ae = 2$ and hence $A = 2e^{-1}$. Therefore the particular solution is $f(x) = 2e^{-1}e^{-\cos x} + 1 = 2e^{-(\cos x + 1)} + 1$.

11.6. The equation $\frac{dy}{dx} - x^2y^2 = 0$ can be rearranged to $\frac{dy}{dx} = x^2y^2$ and hence to $\frac{dy}{y^2} = x^2\,dx$. So $\displaystyle\int y^{-2}\,dy = \int x^2\,dx$. Therefore $-y^{-1} = \frac{1}{3}x^3 + C$ so that the general solution is $y = f(x) = -\frac{1}{\frac{1}{3}x^3 + C}$ for a constant C. The initial condition $f(0) = 8$ means that $8 = -\frac{1}{C}$. So $C = -\frac{1}{8}$. Therefore the particular solution is $f(x) = -\frac{1}{\frac{1}{3}x^3 - \frac{1}{8}} = \frac{8}{1 - \frac{8}{3}x^3}$.

11.7. The equation $x\frac{dy}{dx} = 3y + x^2$ can be rewritten as $\frac{dy}{dx} = \frac{3}{x}\,y + x$ and hence as $y' + p(x)y = q(x)$ with $p(x) = -\frac{3}{x}$ and $q(x) = x$. This is an equation to which the method of integrating factors applies. The function $P(x) = -3\ln x = \ln x^{-3}$ is an antiderivative of $p(x)$ and $e^{P(x)} = e^{\ln x^{-3}} = x^{-3}$. It follows that

$$\int e^{P(x)}q(x)\,dx = \int x^{-3}x\,dx = \int x^{-2}\,dx.$$

Therefore by the method of integrating factors,

$$y\cdot x^{-3} + C = y\cdot e^{P(x)} + C = \int e^{P(x)}q(x)\,dx = \int x^{-2}\,dx = -x^{-1} + C'.$$

So $yx^{-3} + C = -x^{-1}$. Replacing C with $-C$, we get the general solution $y = -x^2 + Cx^3$.

11.8. After dividing through by $3x$, the equation $3xy' - y = \ln x + 1$ is in the form $y' + p(x)y = q(x)$ with $p(x) = -\frac{1}{3x}$ and $q(x) = \frac{\ln x + 1}{3x}$, so that it can be approached with the method of integrating factors. Noting that $P(x) = -\frac{1}{3}\ln x = \ln x^{-\frac{1}{3}}$ is antiderivative of $p(x)$ and that and $e^{P(x)} = x^{-\frac{1}{3}}$, we get

$$\int e^{P(x)}q(x)\,dx = \int (x^{-\frac{1}{3}})\tfrac{\ln x+1}{3x}\,dx = \tfrac{1}{3}\int (x^{-\frac{4}{3}}\ln x + x^{-\frac{4}{3}})\,dx\,.$$

To solve $\int x^{-\frac{4}{3}}\ln x\,dx$ apply integration by parts with $u = \ln x$ and $dv = x^{-\frac{4}{3}}\,dx$. So $du = \frac{1}{x}\,dx, v = -3x^{-\frac{1}{3}}$ and hence

$$\begin{aligned}\int x^{-\frac{4}{3}}\ln x\,dx &= \int u\,dv = uv - \int v\,du = -3x^{-\frac{1}{3}}\ln x + \int 3x^{-\frac{4}{3}}\,dx\\ &= -3x^{-\frac{1}{3}}\ln x - 9x^{-\frac{1}{3}} + C.\end{aligned}$$

Since $\int x^{-\frac{4}{3}}\,dx = -3x^{-\frac{1}{3}} + C'$, we get from the earlier equation that

$$\int e^{P(x)}q(x)\,dx = \tfrac{1}{3}(-3x^{-\frac{1}{3}}\ln x - 9x^{-\frac{1}{3}}) - \tfrac{1}{3}(3x^{-\frac{1}{3}}) + C' = -x^{-\frac{1}{3}}\ln x - 4x^{-\frac{1}{3}} + C'.$$

By step 4 of the method of integrating factors,

$$yx^{-\frac{1}{3}} + C = y\cdot e^{P(x)} + C = \int e^{P(x)}q(x)\,dx = -x^{-\frac{1}{3}}\ln x - 4x^{-\frac{1}{3}} + C',$$

so that $yx^{-\frac{1}{3}} = -x^{-\frac{1}{3}}\ln x - 4x^{-\frac{1}{3}} + C$. So the general solution of the differential equation $3xy' - y = \ln x + 1$ is $y = f(x) = Cx^{\frac{1}{3}} - \ln x - 4$. With the initial condition $f(1) = 5$, we get $5 = C - 0 - 4$ and hence that $C = 9$. Therefore the particular solution is $f(x) = 9x^{\frac{1}{3}} - \ln x - 4$.

11.9. The equation $(t^2+1)y' - (1-t)^2y = te^t$ can be written as $y' + \frac{-(1-t)^2}{1+t^2}y = \frac{te^t}{1+t^2}$. So the method of integrating factors applies with $p(t) = \frac{-(1-t)^2}{1+t^2}$ and $q(t) = \frac{te^t}{1+t^2}$. The equality $p(t) = \frac{-(1-t)^2}{1+t^2} = \frac{-1+2t-t^2}{1+t^2} = \frac{2t}{1+t^2} - 1$ and the fact that $\frac{d}{dt}\ln(1+t^2) = \frac{2t}{1+t^2}$ (see Section 7.11) tells us that $P(t) = \ln(1+t^2) - t$ is an antiderivative of $p(t) = \frac{-(1-t)^2}{t^2+1)}$. Since $e^{P(t)} = e^{\ln(1+t^2)-t} = e^{\ln(1+t^2)}e^{-t} = (1+t^2)e^{-t}$, we get $e^{P(t)}q(t) = (1+t^2)e^{-t}\frac{te^t}{1+t^2} = t$. By step 4 of the integrating factors strategy $\int e^{P(t)}q(t)\,dt = y\cdot e^{P(t)} + C$, so that

$$y(1+t^2)e^{-t} = y\cdot e^{P(t)} = \int e^{P(t)}q(t)\,dt = \tfrac{1}{2}t^2 + C.$$

It follows that the general solution of $(t^2+1)y' - (1-t)^2y = te^t$ is $y = \frac{\frac{1}{2}t^2+C}{e^{-t}(1+t^2)}$.

11.10. After separating variables, we get $\int (y-3)\,dy = \int dt$ and therefore the implicit solution $\frac{1}{2}y^2 - 3y = t + C$. By applying the quadratic formula to $\frac{1}{2}y^2 - 3y - (t - C) = 0$ we get

$$y = 3 \pm \sqrt{9 + 2(t - C)}.$$

The particular solution $y = f(t)$ we need satisfies $f(0) = 4$. So $4 = 3 \pm \sqrt{9 + 2(0 - C)}$ and hence $1 = +\sqrt{9 - 2C}$. So $2C = 8$ and therefore $f(t) = 3 + \sqrt{1 + 2t}$.

11.11. By separating variables, we get $(1 + y)\,dy = (\sin x - \cos x)\,dx$, so that

$$\int (1+y)\,dy = \int (\sin x - \cos x)\,dx.$$

Therefore $y + \frac{1}{2}y^2 = -\cos x - \sin x + C$ and hence $\frac{1}{2}y^2 + y + (\cos x + \sin x - C) = 0$. By the quadratic formula $y = -1 \pm \sqrt{1 - 2(\cos x + \sin x - C)}$. To get the particular solution $y = f(x)$ that satisfies $f(\pi) = 0$, we solve

$$0 = -1 \pm \sqrt{1 - 2(\cos\pi + \sin\pi - C)} = -1 + \sqrt{1 - 2((-1) + 1 - C)} = -1 + \sqrt{1 + 2C}$$

(note that the $-$ option does not arise in this case) so that $2C = 0$. Therefore the particular solution is $y = f(x) = -1 + \sqrt{1 - 2(\cos x + \sin x - C)} = \sqrt{1 - 2(\cos x + \sin x)}$.

11.12. After separating variables, $\int \frac{(\ln y)^2}{y}\,dy = \int x^2\,dx$. To solve the integral on the left we'll use integration by parts and let $u = (\ln y)^2$ and $dv = \frac{1}{y}\,dy$. So $du = 2(\ln y)\frac{1}{y}$ and $v = \ln y$. Therefore $\int \frac{(\ln y)^2}{y}\,dy = uv - \int v\,du = (\ln y)^3 - \int \frac{2(\ln y)^2}{y}\,dy$. So $\int \frac{3(\ln y)^2}{y}\,dy = (\ln y)^3 + C$ and hence $\int \frac{(\ln y)^2}{y}\,dy = \frac{1}{3}(\ln y)^3 + C$. Therefore $\frac{1}{3}(\ln y)^3 = \frac{1}{3}x^3 + C$. So $(\ln y)^3 = x^3 + C$ and hence $\ln y = (x^3 + C)^{\frac{1}{3}}$. We finally arrive at the explicit general solution $y = e^{(x^3+C)^{\frac{1}{3}}}$.

The calculator of the website

https://www.symbolab.com/solver/ordinary-differential-equation-calculator/

presents the solution of the equation $2y' - y = 4\sin 3x$ (the site uses the variable x rather than t) in the form $y = \frac{-24e^{-\frac{x}{2}}\cos(3x) - 4e^{-\frac{x}{2}}\sin(3x) + c_1}{37e^{-\frac{x}{2}}}$.

11.13. We begin "our own" solution of the equation $2y' - y = 4\sin 3t$ by recognizing that it can be put into the form $y' - \frac{1}{2}y = 2\sin 3t$, so that the method of integrating factors applies. Accordingly, we'll let $p(t) = -\frac{1}{2}$ and $q(t) = 2\sin 3t$. Taking the antiderivative $P(t) = -\frac{1}{2}t$ of $p(t)$, we need—according to step 4 of this method—to solve $\int 2e^{-\frac{1}{2}t}\sin 3t\,dt = y \cdot e^{-\frac{1}{2}t} + C_1$.

A look back to Example 9.25 suggests that the integral $\int e^{-\frac{1}{2}t}\sin 3t\,dt$ might yield to integration by parts. With $u = e^{-\frac{1}{2}t}$ and $dv = \sin 3t\,dt$, we get $du = -\frac{1}{2}e^{-\frac{1}{2}t}$ and $v = -\frac{1}{3}\cos 3t$. So

$$\int e^{-\frac{1}{2}t}\sin 3t\,dt = \int u\,dv = uv - \int v\,du = -\frac{1}{3}e^{-\frac{1}{2}t}\cos 3t - \int \frac{1}{6}e^{-\frac{1}{2}t}\cos 3t\,dt.$$

As in Example 9.25 we need to go another round, now with the integral $\int e^{-\frac{1}{2}t}\cos 3t\,dt$. This time $u = e^{-\frac{1}{2}t}$ and $dv = \cos 3t\,dt$, so that $du = -\frac{1}{2}e^{-\frac{1}{2}t}$ and $v = \frac{1}{3}\sin 3t$. Therefore

$$\int e^{-\frac{1}{2}t}\cos 3t\,dt = \int u\,dv = uv - \int v\,du = \tfrac{1}{3}e^{-\frac{1}{2}t}\sin 3t + \int \tfrac{1}{6}e^{-\frac{1}{2}t}\sin 3t\,dt.$$

By combining the solutions of the last two integrals, we get

$$\begin{aligned}\int e^{-\frac{1}{2}t}\sin 3t\,dt &= -\tfrac{1}{3}e^{-\frac{1}{2}t}\cos 3t - \tfrac{1}{6}\int e^{-\frac{1}{2}t}\cos 3t\,dt \\ &= -\tfrac{1}{3}e^{-\frac{1}{2}t}\cos 3t - \tfrac{1}{6}\Big(\tfrac{1}{3}e^{-\frac{1}{2}t}\sin 3t + \int \tfrac{1}{6}e^{-\frac{1}{2}t}\sin 3t\,dt\Big).\end{aligned}$$

It follows that $\frac{37}{36}\int e^{-\frac{1}{2}t}\sin 3t\,dt = -\frac{1}{3}e^{-\frac{1}{2}t}\cos 3t - \frac{1}{18}e^{-\frac{1}{2}t}\sin 3t + C_2$ and therefore that $\int e^{-\frac{1}{2}t}\sin 3t\,dt = -\frac{12}{37}e^{-\frac{1}{2}t}\cos 3t - \frac{2}{37}e^{-\frac{1}{2}t}\sin 3t + C_3$.

By combining all we know,

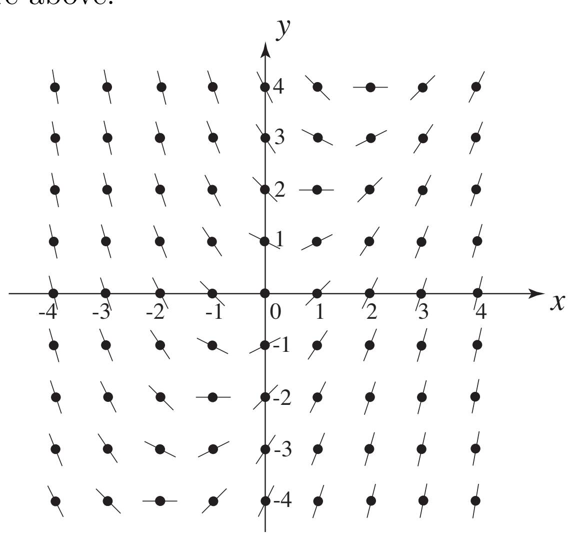
$$y\cdot e^{-\frac{1}{2}t} = 2\int e^{-\frac{1}{2}t}\sin 3t\,dt - C_1 = -\tfrac{24}{37}e^{-\frac{1}{2}t}\cos 3t - \tfrac{4}{37}e^{-\frac{1}{2}t}\sin 3t + 2C_3 - C_1$$

and hence $y = Ce^{\frac{1}{2}t} - \frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t$. This is equivalent to the solution that the calculator provided.

The calculator gives the general solution $y = f(x) = \frac{\pm\sqrt{5}}{\sqrt{5Cx^8+2x^3}}$ for the differential equation $y' + \frac{4}{x}y = x^2y^3$. Inserting the initial condition $f(1) = 2$, we get $2 = \frac{\sqrt{5}}{\sqrt{5C+2}}$, so that $2\sqrt{5C+2} = \sqrt{5}$. Since $4(5C+2) = 5$, we get $C = -\frac{3}{20}$. So the particular solution is $y = \frac{\sqrt{5}}{\sqrt{2x^3 - \frac{3}{4}x^8}}$.

11.14. Throughout this problem $F(x,y) = x - \frac{1}{2}y$.

i. The slope field for the differential equation $y' = x - \frac{1}{2}y$ corresponding to Table 11.2 is depicted in the figure above.



ii. (a) We'll follow the recipe described in Section 11.3 with $h = 1$. The relevant move is to apply $(x_i, y_i) = (x_{i-1}+1, y_{i-1} + 1\cdot F(x_{i-1}, y_{i-1}))$ step by step—note that the equality

$y_i = y_0 + hF(x_{i-1}, y_{i-1})h$ in the text should read $y_i = y_{i-1} + hF(x_{i-1}, y_{i-1})h$—starting with $i = 1$ and the point $(x_0, y_0) = (0, -1)$:

the slope at $(0, -1)$ is $0 - \frac{1}{2}(-1) = \frac{1}{2}$, so that $(x_1, y_1) = (1, -1 + 1 \cdot (\frac{1}{2})) = (1, -\frac{1}{2})$;
the slope at $(1, -\frac{1}{2})$ is $1 - \frac{1}{2}(-\frac{1}{2}) = \frac{5}{4}$, so that $(x_2, y_2) = (2, -\frac{1}{2} + 1 \cdot (\frac{5}{4})) = (2, \frac{3}{4})$;
the slope at $(2, \frac{3}{4})$ is $2 - \frac{1}{2}(\frac{3}{4}) = \frac{13}{8}$, so that $(x_3, y_3) = (3, \frac{3}{4} + 1 \cdot (\frac{13}{8})) = (3, \frac{19}{8})$;
the slope at $(3, \frac{19}{8})$ is $3 - \frac{1}{2}(\frac{19}{8}) = \frac{29}{16}$, so that $(x_4, y_4) = (4, \frac{19}{8} + 1 \cdot (\frac{29}{16})) = (4, \frac{67}{16})$.

Doing a similar thing going from $(\bar{x}_0, \bar{y}_0) = (0, -1)$ in the negative direction using $(\bar{x}_i, \bar{y}_i) = (\bar{x}_{i-1} - 1, \bar{y}_{i-1} - 1 \cdot F(\bar{x}_{i-1}, \bar{y}_{i-1})$ starting with $i = 1$ and the point $(\bar{x}_0, \bar{y}_0) = (0, -1)$ we get:

the slope at $(0, -1)$ is $0 - \frac{1}{2}(-1) = \frac{1}{2}$,
so that $(\bar{x}_1, \bar{y}_1) = (-1, -1 - 1 \cdot (\frac{1}{2})) = (-1, -\frac{3}{2})$;
the slope at $(-1, -\frac{3}{2})$ is $-1 - \frac{1}{2}(-\frac{3}{2}) = -\frac{1}{4}$,
so that $(\bar{x}_2, \bar{y}_2) = (-2, -\frac{3}{2} - 1 \cdot (-\frac{1}{4})) = (-2, -\frac{5}{4})$;
the slope at $(-2, -\frac{5}{4})$ is $-2 - \frac{1}{2}(-\frac{5}{4}) = -\frac{11}{8}$,
so that $(\bar{x}_3, \bar{y}_3) = (-3, -\frac{5}{4} - 1 \cdot (-\frac{11}{8})) = (-3, \frac{1}{8})$;
the slope at $(-3, \frac{1}{8})$ is $-3 - \frac{1}{2}(\frac{1}{8}) = -\frac{49}{16}$,
so that $(\bar{x}_4, \bar{y}_4) = (-4, \frac{1}{8} - 1 \cdot (-\frac{49}{16})) = (-4, \frac{51}{16})$.

The resulting approximate solution of the particular solution $y = f(x)$ of the differential equation with initial condition $f(0) = -1$ is sketched in figure (a) below.

(b) This time $h = 0.5$. Now $(x_i, y_i) = (x_{i-1} + 0.5, y_{i-1} + 0.5 \cdot F(x_{i-1}, y_{i-1})$ step by step starting with $i = 1$ and the point $(x_0, y_0) = (0, -1)$:

the slope at $(0, -1)$ is $0 - \frac{1}{2}(-1) = \frac{1}{2}$,
so that $(x_1, y_1) = (0.5, -1 + 0.5 \cdot (\frac{1}{2})) = (0.5, -\frac{3}{4}) = (0.5, -0.75)$;
the slope at $(0.5, -\frac{3}{4})$ is $0.5 - \frac{1}{2}(-\frac{3}{4}) = \frac{7}{8}$,
so that $(x_2, y_2) = (1, -\frac{3}{4} + 0.5 \cdot (\frac{7}{8})) = (1, -\frac{5}{16}) \approx (1, -0.3125)$;
the slope at $(1, -\frac{5}{16})$ is $1 - \frac{1}{2}(-\frac{5}{16}) = \frac{37}{32}$,
so that $(x_3, y_3) = (1.5, -\frac{5}{16} + 0.5 \cdot (\frac{37}{32})) = (1.5, \frac{17}{64}) \approx (1.5, 0.2656)$;
the slope at $(1.5, \frac{17}{64})$ is $1.5 - \frac{1}{2}(\frac{17}{64}) = \frac{175}{128}$,
so that $(x_4, y_4) = (2, \frac{17}{64} + 0.5 \cdot (\frac{175}{128})) = (2, \frac{243}{256}) \approx (2, 0.9492)$;
the slope at $(2, \frac{243}{256})$ is $2 - \frac{1}{2}(\frac{243}{256}) = \frac{781}{512}$,
so that $(x_5, y_5) = (2.5, \frac{243}{256} + 0.5 \cdot (\frac{781}{512})) = (2.5, \frac{1753}{1024}) \approx (2.5, 1.7119)$;
the slope at $(2.5, \frac{1753}{1024})$ is $2.5 - \frac{1}{2}(\frac{1753}{1024}) = \frac{3367}{2048}$,
so that $(x_6, y_6) = (3, \frac{1753}{1024} + 0.5 \cdot (\frac{3367}{2048})) = (3, \frac{10379}{4096}) \approx (3, 2.5339)$;
the slope at $(3, \frac{10379}{4096})$ is $3 - \frac{1}{2}(\frac{10379}{4096}) = \frac{14197}{8192}$,
so that $(x_7, y_7) = (3.5, \frac{10379}{4096} + 0.5 \cdot (\frac{14197}{8192})) = (3.5, \frac{55713}{16384}) \approx (3.5, 3.4005)$;
the slope at $(3.5, \frac{55713}{16384})$ is $3.5 - \frac{1}{2}(\frac{55713}{16384}) = \frac{58975}{32768}$,
so that $(x_8, y_8) = (4, \frac{55713}{16384} + 0.5 \cdot (\frac{58975}{32768})) = (4, \frac{281827}{65536}) \approx (4, 4.3003)$.

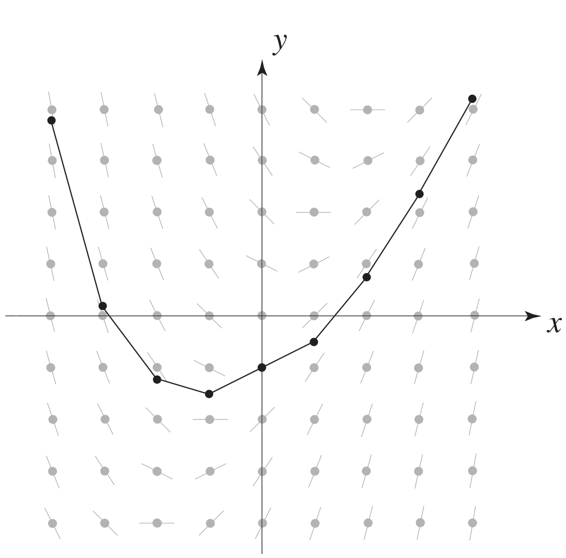
We'll now start from $(\bar{x}_0, \bar{y}_0) = (0, -1)$ and go in the negative direction. Using $(\bar{x}_i, \bar{y}_i) = (\bar{x}_{i-1} - 0.5, \bar{y}_{i-1} - 0.5 \cdot F(\bar{x}_{i-1}, \bar{y}_{i-1})$ and starting with $i = 1$ and the point $(\bar{x}_0, \bar{y}_0) = (0, -1)$, we get:

the slope at $(0,-1)$ is $0-\frac{1}{2}(-1)=\frac{1}{2}$,

so that $(\bar{x}_1,\bar{y}_1)=(-0.5,-1-0.5\cdot(\frac{1}{2}))=(-0.5,-\frac{5}{4})=(-0.5,-1.25)$;

the slope at $(-0.5,-\frac{5}{4})$ is $-0.5-\frac{1}{2}(-\frac{5}{4})=\frac{1}{8}$,

so that $(\bar{x}_2,\bar{y}_2)=(-1,-\frac{5}{4}-0.5\cdot(\frac{1}{8}))=(-1,-\frac{21}{16})\approx(-1,-1.3125)$;



(a) (b)

the slope at $(-1,-\frac{21}{16})$ is $-1-\frac{1}{2}(-\frac{21}{16})=-\frac{11}{32}$,

so that $(\bar{x}_3,\bar{y}_3)=(-1.5,-\frac{21}{16}-0.5\cdot(-\frac{11}{32}))=(-1.5,-\frac{73}{64})\approx(-1.5,-1.1406)$;

the slope at $(-1.5,-\frac{73}{64})$ is $-1.5-\frac{1}{2}(-\frac{73}{64})=-\frac{119}{128}$,

so that $(\bar{x}_4,\bar{y}_4)=(-2,-\frac{73}{64}-0.5\cdot(-\frac{119}{128}))=(-2,-\frac{173}{256})\approx(-2,-0.6758)$;

the slope at $(-2,-\frac{173}{256})$ is $-2-\frac{1}{2}(-\frac{173}{256})=-\frac{851}{512}$,

so that $(\bar{x}_5,\bar{y}_5)=(-2.5,-\frac{173}{256}-0.5\cdot(-\frac{851}{512}))=(-2.5,\frac{159}{1024})\approx(-2.5,0.1553)$;

the slope at $(-2.5,\frac{159}{1024})$ is $-2.5-\frac{1}{2}(\frac{159}{1024})=-\frac{5279}{2048}$,

so that $(\bar{x}_6,\bar{y}_6)=(-3,\frac{159}{1024}-0.5\cdot(-\frac{5279}{2048}))=(-3,\frac{5915}{4096})\approx(-3,1.4448)$;

the slope at $(-3,\frac{5915}{4096})$ is $-3-\frac{1}{2}(\frac{5915}{4096})=-\frac{30491}{8192}$,

so that $(\bar{x}_7,\bar{y}_7)=(-3.5,\frac{5915}{4096}-0.5\cdot(-\frac{30491}{8192}))=(-3.5,\frac{54151}{16384})\approx(-3.5,3.3051)$;

the slope at $(-3.5,\frac{54151}{16384})$ is $-3.5-\frac{1}{2}(\frac{54151}{16384})=-\frac{168839}{32768}$,

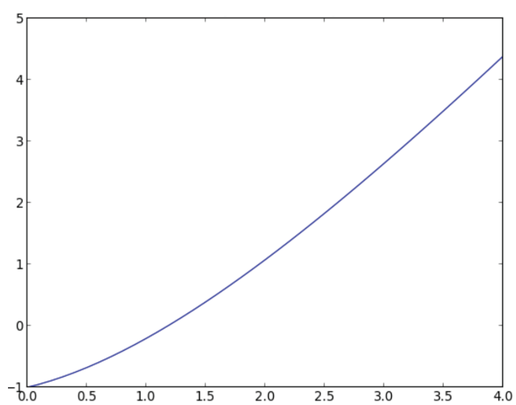
so that $(\bar{x}_8,\bar{y}_8)=(-4,\frac{54151}{16384}-0.5\cdot(-\frac{168839}{32768}))=(-4,\frac{385443}{65536})\approx(-4,5.8814)$.

The approximate solution of the particular solution $y=f(x)$ of the differential equation with initial condition $f(0)=-1$ that this set of points gives rise to is sketched in figure (b) above.

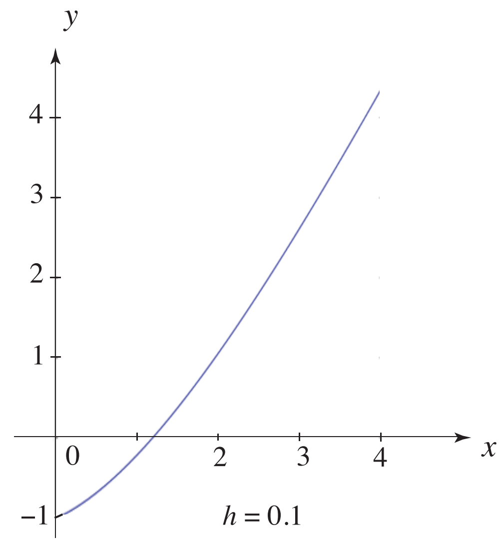
iii. We we now turn to the Euler method calculator on the website

http://www.math-cs.gordon.edu/~senning/desolver/.

The variables the site uses are t (instead of x) and y. To use the calculator, feed $t-0.5y$ into the slot for $dy/dt=f(t,y)=$. (Note that the choices 1/2 or (1/2) in place of 0.5 lead to errors.) Note also that for any initial values t_0 and y_0 of the variables t and y, the calculator requires the assumption that t satisfies $t_0\le t\le t_1$ for some t_1. In the current context this means that the approximate solution $y = f(x)$ of the differential equation $y' = x - \frac{1}{2}y$ with initial condition $f(0) = -1$ that the calculator provides is subject to the restriction $x \geq 0$. The calculator uses the same notation h for the step size as the text. Here the choice is $h = 0.1$ or $h = 0.2$. Finally take "Graph and Data points" as the Output format. Doing this for the current problem with $h = 0.1$ provides graph (c). Graph (d) is obtained by stretching graph (c) vertically so that the units of length of



(c)



(d)

the x- and y-axes are the same. The table of numbers (e) lists the y-coordinates of the points on the graph that correspond to the various x-coordinates between $x = 0$ to $x = 4$

x	y	x	y
0.000000	-1.000000	2.100000	1.221685
0.100000	-0.950000	2.200000	1.370601
0.200000	-0.892500	2.300000	1.522071
0.300000	-0.827875	2.400000	1.675967
0.400000	-0.756481	2.500000	1.832169
0.500000	-0.678657	2.600000	1.990560
0.600000	-0.594724	2.700000	2.151032
0.700000	-0.504988	2.800000	2.313481
0.800000	-0.409739	2.900000	2.477807
0.900000	-0.309252	3.000000	2.643916
1.000000	-0.203789	3.100000	2.811720
1.100000	-0.093600	3.200000	2.981134
1.200000	0.021080	3.300000	3.152078
1.300000	0.140026	3.400000	3.324474
1.400000	0.263025	3.500000	3.498250
1.500000	0.389874	3.600000	3.673338
1.600000	0.520380	3.700000	3.849671
1.700000	0.654361	3.800000	4.027187
1.800000	0.791643	3.900000	4.205828
1.900000	0.932061	4.000000	4.385536
2.000000	1.075458		

(e)

in increasing increments of 0.1. In terms of the procedure involved, the case $h = 0.2$ is identical to the case $h = 0.1$ so that we'll omit it.

iv. So far we have considered three approximations of the solution $y = f(x)$ of the initial value problem $y' = x - \frac{1}{2}y$ with $f(0) = -1$. They are graphed in (a), (b), and (d) above (in the last case for $x \geq 0$). We will now show that the "on the nose" solution of this problem is given by the function $f(x) = 2x - 4 + 3e^{-\frac{x}{2}}$. Rewriting the differential equation as $y' + \frac{1}{2}y = x$ suggests that the method of integrating factors should be applied with $p(x) = \frac{1}{2}$ and $q(x) = x$. This method has already been illustrated in Problems 11.7, 11.8, 11.9, and 11.13. So we'll use the calculator

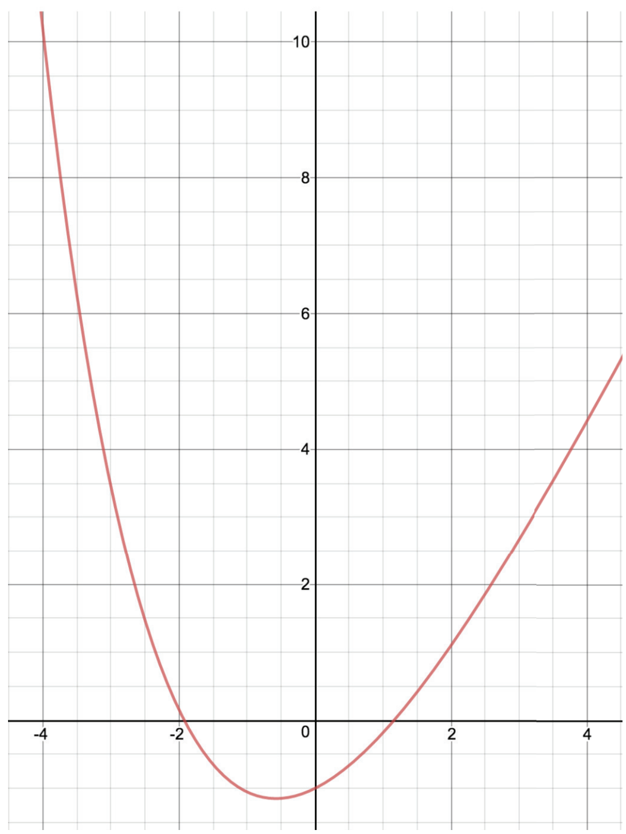
https://www.symbolab.com/solver/ordinary-differential-equation-calculator

this time. It supplies the general solution $y = \frac{2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c_1}{e^{\frac{x}{2}}}$. This is easily rewritten as $y = f(x) = 2x - 4 + c_1 e^{-\frac{1}{2}x}$. Inserting the initial condition $f(0) = -1$, we get $-1 = -4 + c_1$ and hence $c_1 = 3$. So the particular solution is $f(x) = 2x - 4 + 3e^{-\frac{1}{2}x}$.

v. The graphing calculator

https://www.desmos.com/calculator

provides the graph of the solution of (iv) for $-4 \leq x \leq 4$ drawn in the figure below.



We'll consider $x = 1$ and $x = 3$ and compare the corresponding y-coordinates supplied by the approximations for the step sizes $h = 1, 0.5$, and $h = 0.1$ against the y-coordinates of the precise solution $f(x) = 2x - 4 + 3e^{-\frac{1}{2}x}$. A look at the results in (ii) and (iii) above tells us that for $h = 1, 0.5$, and 0.1 the y-coordinates corresponding to $x = 1$ are $y = -\frac{1}{2} = -0.5, y = -\frac{5}{16} \approx -0.3125$ and $y = -0.203789$, respectively. For the precise solution it is $y = -0.180408$ (with an accuracy of six decimal places). For $x = 3$, the corresponding values for y are $y = \frac{19}{8} = 2.375, y \approx 2.5339$ and $y = 2.643916$, respectively. For the precise solution $y = 2.669390$ (accurate up to six decimal places).

11.15. This problem is very similar to Problem 11.14 so that the solution will be skipped.

11.16. To go from polar to Cartesian coordinates we'll use the two transformation equations

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

as well as elementary facts about the sine and cosine. (See Sections 1.4 and 4.4.) Problems viii to x require a calculator (used in radian mode).

i. $x = 3\cos\frac{\pi}{4} = 3\cdot\frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2}$ and $y = 3\sin\frac{\pi}{4} = 3\cdot\frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2}$. So $(x, y) = \left(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$.

ii. $x = -2\cos\left(-\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -2\frac{\sqrt{3}}{2} = -\sqrt{3}$ and $y = -2\sin\left(-\frac{\pi}{6}\right) = (-2)\left(-\sin\frac{\pi}{6}\right) = (-2)\left(-\frac{1}{2}\right) = 1$. So $(x, y) = \left(-\sqrt{3}, 1\right)$.

iii. $x = 3\cos\frac{7\pi}{3} = 3\cos\left(2\pi + \frac{\pi}{3}\right) = 3\cos\frac{\pi}{3} = 3\cdot\frac{1}{2} = \frac{3}{2}$ and $y = 3\sin\frac{7\pi}{3} = 3\sin\left(2\pi + \frac{\pi}{3}\right) = 3\sin\frac{\pi}{3} = 3\cdot\frac{\sqrt{3}}{2} = \frac{3}{2}\sqrt{3}$. So $(x, y) = \left(\frac{3}{2}, \frac{3}{2}\sqrt{3}\right)$.

iv. $x = 5\cos 0 = 5$ and $y = 5\sin 0 = 0$. So $(x, y) = (5, 0)$.

v. $x = -4\cos\frac{7\pi}{2} = -4\cos(4\pi - \frac{\pi}{2}) = -4\cos(-\frac{\pi}{2}) = 0$ and $y = -4\sin\frac{7\pi}{2} = -4\sin(4\pi - \frac{\pi}{2}) = -4\sin(-\frac{\pi}{2}) = 4\sin\frac{\pi}{2} = 4$. So $(x, y) = (0, 4)$.

vi. $x = 5\cos(-\frac{9\pi}{2}) = 5\cos(-4\pi - \frac{\pi}{2}) = 5\cos(-\frac{\pi}{2}) = 0$ and $y = 5\sin(-\frac{9\pi}{2}) = 5\sin(-4\pi - \frac{\pi}{2}) = 5\sin(-\frac{\pi}{2}) = -5$. So $(x, y) = (0, -5)$.

vii. The fact that $r = 0$ means that the point is the origin. So $(x, y) = (0, 0)$.

viii. $x = 3\cos 8 \approx 3(-0.145500) \approx -0.436500$ and $y = 3\sin 8 \approx 3(0.989358) \approx 2.968075$. So $(x, y) \approx (-0.436500, 2.968075)$.

ix. $x = -\cos 23 \approx -(-0.532833) = 0.532833$ and $y = -\sin 23 \approx -(-0.846220) = 0.846220$. Hence $(x, y) \approx (0.532833, 0.846220)$.

x. $x = 3\cos(-32) \approx 3(0.834223) \approx 2.502670$ and $y = 3\sin(-32) \approx 3(-0.551427) \approx -1.654280$. Therefore $(x, y) \approx (2.502670, -1.654280)$.

11.17. **i.** The point P with Cartesian point $(0, 5)$ lies on the y-axis. Going polar we see that P lies on the ray $\theta = \frac{\pi}{2}$. It follows that $(5, \frac{\pi}{2})$ are polar coordinates of P.

ii. The point P with Cartesian coordinates $(-4, 0)$ is on the negative part of the x-axis. Since it lies on the ray $\theta = -\pi$, $(4, -\pi)$ are polar coordinates of P. Alternatively, we can reach the point P by considering the ray $\theta = 0$ and going 4 units in the opposite direction. So $(-4, 0)$ are also polar coordinates of P.

iii. Let P be the point with Cartesian coordinates $(3, -3)$. Note that the ray $\theta = -\frac{\pi}{4}$ goes through P. The distance from the origin to P is $\sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$ so that $(3\sqrt{2}, -\frac{\pi}{4})$ are polar coordinates of P.

For the remaining problems we'll use the fact that for any point P with Cartesian coordinates (x, y) with $x \neq 0$ (not on the y-axis) a set of polar coordinates (r, θ) for P with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is given by $\theta = \tan^{-1}\frac{y}{x}$ and either $r = \sqrt{x^2 + y^2}$ or $r = -\sqrt{x^2 + y^2}$. We will compute with an accuracy of two decimal places.

iv. Let P be the point with Cartesian coordinates $(4, -5)$. A calculator tells us that $\theta = \tan^{-1}(\frac{-5}{4}) = \tan^{-1}(-1.25) \approx -51.34° \approx -0.90$ radians. Since the ray that θ determines runs through the fourth quadrant and P is also in the fourth quadrant, the corresponding r is $\sqrt{4^2 + (-5)^2} = \sqrt{41} \approx 6.40$. So $(\theta, r) = (\sqrt{41}, \tan^{-1}(-1.25)) \approx (6.40, -0.90)$. In the other direction, it is the case that $x = \sqrt{41}\cos(\tan^{-1}(-1.25)) = 4$ and $y = \sqrt{41}\sin(\tan^{-1}(-1.25)) = -5$, respectively. (Given that is subject to roundoff errors, a calculator shows that $\sqrt{41}\cos(\tan^{-1}(-1.25)) \approx 4.00...$ with eleven 0s followed by a 3 and $\sqrt{41}\sin(\tan^{-1}(-1.25)) \approx -4.99...$ with twelve 9s followed by an 8.)

v. For the point P with Cartesian coordinates $(-3, 7)$, $\tan^{-1}(-\frac{7}{3}) \approx -66.80° \approx -1.17$ radians. Since the ray determined by $\theta = \tan^{-1}(-\frac{13}{7})$ runs through the fourth quadrant and P is in the second quadrant, r is given by $r = -\sqrt{(-3)^2 + 7^2} = -\sqrt{58} \approx -7.62$. The corresponding polar coordinates are $(r, \theta) = (-\sqrt{58}, \tan^{-1}(-\frac{7}{3})) \approx (-7.62, -1.17)$.

vi. For P with Cartesian coordinates $(7, -13)$, $\tan^{-1}(-\frac{13}{7}) \approx -61.70° \approx -1.08$ radians. Both the ray and P lie in the fourth quadrant so that $r = \sqrt{7^2 + (-13)^2} = \sqrt{218} \approx 14.76$. So the corresponding polar coordinates are $(r, \theta) = (\sqrt{218}, \tan^{-1}(-\frac{13}{7}) \approx (14.76, -1.08)$.

vii. For P with Cartesian coordinates $(-5, 9)$, we find that $\tan^{-1}(-\frac{9}{5}) \approx -60.95° \approx -1.06$ radians. Since the ray that this angle determines lies in the fourth quadrant and P in the second, r is given by $r = -\sqrt{(-5)^2 + 9^2} = -\sqrt{106} \approx -10.30$. The corresponding polar coordinates are $(r, \theta) = (-\sqrt{106}, \tan^{-1}(-\frac{9}{5})) \approx (-10.30, -1.06)$.

viii. For P with Cartesian coordinates $(-6, -11)$, $\tan^{-1}(\frac{-11}{-6}) = \tan^{-1}(\frac{11}{6}) \approx 61.39° \approx 1.07$ radians. Since the ray that this angle determines lies in the first quadrant and P in the third, r is given by $r = -\sqrt{(-6)^2 + (-11)^2} = -\sqrt{157} \approx -12.53$. So the corresponding polar coordinates are $(r, \theta) = (-\sqrt{157}, \tan^{-1}(-\frac{11}{6})) \approx (-12.53, 1.07)$.

ix. With P the point that has Cartesian coordinates $(8, 23)$, $\tan^{-1}(\frac{23}{8}) \approx 70.82° \approx 1.24$ radians. Since the ray that this angle determines and the point P both lie in the first quadrant, $r = \sqrt{8^2 + 23^2} = \sqrt{593} \approx 24.35$. So the corresponding polar coordinates are $(r, \theta) = (\sqrt{593}, \tan^{-1}(\frac{23}{8})) \approx (24.35, 1.24)$.

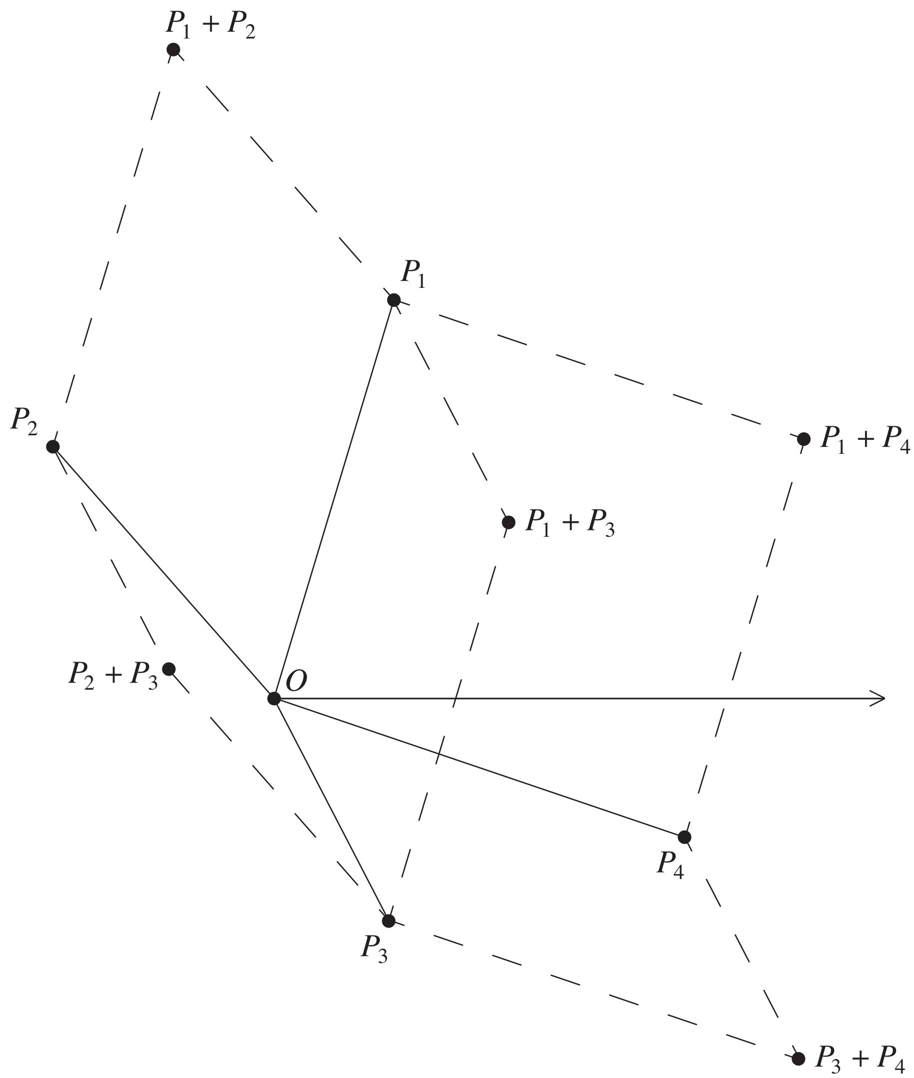
x. For P with Cartesian coordinates $(9, -36)$, $\tan^{-1}(-\frac{36}{9}) \approx -75.96° \approx -1.33$ radians. The ray $\theta = \tan^{-1}(-\frac{36}{9})$ and P both lie in the fourth quadrant so that the corresponding r is $r = \sqrt{9^2 + (-36)^2} = \sqrt{1377} \approx 37.11$. The corresponding polar coordinates of P are $(r, \theta) = (\sqrt{1377}, \tan^{-1}(-\frac{36}{9}) \approx (37.11, -1.33)$.

11.18. Note first that all sets of polar coordinates of the origin O have the form $(0, \theta)$ where any θ can arise. Let P be any point other than O and let (r, θ) with $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$ be a set of polar coordinates of P. Then

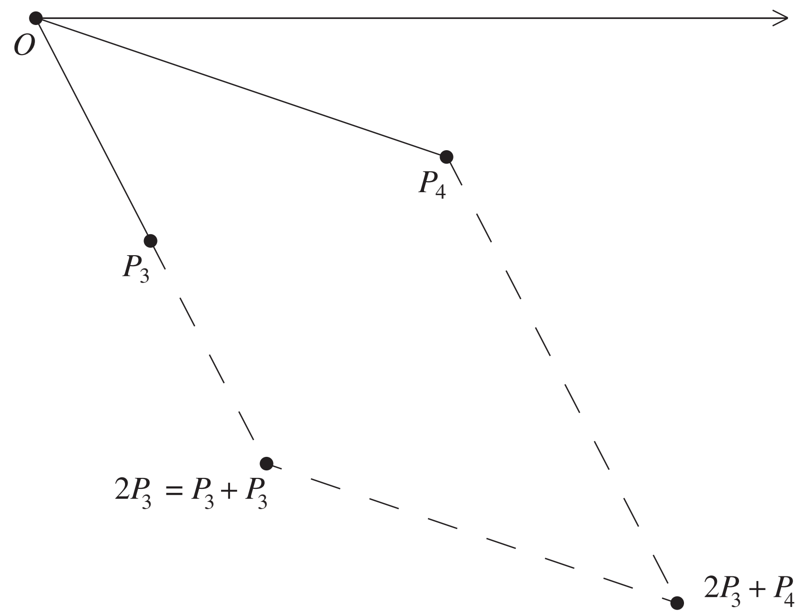
$$(r, \theta + 2\pi), (r, \theta - 2\pi), (r, \theta + 4\pi), (r, \theta - 4\pi),$$

and more generally $(r, \theta + 2k\pi)$, where k can be any integer (positive or negative), are polar coordinates of P. Observe that any set of polar coordinates of P with first coordinate r has the form $(r, \theta + 2k\pi)$. Note that $(-r, \theta + \pi)$ also represents P and hence that any set polar coordinates of P with first coordinate $-r$ has the form $(-r, (\theta+\pi)+2k\pi) = (-r, \theta+(2k+1)\pi)$, where, as before, k can be any positive or negative integer. If a single set of polar coordinates can be determined for P, then all others are given by the "recipe" above.

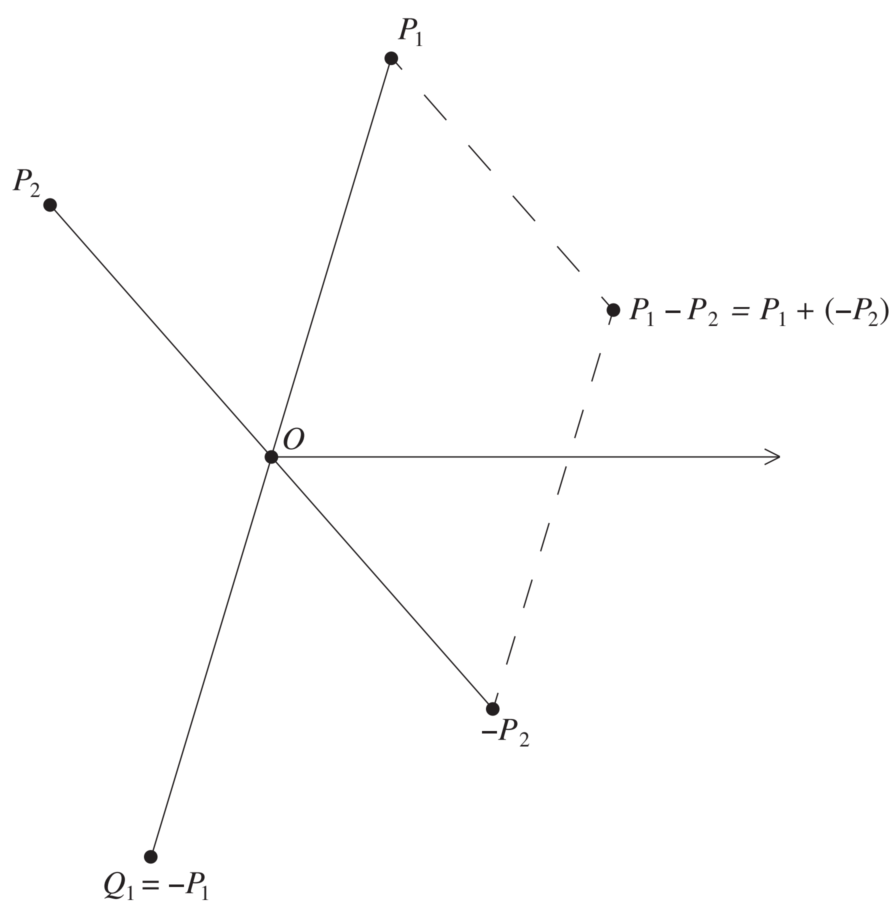
11.19. The addition of each pair of points is carried out in the diagram below by completing a parallelogram with the insertion of a pair of dashed lines. For $2P_3 = P_3 + P_3$ and $2P_3 + P_4$



this is done with the figure below. Notice that the parallelogram for $P_3 + P_3 = 2P_3$ is completely flat.



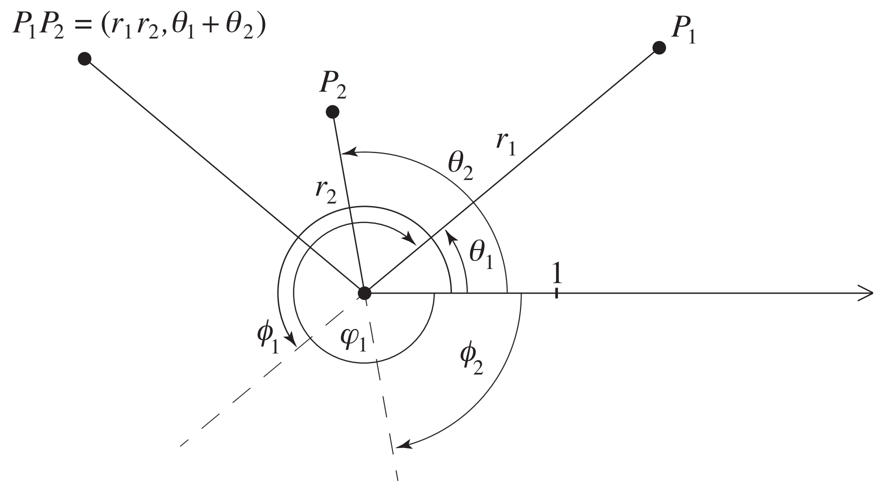
11.20. The points $-P_1, -P_2$, and $P_1 - P_2 = P_1 + (-P_2)$ are located in the diagram below.



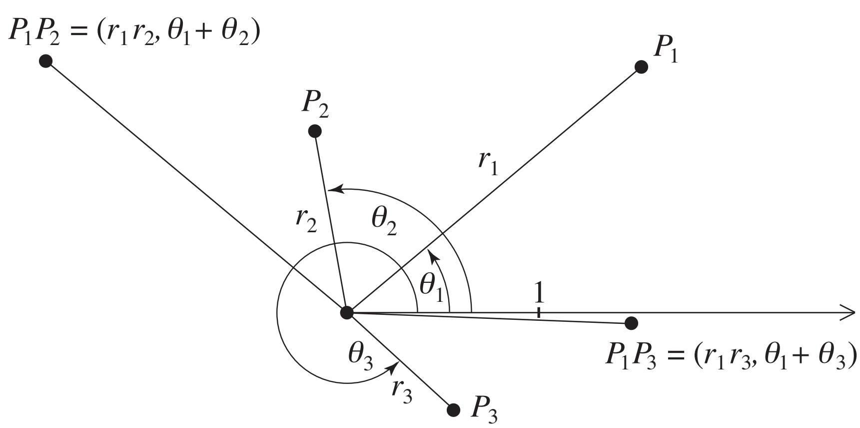
11.21. Let's concentrate on the product P_1P_2 first. We know that any point in the polar plane can be expressed in terms of polar coordinate in many (in fact infinitely many) ways. In the figure below $(r_1, \theta_1), (-r_1, \phi_1)$, and (r_1, φ_1) are three pairs of coordinates for P_1 and (r_2, θ_2) and $(-r_2, \phi_2)$ are two pairs of coordinates for P_2. For the definition of the product P_1P_2 to make sense it must give the same result no matter how the coordinates for P_1 and P_2 are chosen. So

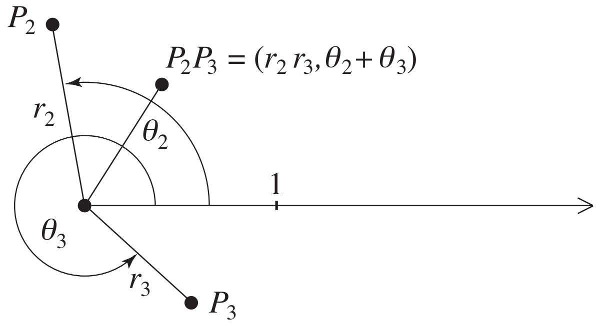
$$(r_1r_2, \theta_1 + \theta_2) = (r_1(-r_2), \theta_1 + \phi_2) = ((-r_1)r_2, \phi_1 + \theta_2)$$
$$= ((-r_1)(-r_2), \phi_1 + \phi_2) = (r_1r_2, \varphi_1 + \theta_2) = (r_1(-r_2), \varphi_1 + \phi_2)$$

should all represent the same point. Notice that $\phi_2 = \theta_2 - \pi$, so that $(r_1(-r_2), \theta_1 + \phi_2) = (-r_1r_2, (\theta_1 + \theta_2) - \pi)$. Since the ray given by $(\theta_1 + \theta_2) - \pi$ points in the direction opposite to that of $\theta_1 + \theta_2$, it follows that $((-r_1)r_2, \theta_1 + \phi_2) = (r_1r_2, \theta_1 + \theta_2)$. Also $\phi_1 = \theta_1 + \pi$, so that $((-r_1)r_2, \phi_1 + \theta_2) = (-r_1r_2, (\theta_1 + \theta_2) + \pi)$. As before this is equal to $(r_1r_2, \theta_1 + \theta_2)$. The other equalities are verified similarly. For the last one, $\varphi_1 = \theta_1 - 2\pi$ and hence $(r_1(-r_2), \varphi_1 + \phi_2) = (-r_1r_2, (\theta_1 - 2\pi) + (\theta_2 - \pi)) = (-r_1r_2, (\theta_1 + \theta_2 - 3\pi)) = (-r_1r_2, (\theta_1 + \theta_2 - \pi)) = (r_1r_2, \theta_1 + \theta_2)$.

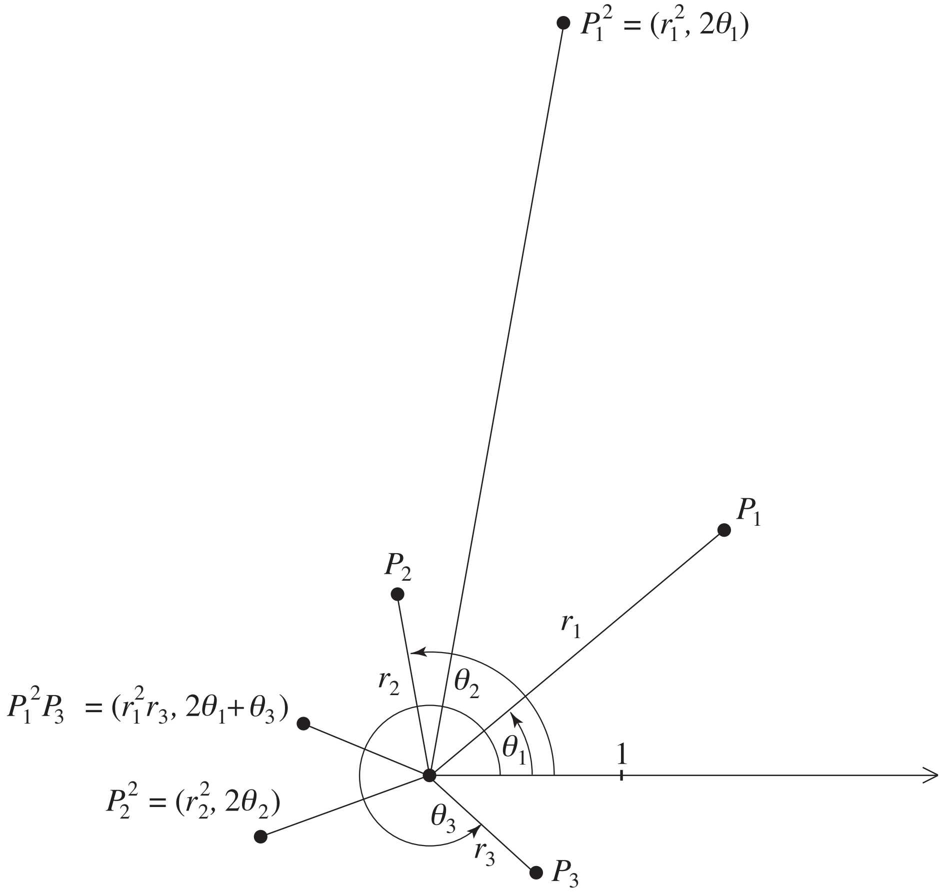


The points P_1P_3 and P_2P_3, are located in the two diagrams that follow. The fact that the point 1 (and hence the length 1) is specified tells us that $r_1 \approx 2, r_2 \approx 1$, and (in the diagrams below) that $r_3 \approx 0.75$. The angles θ_1, θ_2, and θ_3 are approximately equal to $40°, 100°$ and $320°$, respectively.

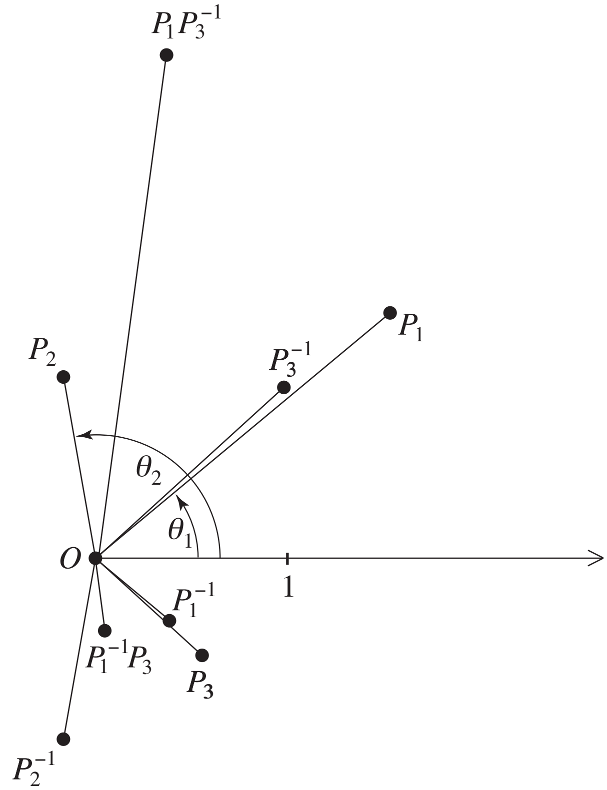




11.22. Check the placement of the points P_1^2, P_2^2, and $P_1^2 P_3$ in the diagram below.

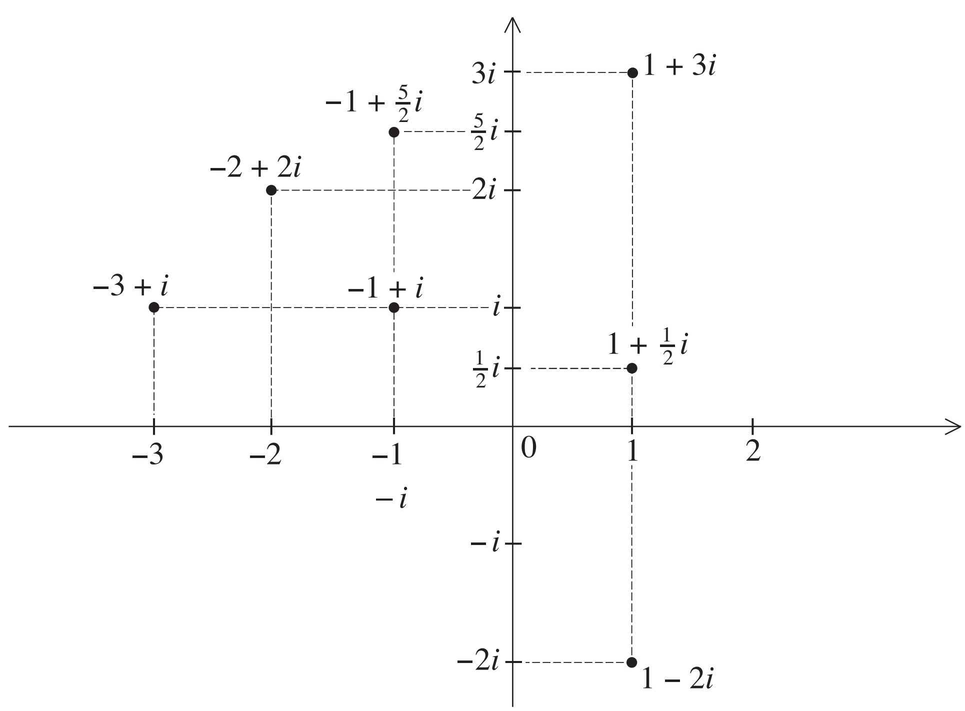


11.23. Consider the point in the complex plane with polar (or Cartesian) coordinates $(1, 0)$. For any point $P = (r, \theta)$, the product $P(1, 0)$ is equal to $(r, \theta)(1, 0) = (r \cdot 1, \theta + 0) = (r, \theta) = P$. So $1 = (1, 0)$ is the multiplicative identity in the complex plane. Let (s_1, ϕ_1) be the coordinates of Q_1. For $P_1 = (r_1, \theta_1)$, the requirement $P_1Q_1 = (r_1, \theta_1)(s_1, \phi_1) = (r_1 s_1, \theta_1 + \phi_1) = (1, 0)$ is met by taking $s_1 = r_1^{-1} = \frac{1}{r_1}$ and $\phi_1 = -\theta_1$. The resulting $Q_1 = (r_1^{-1}, -\theta_1)$ is the multiplicative inverse P^{-1} of P_1. For any $P_1 \neq O$ (the origin) $r_1 \neq 0$, so that P_1 has such an inverse. The inverses $P_1^{-1}, P_2^{-1}, P_3^{-1}$ and the products $P_1P_3^{-1}$ and $P_1^{-1}P_3$ are placed in the diagram below. Since $r_1 \approx 2$ and $\theta_1 \approx 40° \approx 0.70$ radians, the point $P_1^{-1} \approx (\frac{1}{2}, -0.70)$. Since $r_2 \approx 1$ and $\theta_2 \approx 100° \approx 1.75$ radians, the point $P_2^{-1} \approx (1, -1.75)$. Find similar approximations for P_3^{-1} and $P_1P_3^{-1}$.



11.24. If $b > 0$, then $(b, \frac{\pi}{2})$ is the point on the ray $\theta = \frac{\pi}{2}$ that lies b units above the origin. If $b = 0$, $(b, \frac{\pi}{2}) = (0, \frac{\pi}{2}$ is the origin. If $b < 0$ then $(b, \frac{\pi}{2})$ is on the ray $\theta = -\frac{\pi}{2}$ a distance $-b$ below the origin. So in all cases $(b, \frac{\pi}{2})$ is on the imaginary axis and b is its imaginary coordinate.

11.25. These points (and others) are placed in the figure below.



11.26. The sum and product of $c_1 = -1 + i$ and $c_2 = 1 - 2i$ are $c_1 + c_2 = (-1+1) + (i - 2i) = -i$ and $c_1c_2 = -1 \cdot 1 + (-1)(-2i) + i \cdot 1 + i(-2i) = -1 + 2i + i - 2i^2 = 1 + 3i$. For $c_1 = -2 + 2i$ and $c_2 = 1 + \frac{1}{2}i$, we get the sum $c_1 + c_2 = (-2+1) + (2i + \frac{1}{2}i) = -1 + \frac{5}{2}i$ and the product $c_1c_2 = -2 \cdot 1 + (-2)\frac{1}{2}i + 2i \cdot 1 + 2i(\frac{1}{2}i) = -2 + i^2 + i = -3 + i$. In each case, the points $c_1 + c_2$ and c_1c_2 are located in the figure above.

11.27. For $c_1 = 1 + i$ and $c_2 = a_2 + b_2i$ we get $c_1c_2 = a_2 + b_2i + a_2i + b_2i^2 = (a_2 - b_2) + (a_2 + b_2)i$. Since $1 = (1, 0)$, we need to have $a_2 - b_2 = 1$ and $a_2 + b_2 = 0$. It follows that $b_2 = -a_2$ and $2a_2 = 1$. Therefore $a_2 = \frac{1}{2}, b_2 = -\frac{1}{2}$ and hence $c_2 = c_1^{-1} = \frac{1}{2} - \frac{1}{2}i$.

For $c_1 = 2 - i$ we get $c_1c_2 = 2a_2 + 2b_2i - i(a_2 + b_2i) = (2a_2 + b_2) + (-a_2 + 2b_2)i$. So $2a_2 + b_2 = 1$ and $-a_2 + 2b_2 = 0$. Therefore $a_2 = 2b_2$, hence $5b_2 = 1, b_2 = \frac{1}{5}$, and $a_2 = \frac{2}{5}$. Therefore $c_2 = c_1^{-1} = \frac{2}{5} + \frac{1}{5}i$.

11.28. For $c = 3$, we have $a = 3$ and $b = 0$, so that $\bar{c} = 3$ and $N(c) = c\bar{c} = 3 \cdot 3 = 9$. For $c = -i$, we get $a = 0, b = -1, \bar{c} = -(-1)i = i$ and $N(c) = (-i)i = 1$. With $c = 3 + i$, we see that $a = 3$ and $b = 1$, so that $\bar{c} = 3 - i$ and $N(c) = c\bar{c} = 3^2 - i^2 = 10$. For $c = 5 - i$ finally, $a = 5$ and $b = -1$, so that $\bar{c} = 5 + i$ and $N(c) = 5^2 - i^2 = 26$.

11.29. Since $c = a + bi \neq 0$, one of a or b is not zero. So $c\bar{c} = N(c) = a^2 + b^2$ is a nonzero real number. Therefore $(c\bar{c})\frac{1}{N(c)} = 1$, and hence $c^{-1} = \frac{1}{N(c)}\bar{c}$. In terms of a and b, $c^{-1} = \frac{1}{N(c)}\bar{c} = \frac{1}{a^2+b^2}(a - bi) = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. The computation

$$\begin{aligned}(a+bi)\big(\tfrac{a}{a^2+b^2} + \tfrac{-b}{a^2+b^2}i\big) &= a\big(\tfrac{a}{a^2+b^2} + \tfrac{-b}{a^2+b^2}i\big) + bi\big(\tfrac{a}{a^2+b^2} + \tfrac{-b}{a^2+b^2}i\big) \\ &= \big(\tfrac{a^2}{a^2+b^2} + \tfrac{b^2}{a^2+b^2}\big) + \big(\tfrac{-ab}{a^2+b^2} + \tfrac{ab}{a^2+b^2}\big)i = (1, 0)\end{aligned}$$

confirms this conclusion.

11.30. The solution of $x + 7 = 0$ requires the negative integer -7. The solutions of $2x - 3 = 0$ and $3x + 5 = 0$ require the rational numbers $\frac{2}{3}$ and $-\frac{3}{5}$. The solution of $x^2 - 3$ requires the real (but irrational) number $\sqrt{3}$. The solutions of the equations $6x^2 + 5 = 0, x^2 + 1 = 0$ and $x^2 + 5 = 0$ require the complex (and imaginary) numbers $\sqrt{\frac{5}{6}}i$, i, and $\sqrt{5}i$. The enlargements that are required are: from the positive integers, to all integers, to the rational numbers, to the real numbers, and finally to the complex numbers.

11.31. This product is most easily computed in two basic ways. One way is

$$\begin{aligned}&(1 + 2i + 3j + 4k)(-3 - 2i - 3k) \\ &\quad = 1(-3 - 2i - 3k) + 2i(-3 - 2i - 3k) + 3j(-3 - 2i - 3k) + 4k(-3 - 2i - 3k) \\ &\quad = (-3 - 2i - 3k) + (-6i + 4 + 6j) + (-9j + 6k - 9i) + (-12k - 8j + 12) \\ &\quad = (-3 + 4 + 12) + (-2i - 6i - 9i) + (6j - 9j - 8j) + (-3k + 6k - 12k) \\ &\quad = 13 - 17i - 11j - 9k.\end{aligned}$$

Another way is

$$\begin{aligned}&(1 + 2i + 3j + 4k)(-3 - 2i - 3k) \\ &\quad = (1 + 2i + 3j + 4k)(-3) + (1 + 2i + 3j + 4k)(-2i) + (1 + 2i + 3j + 4k)(-3k) \\ &\quad = (-3 - 6i - 9j - 12k) + (-2i + 4 + 6k - 8j) + (-3k + 6j - 9i + 12) \\ &\quad = 13 - 17i - 11j - 9k.\end{aligned}$$

11.32. Using the first approach of the problem above we get

$$(a+bi+cj+dk)(a'+b'i+c'j+d'k)$$
$$= a(a'+b'i+c'j+d'k)+bi(a'+b'i+c'j+d'k)+cj(a'+b'i+c'j+d'k)+dk(a'+b'i+c'j+d'k)$$
$$= (aa'+ab'i+ac'j+ad'k)+(ba'i-bb'+bc'k-bd'j)+(ca'j-cb'k-cc'+cd'i)+(da'k+db'j-dc'i-dd')$$
$$= (aa'-bb'-cc'-dd')+(ab'+ba'+cd'-dc')i+(ac'-bd'+ca'+db')j+(ad'+bc'-cb'+da')k.$$

So a'', b'', c'', and d'' are given by $a'' = aa'-bb'-cc'-dd', b'' = ab'+ba'+cd'-dc', c'' = ac'-bd'+ca'+db'$, and $d'' = ad'+bc'-cb'+da'$.

11.33. (Note first, that "$= q$" on the first line should be deleted.) By the formula of the previous problem

$$q\bar{q} = (a^2+b^2+c^2+d^2)+(-ab+ba-cd+dc)i+(-ac+bd+ca-db)j+(-ad-bc+cb+da)k$$
$$= (a^2+b^2+c^2+d^2)+0i+0j+0d = a^2+b^2+c^2+d^2.$$

If $q \neq 0$, then at least one of the coefficients a, b, c, and d is not zero, so that $N(q) = q\bar{q} = a^2+b^2+c^2+d^2$ is a nonzero real number. It follows that $(q\bar{q})\frac{1}{N(q)} = 1$ and that $\frac{1}{N(q)}\bar{q}$ is a multiplicative inverse for q. In terms of a, b, c, and d,

$$q^{-1} = \tfrac{1}{a^2+b^2+c^2+d^2}(a-bi-cj-dk)$$
$$= \tfrac{a}{a^2+b^2+c^2+d^2} - \tfrac{b}{a^2+b^2+c^2+d^2}i - \tfrac{c}{a^2+b^2+c^2+d^2}j - \tfrac{d}{a^2+b^2+c^2+d^2}k.$$

11.34. Since $y' = 2\cos 2x + 2\sin 2x$ and $y'' = -4\sin 2x + 4\cos 2x$, we see that

$$y'' + 4y = -4\sin 2x + 4\cos 2x + 4(\sin 2x - \cos 2x) = 0.$$

Since there are no restrictions on x we can take any x in $(-\infty, \infty)$.

11.35. By the product rule $y' = e^{-2x}+x(e^{-2x}(-2)) = (1-2x)e^{-2x}$ and $y'' = -2e^{-2x}+(1-2x)(-2)e^{-2x} = -2(2-2x)e^{-2x}$. So for any x in $(-\infty, \infty)$

$$y'' + 4y' + 4y = -2(2-2x)e^{-2x} + 4(1-2x)e^{-2x} + 4xe^{-2x}$$
$$= -4e^{-2x} + 4xe^{-2x} + 4e^{-2x} - 8xe^{-2x} + 4xe^{-2x} = 0.$$

11.36. Since $A(a+bi)^2 + B(a+bi) + C = 0$, we get

$$A(a^2+2abi-b^2) + B(a+bi) + C = \big(A(a^2-b^2)+Ba+C\big) + (2Aab+Bb)i = 0,$$

so that $A(a^2-b^2)+Ba+C = 0$ and $2Aab+Bb = 0$.

By the product rule

$$y' = ae^{ax}\sin bx + e^{ax}(\cos bx)b = ae^{ax}\sin bx + be^{ax}\cos bx \text{ and}$$
$$y'' = a^2e^{ax}\sin bx + ae^{ax}(\cos bx)b + \big(abe^{ax}\cos bx + be^{ax}(-\sin bx)b\big)$$
$$= (a^2e^{ax} - b^2e^{ax})\sin bx + (abe^{ax} + abe^{ax})\cos bx$$
$$= (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx.$$

Therefore $Ay'' + By' + Cy$

$$= A[(a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx] + B[ae^{ax}\sin bx + be^{ax}\cos bx] + Ce^{ax}\sin bx$$
$$= A[(a^2-b^2)+Ba+C]e^{ax}\sin bx + [2Aab+Bb]e^{ax}\cos bx = 0.$$

11.37. **i.** The characteristic polynomial x^2-6x+8 factors as $(x-2)(x-4)$. By Case 1, the general solution of $y''-6y'+8y=0$ is $y=D_1e^{2x}+D_2e^{4x}$.

ii. The characteristic polynomial x^2+2x+1 is equal to $(x+1)^2$. So Case 2 applies to tell us that the general solution of $y''+2y'+y=0$ is $y=D_1e^{-x}+D_2xe^{-x}$.

iii. The characteristic polynomial x^2-4x+3 is equal to $(x-1)(x-3)$. So by Case 1 the general solution of $y''-4y'+3y=0$ is $y=D_1e^{x}+D_2e^{3x}$.

iv. The characteristic polynomial $9x^2+1$ has the two complex roots $\frac{1}{3}i$ and $-\frac{1}{3}i$. So Case 3 applies with $a=0$ and $b=\frac{1}{3}$. It follows that the general solution of $9y''+y=0$ is $y=D_1\cos\frac{1}{3}x+D_2\sin\frac{1}{3}x$.

v. The characteristic polynomial x^2+1 has the roots i and $-i$. So Case 3 applies with $a=0$ and $b=1$. Therefore the general solution of $y''+y=0$ is $y=D_1\cos x+D_2\sin x$.

vi. The characteristic polynomial $Ax^2+Bx+C=2x^2+4x+7$ has $B^2-4AC=4^2-(4)(2)(7)=-40<0$. So it has the complex roots $a\pm bi$ with $a=\frac{-B}{2A}=\frac{-4}{4}=-1$ and $b=\frac{\sqrt{4AC-B^2}}{2A}=\frac{\sqrt{4(2)(7)-4^2}}{4}=\frac{\sqrt{40}}{4}=\frac{\sqrt{10}}{2}$. Case 3 applies to tell us that the general solution of $2y''+4y'+7=0$ is $y=e^{-x}(D_1\cos\sqrt{10}\,x+D_2\sin\sqrt{10}\,x)$.

11.38. If $A=0$ in the equation $Ay''+By'=0$, then either $B=0$ or $y'=0$. If B is zero, then any function $y=f(x)$ is a solution. If $y'=0$, the $y=f(x)$ is a constant function.

So we'll assume that $A\neq 0$. Let's first apply the discussion of Section 11.6 to $Ay''+By'=0$. The characteristic polynomial Ax^2+Bx has the roots $x=0$ and $x=\frac{-B}{A}$. If $B=0$, then 0 is a double root and by Case 2, the general solution of $Ay''+By'=0$ is D_1+D_2x. If $B\neq 0$, then Case 1 applies and the general solution is $D_1+D_2e^{\frac{-B}{A}x}$.

Let's solve $Ay''+By'=0$ again in a different way. Letting x be the variable and $z=y'$ this equation becomes $A\frac{dz}{dx}=-Bz$. Since $A\neq 0$, this separable equation can be written as $\frac{dz}{z}=\frac{-B}{A}\,dx$. It follows that $\ln z=\frac{-B}{A}x+c$ with c a constant, and hence that $z=e^{\ln z}=e^{\frac{-B}{A}x+c}=e^c(e^{\frac{-B}{A}x})$. Therefore $\frac{dy}{dx}=e^c(e^{\frac{-B}{A}x})$. If $B=0$, then $\frac{dy}{dx}=e^c=D_2$ is a constant and $y=D_1+D_2x$ for a constant D_1. If $B\neq 0$, then $y=-e^c(\frac{A}{B})e^{\frac{-B}{A}x}+D_1$, again D_1 a constant. Letting $D_2=-e^c(\frac{A}{B})$, $y=D_1+D_2e^{\frac{-B}{A}x}$. In either case we get what we had before.

11.39. Let $y=\sin 2x$ and note that $y'=2\cos 2x$ and $y''=-4\sin 2x$. So $y''+4y=0$. A similar computation shows that $y=\cos 2x$ is also a solution of this equation. Let $y=f(x)$ be any solution of $y''+4y=0$ and notice that $f''(x)=-4f(x)$. Now set

$$D_1(x)=(\sin 2x)f(x)+(\tfrac{1}{2}\cos 2x)f'(x) \quad\text{and}\quad D_2(x)=(\cos 2x)f(x)-(\tfrac{1}{2}\sin 2x)f'(x).$$

It follows that

$$\begin{aligned}&D_1(x)\sin 2x+D_2(x)\cos 2x\\&\qquad=(\sin^2 2x)f(x)+(\tfrac{1}{2}\cos 2x)(\sin 2x)f'(x)+(\cos^2 2x)f(x)-(\tfrac{1}{2}\sin 2x)(\cos 2x)f'(x)\\&\qquad=f(x)(\sin^2 2x+\cos^2 2x)=f(x).\end{aligned}$$

We have shown that $f(x)=D_1(x)\sin 2x+D_2(x)\cos 2x$. If we can verify that $D_1'(x)=0$ and $D_2'(x)=0$, then both $D_1(x)$ and $D_2(x)$ are constants and the solution of the problem is complete. By the product rule

$$\begin{aligned} D_1'(x) &= (\cos 2x)2f(x) + (\sin 2x)f'(x) - (\sin 2x)f'(x) + (\tfrac{1}{2}\cos 2x)f''(x) \\ &= (\cos 2x)2f(x) + (\tfrac{1}{2}\cos 2x)(-4f(x)) = 0 \text{ and} \end{aligned}$$

$$\begin{aligned} D_2'(x) &= (-\sin 2x)2f(x) + (\cos 2x)f'(x) - (\cos 2x)f'(x) - (\tfrac{1}{2}\sin 2x)f''(x) \\ &= (-\sin 2x)2f(x) - (\tfrac{1}{2}\sin 2x)(-4f(x)) = 0. \end{aligned}$$

11.40. For $y = \cos 3x$, we get $y' = -3\sin 3x$ and $y'' = -9\cos 3x$. So $y'' + 9y = 0$. In the same way, $y = \sin 3x$ is also a solution. Let $y = f(x)$ be any solution of $y'' + 9y = 0$ and notice that $f''(x) = -9f(x)$. Set

$$D_1(x) = (\sin 3x)f(x) + (\tfrac{1}{3}\cos 3x)f'(x) \quad\text{and}\quad D_2(x) = (\cos 3x)f(x) - (\tfrac{1}{3}\sin 3x)f'(x).$$

Check that

$$\begin{aligned} &D_1(x)\sin 3x + D_2(x)\cos 3x \\ &\quad = (\sin^2 3x)f(x) + (\tfrac{1}{3}\cos 3x)(\sin 3x)f'(x) + (\cos^2 3x)f(x) - (\tfrac{1}{3}\sin 3x)(\cos 3x)f'(x) \\ &\quad = f(x)(\sin^2 3x + \cos^2 3x) = f(x). \end{aligned}$$

So $f(x) = D_1(x)\sin 3x + D_2(x)\cos 3x$. We need to show that $D_1'(x) = 0$ and $D_2'(x) = 0$, for then $D_1(x)$ and $D_2(x)$ are constants and the solution is complete. By the product rule

$$\begin{aligned} D_1'(x) &= (\cos 3x)3f(x) + (\sin 3x)f'(x) - (\sin 3x)f'(x) + (\tfrac{1}{3}\cos 3x)f''(x) \\ &= (\cos 3x)3f(x) + (\tfrac{1}{3}\cos 3x)(-9f(x)) = 0 \text{ and} \\ D_2'(x) &= (-\sin 3x)3f(x) + (\cos 3x)f'(x) - (\cos 3x)f'(x) - (\tfrac{1}{3}\sin 3x)f''(x) \\ &= (-\sin 3x)3f(x) - (\tfrac{1}{3}\sin 3x)(-9f(x)) = 0. \end{aligned}$$

11.41. Withr $y = \cos\sqrt{C}x$, we see that $y' = -\sqrt{C}\sin\sqrt{C}x$ and $y'' = -C\cos\sqrt{C}x$. So $y'' + Cy = 0$. Similarly, $y = \sin\sqrt{C}x$ is also a solution. Let $y = f(x)$ be any solution of $y'' + Cy = 0$ and observe that $f''(x) = -Cf(x)$. Set

$$\begin{aligned} D_1(x) &= (\sin\sqrt{C}x)f(x) + (\tfrac{1}{\sqrt{C}}\cos\sqrt{C}x)f'(x) \quad\text{and} \\ D_2(x) &= (\cos\sqrt{C}x)f(x) - (\tfrac{1}{\sqrt{C}}\sin\sqrt{C}x)f'(x). \end{aligned}$$

It follows that

$$\begin{aligned} &D_1(x)\sin\sqrt{C}x + D_2(x)\cos\sqrt{C}x \\ &\quad = (\sin^2\sqrt{C}x)f(x) + (\tfrac{1}{\sqrt{C}}\cos\sqrt{C}x)(\sin\sqrt{C}x)f'(x) \\ &\qquad\qquad + (\cos^2\sqrt{C}x)f(x) - (\tfrac{1}{\sqrt{C}}\sin\sqrt{C}x)(\cos\sqrt{C}x)f'(x) \\ &\quad = f(x)(\sin^2\sqrt{C}x + \cos^2\sqrt{C}x) = f(x). \end{aligned}$$

So $f(x) = D_1(x)\sin\sqrt{C}x + D_2(x)\cos\sqrt{C}x$. It remains to verify that $D_1'(x) = 0$ and $D_2'(x) = 0$ and hence that $D_1(x)$ and $D_2(x)$ are constants. By the product rule

$$\begin{aligned} D_1'(x) &= (\cos\sqrt{C}x)\sqrt{C}f(x) + (\sin\sqrt{C}x)f'(x) - (\sin\sqrt{C}x)f'(x) + (\tfrac{1}{\sqrt{C}}\cos\sqrt{C}x)f''(x) \\ &= (\cos\sqrt{C}x)\sqrt{C}f(x) + (\tfrac{1}{\sqrt{C}}\cos\sqrt{C}x)(-Cf(x)) = 0 \text{ and} \end{aligned}$$

$$\begin{aligned} D_2'(x) &= (-\sin\sqrt{C}x)\sqrt{C}f(x) + (\cos\sqrt{C}x)f'(x) - (\cos\sqrt{C}x)f'(x) - (\tfrac{1}{\sqrt{C}}\sin\sqrt{C}x)f''(x) \\ &= (-\sin\sqrt{C}x)\sqrt{C}f(x) - (\tfrac{1}{\sqrt{C}}\sin\sqrt{C}x)(-Cf(x)) = 0. \end{aligned}$$

11.42. Following the strategy of the example, we'll let $y = f(x) = E_1\sin(3t) + E_2\cos(3t)$ where E_1 and E_2 are constants. The assumption that $y = f(t)$ is a solution of $2y'' - 3y' + 5y = 4\sin(3t)$ lets us solve for E_1 and E_2. Computing y' and y'', we get

$$y' = 3E_1\cos(3t) - 3E_2\sin(3t) \quad\text{and}\quad y'' = -9E_1\sin(3t) - 9E_2\cos(3t).$$

It follows that

$$\begin{aligned} &2y'' - 3y' + 5y \\ &\quad= 2\big(-9E_1\sin(3t) - 9E_2\cos(3t)\big) - 3\big(3E_1\cos(3t) - 3E_2\sin(3t)\big) + 5\big(E_1\sin(3t) + E_2\cos(3t)\big) \\ &\quad= (-18E_1 + 9E_2 + 5E_1)\sin(3t) + (-18E_2 - 9E_1 + 5E_2)\cos(3t). \\ &\quad= (-13E_1 + 9E_2)\sin(3t) + (-9E_1 - 13E_2)\cos(3t). \end{aligned}$$

Since $2y'' - 3y' + 5y = 4\sin(3t)$, we get $-13E_1 + 9E_2 = 4$ and $-9E_1 - 13E_2 = 0$. Therefore $E_2 = -\frac{9}{13}E_1$ and $-13E_1 - \frac{81}{13}E_1 = 4$. Hence $\frac{169+81}{13}E_1 = -4$ so that $E_1 = \frac{-4\cdot 13}{250} = -\frac{26}{125}$ and $E_2 = (-\frac{9}{13})(-\frac{26}{125}) = \frac{18}{125}$. The specific solution of $2y'' - 3y' + 5y = 4\sin(3t)$ that we have found is

$$y = -\tfrac{26}{125}\sin(3t) + \tfrac{18}{125}\cos(3t).$$

To find the general solution of $2y'' - 3y' + 5y = 4\sin(3t)$ we still need to find—as the earlier example demonstrated—the general solution of $2y'' - 3y' + 5y = 0$ and add the specific solution $-\frac{26}{125}\sin(3t) + \frac{18}{125}\cos(3t)$ to it. This general solution can be obtained by applying the results of Section 11.6. Having already illustrated these results, we'll use the differential equations solver of the site

https://www.symbolab.com/solver/ordinary-differential-equation-calculator/

instead. Turn to this site and start by clicking on $\dfrac{d}{dx}$ twice to place $\dfrac{d}{dx}\left(\dfrac{d}{dx}(\ )\right)$ into the box provided. Then use the cursor and insert a 2 and a y to get $2\dfrac{d}{dx}\left(\dfrac{d}{dx}(y)\right)$. This is an expression for $2\dfrac{d^2}{dx^2}(y) = 2y''$. Continue in this way to write $2y'' - 3y' + 5y = 0$ as

$$2\frac{d}{dx}\left(\frac{d}{dx}(y)\right) - 3\frac{d}{dx}(y) + 5y = 0.$$

Then push on the white Go in the red field. Very quickly the calculator will come up with the answer $y = e^{\frac{3}{4}x}\left(c_1\cos\left(\frac{\sqrt{31}}{4}x\right) + c_2\sin\left(\frac{\sqrt{31}}{4}x\right)\right)$. (The default variable of the calculator is x but it is possible to replace x by t.)

With regard to the next several problems recall that the radius of convergence R of a power series $\displaystyle\sum_{k=1}^{\infty} a_k(x - x_0)^k$ is determined by the limit $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ that the ratio test provides. If this limit is L (finite or infinite), then $R = \frac{1}{L}$ (with the understanding that R is infinite, if $L = 0$).

11.43. **i.** Here $a_k = \frac{1}{k^2}$ so that $\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)^2} \cdot \frac{k^2}{1} = \left(\frac{k}{k+1}\right)^2$. After dividing the numerator and denominator by k we get $\left|\frac{a_{k+1}}{a_k}\right| = \left(\frac{1}{1+\frac{1}{k}}\right)^2$. Therefore $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left(\frac{1}{1+\frac{1}{k}}\right)^2 = 1$. It follows that $R = 1$.

ii. For this power series $a_k = \frac{1}{k}$ so that $\frac{a_{k+1}}{a_k} = \frac{1}{k+1} \cdot \frac{k}{1} = \frac{k}{k+1}$. Dividing the numerator and denominator by k we get $\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{1+\frac{1}{k}}$. Therefore $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\frac{1}{1+\frac{1}{k}} = 1$ and hence $R = 1$.

iii. Here $a_k = \frac{(-1)^k k}{3^k}$ so that $\frac{a_{k+1}}{a_k} = \frac{(-1)^{k+1}(k+1)}{3^{k+1}} \cdot \frac{3^k}{(-1)^k k} = \frac{(-1)(k+1)}{3k} = -\frac{1}{3}(1 + \frac{1}{k})$. It follows that $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{3}$ and hence that $R = 3$.

iv. Since $\frac{2^k}{k}(2x-4)^k = \frac{2^k}{k}(2(x-2))^k = \frac{2^k}{k}2^k(x-2)^k = \frac{(2^k)^2}{k}(x-2)^k = \frac{2^{2k}}{k}(x-2)^k = \frac{4^k}{k}(x-2)^k$, the series $\sum\limits_{k=0}^{\infty}\frac{2^k}{k}(2x-4)^k$ (which is not formally a power series) can be written as the power series $\sum\limits_{k=0}^{\infty}\frac{4^k}{k}(x-2)^k$. With $a_k = \frac{4^k}{k}$, we get $\frac{a_{k+1}}{a_k} = \frac{4^{k+1}}{k+1} \cdot \frac{k}{4^k} = 4\frac{k}{k+1} = 4\frac{1}{1+\frac{1}{k}}$. It follows that $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 4$ and hence that $R = \frac{1}{4}$.

v. The fact that $k!(2x+3)^k = k!(2(x+\frac{3}{2}))^k = 2^k k!(x+\frac{3}{2})^k$ means that the series can be written as the power series $\sum\limits_{k=0}^{\infty}2^k k!(x+\frac{3}{2})^k$. With $a_k = 2^k k!$ we get $\frac{a_{k+1}}{a_k} = \frac{2^{k+1}(k+1)!}{2^k k!} = 2(k+1)$. So $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = +\infty$ and hence $R = 0$.

vi. For this power series, $a_k = \frac{1}{k^k}$. So $\frac{a_{k+1}}{a_k} = \frac{k^k}{(k+1)^{k+1}} = \frac{k^k}{(k+1)(k+1)^k} = \frac{1}{k+1}\left(\frac{k}{k+1}\right)^k = \frac{\frac{1}{k+1}}{(\frac{k+1}{k})^k}$. Since $\lim\limits_{k\to\infty}\left(\frac{k+1}{k}\right)^k = e$ (see Section 7.10) and $\lim\limits_{k\to\infty}\frac{1}{k+1} = 0$, we see that $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 0$. Therefore $R = \infty$.

vii. Here $a_k = \frac{k^k}{k!}$ so that so that $\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \frac{(k+1)^k}{k!} \cdot \frac{k!}{k^k} = \frac{(k+1)^k}{k^k} = (\frac{k+1}{k})^k$. From the definition of e (in Section 7.10) $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}(\frac{k+1}{k})^k = e$. Therefore $R = \frac{1}{e}$.

11.44. For the power series $S = \sum\limits_{k=1}^{\infty}\frac{(-1)^k(k+1)}{5^k}(x-2)^k$ the kth coefficient is $a_k = \frac{(-1)^k(k+1)}{5^k}$. So $\frac{a_{k+1}}{a_k} = \frac{(-1)^{k+1}(k+2)}{5^{k+1}} \cdot \frac{5^k}{(-1)^k(k+1)} = \frac{-(k+2)}{5(k+1)}$. After dividing the numerator and denominator by k, we get $\frac{a_{k+1}}{a_k} = \frac{-(1+\frac{2}{k})}{5(1+\frac{1}{k})}$. So $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\frac{1+\frac{2}{k}}{5(1+\frac{1}{k})} = \frac{1}{5}$ and hence $R = 5$.

i. This is the power series $\sum\limits_{k=1}^{\infty}a_k(x-2)^k$ where $a_k = \frac{(-1)^k(k+1)}{5^k}$ for k odd and $a_k = 0$ for k even. Consider the power series

$$\sum_{j=0}^{\infty}\frac{(-1)^{2j+1}(2j+2)}{5^{2j+1}}(x-2)^{2j+1} = \frac{(-1)^1(2)}{5^1}(x-2)^1 + \frac{(-1)(4)}{5^3}(x-2)^3 + \frac{(-1)(6)}{5^5}(x-2)^5 + \cdots$$

and notice that it is the series just described. The ratio of consecutive terms is

$$\frac{\frac{(-1)^{2j+3}(2j+4)}{5^{2j+3}}(x-2)^{2j+3}}{\frac{(-1)^{2j+1}(2j+2)}{5^{2j+1}}(x-2)^{2j+1}} = \frac{(2j+4)}{5^{2j+3}}\frac{5^{2j+1}}{(2j+2)}(x-2)^2 = \frac{(j+2)}{(j+1)}\frac{1}{5^2}(x-2)^2 = \frac{(1+\frac{2}{j})}{(1+\frac{1}{j})}\frac{1}{5^2}(x-2)^2.$$

Since

$$\lim_{j\to\infty}\left|\frac{(1+\frac{2}{j})}{(1+\frac{1}{j})}\frac{1}{5^2}(x-2)^2\right| = \frac{1}{5^2}(x-2)^2$$

we know that this power series converges if $\frac{1}{5^2}(x-2)^2 < 1$ and diverges if $\frac{1}{5^2}(x-2)^2 > 1$. So it converges for $|x-2| < 5$ and diverges for $|x-2| > 5$. Therefore its radius of convergence is 5.

ii. This is the power series $\displaystyle\sum_{k=1}^{\infty} a_k(x-2)^k$ where $a_k = \frac{(-1)^k(k+1)}{5^k}$ for k equal to $1, 4, 7, \ldots$ and $a_k = 0$ for all other k. Similarly to the previous situation, the power series

$$\sum_{j=0}^{\infty}\frac{(-1)^{3j+1}(3j+2)}{5^{3j+1}}(x-2)^{3j+1} = \frac{(-1)^1(2)}{5^1}(x-2)^1 + \frac{(-1)^4(5)}{5^4}(x-2)^4 + \frac{(-1)^7(8)}{5^7}(x-2)^7 + \cdots$$

is exactly the series being considered. The ratio of consecutive coefficients is

$$\frac{\frac{(-1)^{3j+4}(3j+5)}{5^{3j+4}}(x-2)^{3j+4}}{\frac{(-1)^{3j+1}(3j+2)}{5^{3j+1}}(x-2)^{3j+1}} = \frac{-(3j+5)}{5^{3j+4}}\frac{5^{3j+1}}{(3j+2)}(x-2)^3 = \frac{-(3j+5)}{(3j+2)}\frac{1}{5^3}(x-2)^3 = \frac{-(3+\frac{5}{j})}{(3+\frac{2}{j})}\frac{1}{5^3}(x-2)^3.$$

The fact that

$$\lim_{j\to\infty}\left|\frac{(3+\frac{5}{j})}{(3+\frac{2}{j})}\frac{1}{5^3}(x-2)^3\right| = \frac{1}{5^3}|x-2|^3$$

tells us that the power series converges if $\frac{1}{5^3}|x-2|^3 < 1$ and diverges if $\frac{1}{5^3}|x-2|^3 > 1$. Therefore it converges for $|x-2| < 5$ and diverges for $|x-2| > 5$. So as before, the radius of convergence is 5.

11.45. The interval of convergence of a power series is centered around the center x_0 of the series. In the case of the series

$$\frac{x^{\frac{1}{2}}}{1+x} = x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}} + x^{\frac{9}{2}} - \ldots$$

and its interval of convergence $[0,1)$ this means that $x_0 = \frac{1}{2}$. But when the center of a power series is substituted into the series a sum of zeros $0+0+0+\cdots$ results. This is not what happens when $x = \frac{1}{2}$ is substituted into the series above. What has gone wrong is simple. The series above is not a power series.

11.46. The Taylor series of a function $y = f(x)$ centered at $x_0 = 0$ is given by

$$T_\infty(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{3!}x^4 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots.$$

Let $f(x) = \sin x$ and $x_0 = 0$. Since $f'(x) = \cos x, f^{(2)}(x) = -\sin x, f^{(3)}(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = \cos x, \ldots$ we get $f(0) = 0, f'(0) = 1, f^{(2)}(0) = 0, f^{(3)}(0) = -1, f^{(4)}(0) = 0, \ldots$ with the pattern $0, 1, 0, -1, 0, 1, \ldots$ repeating. So

$$T_\infty(x) = x + \frac{-1}{2!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + \cdots.$$

Given the pattern, we see that the general term is $\frac{(-1)^k}{(2k+1)!}x^{2k+1}$. Taylor's remainder theorem tells us for any n that the nth remainder $R_n(x) = f(x) - T_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$ for some z between x and 0. Since $f^{(n+1)}(z)$ is equal to $\pm\sin z$ or $\pm\cos z$, we know that

$|f^{(n+1)}(z)| \leq 1$ and hence that $|R_n(x)| \leq \frac{1}{(n+1)!}\,|x^{n+1}|$. Since $\lim\limits_{n\to\infty} \frac{|x^n|}{n!} = 0$ for any x (see the discussion preceding Example 11.23), it follows that $\lim\limits_{n\to\infty} R_n(x) = 0$ for any x. Therefore the Taylor series of $f(x) = \sin x$ centered at $x_0 = 0$ converges to $\sin x$ for all x in $(-\infty, \infty)$.

11.47. The kth coefficient of the binomial series is $a_k = \binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!}$. So $\frac{a_{k+1}}{a_k} = \frac{\frac{r(r-1)(r-2)\cdots(r-k)}{(k+1)!}}{\frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!}} = \frac{r(r-1)(r-2)\cdots(r-k)}{(k+1)!} \cdot \frac{k!}{r(r-1)(r-2)\cdots(r-(k-1))} = \frac{r-k}{k+1}$. It follows that

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{k-r}{k+1}\right| = \left|\frac{1-\frac{r}{k}}{1+\frac{1}{k}}\right| = 1.$$

11.48. For $r = -1$, the binomial coefficient $\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!}$ is equal to

$$\binom{-1}{k} = \frac{(-1)((-1)-1)((-1)-2)\cdots((-1)-(k-1))}{k!} = \frac{(-1)(-2)(-3)\cdots(-k)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k.$$

11.49. The function $g(x)$ is defined to be equal to $\sum\limits_{k=1}^{\infty}\binom{r}{k}x^k$ for all x in the interval of convergence of this series. The fact that $h(x) = (1-x)^{-r}g(x) = 1$ tells us that $g(x) = (1-x)^r$ all such x. So $\sum\limits_{k=1}^{\infty}\binom{r}{k}x^k = (1-x)^{-r}$ for all x in the interval of convergence.

11.50. Let $\sum\limits_{k=0}^{\infty} a_k x^k$ be any power series centered at 0 with a positive radius of convergence R. Define the function $y = f(x)$ by setting $y = f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ for any x satisfying $|x| < R$. (Note that any such x is in the interval of convergence of the series.) By Theorem 4, $f'(x) = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$. Now assume that $y = f(x)$ is a solution of the differential equation $y' = y$. Notice that $y = f(x)$ satisfies the initial condition $f(0) = a_0$. Since $f'(x) = f(x)$,

$$\sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = f'(x) - f(x) = 0$$

for all x with $|x| < R$. Let $i = k-1$ and rewrite the first of these series as $\sum\limits_{i=0}^{\infty}(i+1)a_{i+1}x^i$. After changing the notation for the index back to k, we get

$$\sum_{k=0}^{\infty}[(k+1)a_{k+1} - a_k]x^k = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - \sum_{k=0}^{\infty} a_k x^k = 0.$$

It follows by the corollary of Theorem 4 that $(k+1)a_{k+1} - a_k = 0$ for all $k \geq 0$. For $k = 0, 1, 2, 3, \ldots$, we get $a_1 = a_0, 2a_2 = a_1, 3a_3 = a_2, 4a_4 = a_3, 5a_5 = a_4, \ldots$, so that $a_1 = a_0, a_2 = \frac{1}{2}a_0, a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}a_0, a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}a_0, a_5 = \frac{1}{5}a_4 = \frac{1}{5\cdot 4\cdot 3\cdot 2}a_0, \ldots$. Assuming that $a_k = \frac{1}{k!}a_0$, we get $a_{k+1} = \frac{1}{k+1}a_k = \frac{1}{(k+1)k!}a_0 = \frac{1}{(k+1)!}a_0$ and therefore by the principle of induction (refer to Section 3.8) that $a_k = \frac{1}{k!}a_0$ for all $k \geq 0$.

Therefore the solution of $y' = y$ is given by $f(x) = \sum\limits_{k=0}^{\infty}(\frac{1}{k!}a_0)x^k = a_0\sum\limits_{k=0}^{\infty}\frac{1}{k!}x^k$. By Example 11.23, $f(x) = a_0 e^x$. (This problem was already solved at the end of Section 7.10.)

11.51. As in the solution of the previous problem, we'll et $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ be a power series centered at 0 with a positive radius of convergence R and define $y = f(x)$ by setting $y = f(x) = \displaystyle\sum_{k=0}^{\infty} a_k x^k$ for any x with $|x| < R$ (in particular, any such x is in the interval of convergence of the series).

Let's suppose that $y = f(x)$ satisfies the differential equation $y'' - 2xy' + y = 0$. By two applications of Theorem 4,

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{and} \quad f''(x) = \sum_{k=2}^{\infty} (k-1) k a_k x^{k-2}$$

both for all x with $|x| < R$. Note the initial conditions $f(0) = a_0$ and $f'(0) = a_1$. The fact that $y = f(x)$ satisfies $y'' - 2xy' + y = 0$ tells us that

$$\sum_{k=2}^{\infty} (k-1) k a_k x^{k-2} - 2x \sum_{k=1}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k = 0$$

and hence that $\displaystyle\sum_{k=2}^{\infty} (k-1) k a_k x^{k-2} - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$. In order to extract the implications of this, we'll rewrite the first power series to align it with the other two. With $i = k-2$ this series becomes $\displaystyle\sum_{i=0}^{\infty} (i+1)(i+2) a_{i+2} x^i$. Switching back to k we get

$$\sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

After pulling out the constant term (it corresponds to $k = 0$) and combining the rest, we get

$$(2a_2 + a_0) + \sum_{k=1}^{\infty} \big[(k+1)(k+2) a_{k+2} - (2k-1) a_k\big] x^k = 0.$$

By the corollary to Theorem 4, we get $2a_2 + a_0 = 0$ and $(k+1)(k+2)a_{k+2} - (2k-1)a_k = 0$ for all $k \geq 1$. So $a_2 = -\frac{1}{2} a_0$ and $a_{k+2} = \frac{2k-1}{(k+1)(k+2)} a_k$ for $k \geq 1$. Making use of this last formula again and again, we see that

$$a_3 = \tfrac{1}{2\cdot 3} a_1, a_4 = \tfrac{3}{3\cdot 4} a_2, a_5 = \tfrac{5}{4\cdot 5} a_3, a_6 = \tfrac{7}{5\cdot 6} a_4, a_7 = \tfrac{9}{6\cdot 7} a_5, \ldots.$$

We now get

$$a_2 = -\tfrac{1}{2} a_0, a_3 = \tfrac{1}{2\cdot 3} a_1, a_4 = -\tfrac{1\cdot 3}{2\cdot 3\cdot 4} a_0, a_5 = \tfrac{1\cdot 5}{2\cdot 3\cdot 4\cdot 5} a_1, a_6 = -\tfrac{1\cdot 3\cdot 7}{3\cdot 4\cdot 5\cdot 6} a_0, a_7 = \tfrac{1\cdot 5\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} a_1,$$
$$a_8 = -\tfrac{1\cdot 3\cdot 7\cdot 11}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} a_0, a_9 = \tfrac{1\cdot 5\cdot 9\cdot 13}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9} a_1, a_{10} = -\tfrac{1\cdot 3\cdot 7\cdot 11\cdot 15}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10} a_0, a_{11} = \tfrac{1\cdot 5\cdot 9\cdot 13\cdot 17}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11} a_1, \ldots.$$

The emerging pattern suggests that when k is even $a_k = -\frac{3\cdot 7\cdot 11 \cdots (2k-5)}{k!} a_0$ and when k is odd $a_k = \frac{5\cdot 9\cdot 13 \cdots (2k-5)}{k!} a_1$. That this is so, namely that for any even index $2j$ and any odd index $2j+1$ both with $j \geq 1$,

$$a_{2j} = -\tfrac{1\cdot 3\cdot 7\cdot 11 \cdots (4j-5)}{(2j)!} a_0 \quad \text{and} \quad a_{2j+1} = \tfrac{1\cdot 5\cdot 9\cdot 13 \cdots (4j-3)}{(2j+1)!} a_1$$

respectively, can be verified using the principle of induction (of Section 3.8). Substituting this back into the series for $f(x)$ we get

$$\begin{aligned}
f(x) &= \sum_{j=0}^{\infty} a_{2j}x^{2j} + \sum_{j=0}^{\infty} a_{2j+1}x^{2j+1} \\
&= a_0 - \sum_{j=1}^{\infty} \tfrac{1\cdot 3\cdot 7\cdot 11\cdots(4j-5)}{(2j)!}\, a_0 x^{2j} + a_1 x + \sum_{j=1}^{\infty} \tfrac{1\cdot 5\cdot 9\cdot 13\cdots(4j-3)}{(2j+1)!}\, a_1 x^{2j+1} \\
&= a_0 - a_0\sum_{j=1}^{\infty} \tfrac{1\cdot 3\cdot 7\cdot 11\cdots(4j-5)}{(2j)!}\, x^{2j} + a_1 x + a_1\sum_{j=1}^{\infty} \tfrac{1\cdot 5\cdot 9\cdot 13\cdots(4j-3)}{(2j+1)!}\, x^{2j+1}.
\end{aligned}$$

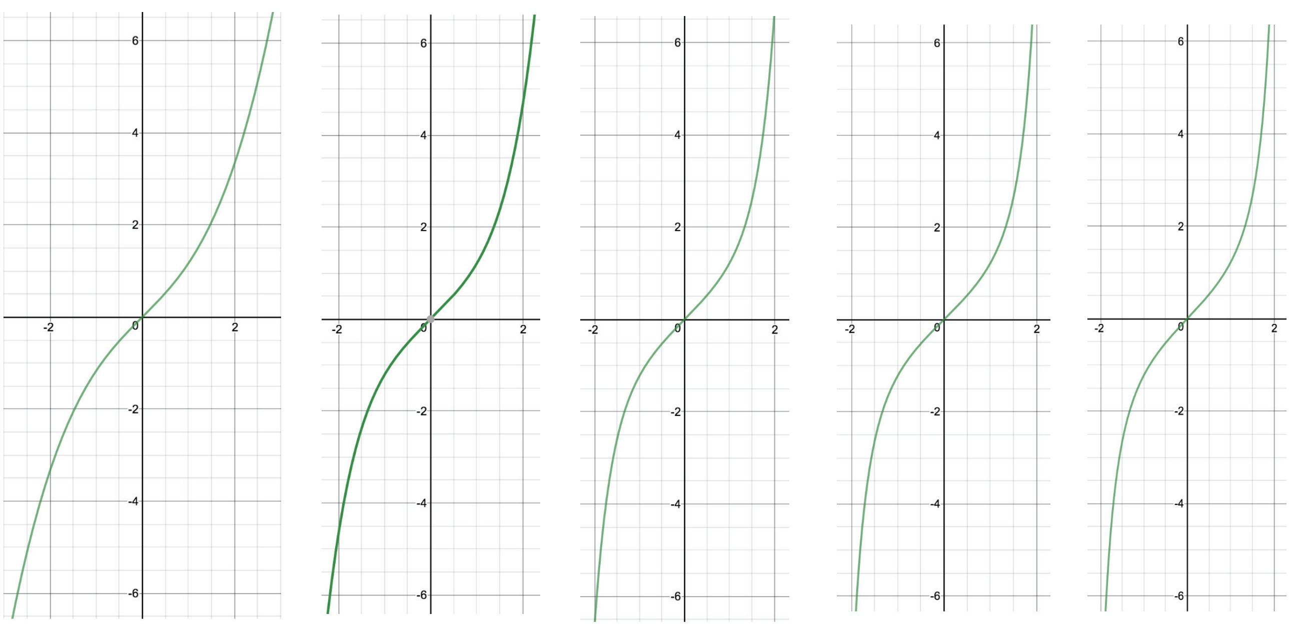
Let's consider the special case $f(0) = a_0 = 0$ and $f'(0) = a_1 = 1$. Now

$$f(x) = x + \sum_{j=1}^{\infty} \tfrac{1\cdot 5\cdot 9\cdot 13\cdots(4j-3)}{(2j+1)!}\, x^{2j+1}.$$

By the concluding remark of Section 11.8 this is the Taylor series for the function $f(x)$. Let's consider the approximate graphs of $y = f(x)$ that the first few Taylor polynomials provide. The polynomial $T_{17}(x)$ is

$$T_{17}(x) = x + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^5 + \tfrac{1}{112}x^7 + \tfrac{13}{8064}x^9 + \tfrac{221}{887040}x^{11} + \tfrac{17}{126720}x^{13} + \tfrac{17}{1064448}x^{15} + \tfrac{29}{17031168}x^{17}.$$

Dropping terms provides $T_{13}(x), T_9(x), T_5(x)$, and $T_3 = x + \frac{1}{6x^3}$. Their graphs are sketched below. Notice the essential shape of the graph is retained from one Taylor polynomial to the



$T_3(x)$ $T_5(x)$ $T_9(x)$ $T_{13}(x)$ $T_{17}(x)$

next. Not surprisingly, the graphs get steeper as more terms are added in.

It is worth noting that the solution of $y'' - 2xy' + y = 0$ (or $\frac{d}{dx}\left(\frac{d}{dx}(y)\right) - 2x\frac{d}{dx}(y) + y = 0$) is beyond the capacity of the differential equation calculator

https://www.symbolab.com/solver/ordinary-differential-equation-calculator/ .

11.52. We'll start by solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y to get the function $y = f(x)$ that has the upper half of the ellipse as its graph. Since $\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$ and hence $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, it follows that

$f(x) = \frac{b}{a}\sqrt{a^2 - x^2} = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$. It follows from Section 9.3 and the symmetry of the graph of the ellipse that the length of the upper half of the ellipse is given by

$$\int_{-a}^{a} \sqrt{1 + f'(x)^2}\, dx = 2\int_0^a \sqrt{1 + f'(x)^2}\, dx.$$

Note that $f'(x) = \frac{b}{a}\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x) = \frac{-bx}{a}(a^2 - x^2)^{-\frac{1}{2}}$ and hence that $f'(x)^2 = \frac{b^2x^2}{a^2(a^2-x^2)}$. Therefore

$$\int_0^a \sqrt{1 + f'(x)^2}\, dx = \int_0^a \sqrt{1 + \tfrac{b^2x^2}{a^2(a^2-x^2)}}\, dx = \int_0^a \sqrt{\tfrac{a^4-a^2x^2+b^2x^2}{a^2(a^2-x^2)}}\, dx = \int_0^a \sqrt{\tfrac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}\, dx.$$

A review of the basic facts about the ellipse from Section 4.4 tells us that $a^2 - b^2 = c^2$ with $c = a\varepsilon$ and ε the eccentricity of the ellipse. It follows that

$$\int_0^a \sqrt{1 + f'(x)^2}\, dx = \int_0^a \sqrt{\tfrac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}\, dx = \int_0^a \sqrt{\tfrac{a^4-a^2\varepsilon^2x^2}{a^2(a^2-x^2)}}\, dx = \int_0^a \sqrt{\tfrac{a^2-\varepsilon^2x^2}{a^2-x^2}}\, dx.$$

We now let $x = a\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Since $\sin^2\theta + \cos^2\theta = 1$ and $a\cos\theta \ge 0$ over $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we see that $\sqrt{a^2 - a^2\sin^2\theta^2} = a\cos\theta$. The fact that $\frac{dx}{d\theta} = a\cos\theta$ and the equalities $a\sin 0 = 0$ and $a = a\sin\frac{\pi}{2}$ imply that

$$\int_0^a \sqrt{1 + f'(x)^2}\, dx = \int_0^a \sqrt{\tfrac{a^2-\varepsilon^2a^2\sin^2\theta}{a^2-a^2\sin^2\theta^2}}\, a\cos\theta\, d\theta = \int_0^a \sqrt{a^2 - \varepsilon^2a^2\sin^2\theta}\, d\theta$$
$$= a\int_0^a \sqrt{1 - \varepsilon^2\sin^2\theta}\, d\theta.$$

So the length of the upper half of the ellipse is and $2a\displaystyle\int_0^a \sqrt{1 - \varepsilon^2\sin^2\theta}\, d\theta$ and the circumference of the full ellipse is $4a\displaystyle\int_0^a \sqrt{1 - \varepsilon^2\sin^2\theta}\, d\theta$.

11.53. By integration by parts with $u = \sin^{n-1}\theta$ and $dv = \sin\theta\, d\theta$, we get $du = (n–1)(\sin^{n-2}\theta)\cos\theta\, d\theta$ and $v = -\cos\theta$, and therefore

$$\int \sin^n\theta\, d\theta = \int u\, dv = uv - \int v\, du = -(\sin^{n-1}\theta)\cos\theta + \int (n-1)(\sin^{n-2}\theta)\cos^2\theta\, d\theta.$$

Since $\cos^2\theta = 1 - \sin^2\theta$, it follows that

$$\int \sin^n\theta\, d\theta = -(\sin^{n-1}\theta)\cos\theta + \int (n-1)(\sin^{n-2}\theta)(1 - \sin^2\theta)\, d\theta$$
$$= -\cos\theta\cdot\sin^{n-1}\theta + (n-1)\int \sin^{n-2}\theta\, d\theta - (n-1)\int \sin^n\theta\, d\theta,$$

and therefore that $n\displaystyle\int \sin^n\theta\, d\theta = -\cos\theta\cdot\sin^{n-1}\theta + (n-1)\int \sin^{n-2}\theta\, d\theta$. The formula

$$\int \sin^n\theta\, d\theta = -\tfrac{1}{n}\cos\theta\cdot\sin^{n-1}\theta + \tfrac{n-1}{n}\int \sin^{n-2}\theta\, d\theta$$

follows.

Let's turn to the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\, d\theta$. If $k = 0$, then $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\, d\theta = \int_0^{\frac{\pi}{2}} 1\, d\theta = \theta\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$. If $k \ge 1$, then $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\, d\theta = -\frac{1}{2k}\Big[\cos\theta\cdot\sin^{2k-1}\theta\Big|_0^{\frac{\pi}{2}}\Big] + \frac{2k-1}{2k}\int_0^{\frac{\pi}{2}} \sin^{2k-2}\theta\, d\theta$. Because

$$-\tfrac{1}{2k}\Big[\cos\theta\cdot\sin^{2k-1}\theta\Big|_0^{\frac{\pi}{2}}\Big] = -\tfrac{1}{2k}\Big[\cos\tfrac{\pi}{2}\cdot\sin^{2k-1}\tfrac{\pi}{2} - \cos 0\cdot\sin^{2k-1}0\Big] = -\tfrac{1}{2k}(0-0) = 0,$$

we get

$$\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\,d\theta = \tfrac{2k-1}{2k}\int_0^{\frac{\pi}{2}} \sin^{2k-2}\theta\,d\theta.$$

By applying this formula k times we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\,d\theta = \tfrac{2k-1}{2k}\,\tfrac{2k-3}{2k-2}\cdots\tfrac{3}{4}\cdot\tfrac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{0}\theta\,d\theta = \tfrac{2k-1}{2k}\,\tfrac{2k-3}{2k-2}\cdots\tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{\pi}{2}.$$

11.54. Combining the equality

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\varepsilon^2\sin^2\theta}\,d\theta = \tfrac{\pi}{2} - \sum_{k=1}^{\infty} \tfrac{1\cdot3\cdot5\cdots(2k-3)\varepsilon^{2k}}{2^k k!}\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\,d\theta$$

derived in the text prior to Problem 11.53 with the conclusion $\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\,d\theta = \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdots\tfrac{2k-1}{2k}\cdot\tfrac{\pi}{2}$ of Problem 11.53, we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{1-\varepsilon^2\sin^2\theta}\,d\theta &= \tfrac{\pi}{2} - \tfrac{\pi}{2}\sum_{k=1}^{\infty}\Big[\tfrac{1\cdot3\cdot5\cdots(2k-3)\varepsilon^{2k}}{2^k k!}\Big]\Big[\tfrac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}\Big]\\ &= \tfrac{\pi}{2}\Big(1 - \sum_{k=1}^{\infty}\Big[\tfrac{1\cdot3\cdot5\cdots(2k-3)(2k-1)\varepsilon^{2k}}{(2k-1)2^k k!}\Big]\Big[\tfrac{1\cdot3\cdot5\cdots(2k-1)}{2^k k!}\Big]\Big)\\ &= \tfrac{\pi}{2}\Big(1 - \sum_{k=1}^{\infty}\tfrac{[1\cdot3\cdot5\cdots(2k-1)]^2}{[2^k k!]^2}\cdot\tfrac{\varepsilon^{2k}}{2k-1}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}4a\int_0^{\frac{\pi}{2}} \sqrt{1-\varepsilon^2\sin^2\theta}\,d\theta &= 2\pi a\Big[1 - \sum_{k=1}^{\infty}\tfrac{[1\cdot3\cdots(2k-1)]^2}{[2^k(k!)]^2}\,\tfrac{\varepsilon^{2k}}{2k-1}\Big]\\ &= 2\pi a\Big[1 - \big(\tfrac{1}{2}\big)^2\tfrac{\varepsilon^2}{1} - \big(\tfrac{1\cdot3}{2\cdot4}\big)^2\tfrac{\varepsilon^4}{3} - \big(\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\big)^2\tfrac{\varepsilon^6}{5} - \big(\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\big)^2\tfrac{\varepsilon^8}{7} - \cdots\Big].\end{aligned}$$

11.55. This is a matter of plugging $a \approx 149{,}598{,}000\,\mathrm{km}$ and $\varepsilon \approx 0.016711$ into the formula of Problem 11.54.

11.56. Simply plug $a \approx 17.83\,\mathrm{au}$ and $\varepsilon \approx 0.967$ into the formula of Problem 11.54.

11.57. Equalities (i) and (ii) are derived in exactly the same way as the analogous equalities for $\sin\theta$ in the discussion that precedes Problem 11.53.

11.58. Integrating by parts with $u = \cos^{n-1}\theta$ and $dv = \cos\theta\,d\theta$, we get $du = -(n\text{–}1)(\cos^{n-2}\theta)\sin\theta\,d\theta$ and $v = \sin\theta$, and therefore

$$\int \cos^n\theta\,d\theta = \int u\,dv = uv - \int v\,du = (\cos^{n-1}\theta)\sin\theta + \int (n-1)(\cos^{n-2}\theta)\sin^2\theta\,d\theta.$$

Since $\sin^2\theta = 1-\cos^2\theta$, we see that

$$\begin{aligned}\int \cos^n\theta\,d\theta &= (\cos^{n-1}\theta)\sin\theta + \int (n-1)(\cos^{n-2}\theta)(1-\cos^2\theta)\,d\theta\\ &= \sin\theta\cdot\cos^{n-1}\theta + (n-1)\int \cos^{n-2}\theta\,d\theta - (n-1)\int \cos^n\theta\,d\theta,\end{aligned}$$

so that $n\displaystyle\int \cos^n\theta\,d\theta = \sin\theta\cdot\cos^{n-1}\theta + (n-1)\int \cos^{n-2}\theta\,d\theta$. Hence

$$\int \cos^n\theta\,d\theta = \tfrac{1}{n}\sin\theta\cdot\cos^{n-1}\theta + \tfrac{n-1}{n}\int \cos^{n-2}\theta\,d\theta.$$

The second equality follows quickly from this (refer to the solution of Problem 11.53).

That the values of the integrals for $\cos^{2k}\theta$ and $\sin^{2k}\theta$ over $0 \le \theta \le \frac{\pi}{2}$ are the same should not come as a surprise. A look at Figures 4.23 and 4.24 tells us how tightly the graphs of $y = \sin\theta$ and $y = \cos\theta$ over $0 \le \theta \le \frac{\pi}{2}$ are related. One graph is simply a reflection of the other about the line $x = \frac{\pi}{4}$.

11.59. Compare the conclusions of Problems 11.53 and 11.58.

11.60. There is not much to do here. Just plug in the numbers.

11.61. Here too, just plug in the numbers.

11.62. This is a continuation of the "Galileo's Cannonballs" segment. The cannonball is the same. The only difference is the height y_0 from which it is dropped. For the fall in a vacuum the relevant equations are $v_{\text{vac}}(t) = -gt$ and $y_{\text{vac}}(t) = -\frac{1}{2}gt^2 + y_0$. For the fall in air they are $v_{\text{air}}(t) = -s_\infty \tanh\frac{g}{s_\infty}t$ and $y_{\text{air}}(t) = y_0 - \frac{s_\infty^2}{g}\ln\cosh(\frac{g}{s_\infty}t)$. In each case, $g = 9.81\,\text{m/sec}^2$. As in the previous segment $s_\infty \approx 178.868827\,\text{m/sec}$, $\frac{g}{s_\infty} \approx 0.054845\,\frac{1}{\text{s}}$, and $\frac{s_\infty^2}{g} \approx 3261.371791\,\text{m}$.

i. Here $y_0 = 381\,\text{m}$. For the fall in a vacuum the time t_1 of impact satisfies $\frac{1}{2}(9.81)t_1^2 = 381$ and hence $t_1 \approx \sqrt{\frac{2\cdot 381}{9.81}} \approx 8.813390\,\text{sec}$. At impact $v_{\text{vac}}(t_1) \approx (-9.81)(8.813390) \approx -86.46\,\text{m/sec}$, so that the speed at impact is $86.46\,\text{m/sec}$.

For the fall in air the time t_1 of impact satisfies $y_0 - \frac{s_\infty^2}{g}\ln\cosh(\frac{g}{s_\infty}t_1) = 0$ so that $\ln\cosh(\frac{g}{s_\infty}t_1) = \frac{g}{s_\infty^2}y_0$. Hence $\frac{g}{s_\infty}t_1 = \cosh^{-1} e^{\frac{g}{s_\infty^2}y_0}$ and $t_1 = \frac{s_\infty}{g}\cosh^{-1} e^{\frac{g}{s_\infty^2}y_0}$. (This derives the formula for t_{imp} anew.) Since $\frac{g}{s_\infty^2} \approx (3261.371791)^{-1} \approx 3.066194\times 10^{-4}$ we get $t_1 \approx (0.054845)^{-1}\cosh^{-1}(e^{(3.066194\times 10^{-4})381}) \approx 8.985893$ sec. The velocity at impact is $v_{\text{air}}(t_1) = -s_\infty\tanh\frac{g}{s_\infty}t_1 = (-178.868827)\tanh(0.054845\cdot 8.985893) \approx -81.65\,\text{m/sec}$, so that the speed at impact is $81.65\,\text{m/sec}$.

ii. Now $y_0 = 2000\,\text{m}$. For the fall in a vacuum the time t_1 of impact satisfies $\frac{1}{2}(9.81)t_1^2 = 2000$ and hence $t_1 \approx \sqrt{\frac{2\cdot 2000}{9.81}} \approx 20.192751\,\text{sec}$. At impact $v_{\text{vac}}(t_1) \approx (-9.81)(20.192751) \approx -198.09\,\text{m/sec}$.

For the fall in air the time t_1 of impact satisfies $t_1 = \frac{s_\infty}{g}\cosh^{-1} e^{\frac{g}{s_\infty^2}y_0}$. Hence $t_1 \approx (0.054845)^{-1}\cosh^{-1}(e^{(3.066194\times 10^{-4})2000}) \approx 22.305630$ sec. The velocity at impact is $v_{\text{air}}(t_1) = -s_\infty\tanh\frac{g}{s_\infty}t_1 = (-178.868827)\tanh(0.054845\cdot 22.305630) \approx -150.36\,\text{m/sec}$. The cannonball's speed at impact is $150.36\,\text{m/sec}$.

11.63. Since $y_\downarrow(t_{\text{ret}}) = y_0$ we get that $\frac{s_\infty^2}{g}\ln\left[\frac{\sqrt{\left(\frac{v_0}{s_\infty}\right)^2+1}}{\cosh\left(\frac{g}{s_\infty}(t_{\text{ret}}-t_{\text{top}})\right)}\right] = 0$ by evaluating the expression $y_\downarrow(t)$ at $t = t_{\text{ret}}$. Hence $\ln\left[\frac{\sqrt{\left(\frac{v_0}{s_\infty}\right)^2+1}}{\cosh\left(\frac{g}{s_\infty}(t_{\text{ret}}-t_{\text{top}})\right)}\right] = 0$. Since $\ln x$ is zero only for $x = 1$ (refer to Section 7.11), $\frac{\sqrt{\left(\frac{v_0}{s_\infty}\right)^2+1}}{\cosh\left(\frac{g}{s_\infty}(t_{\text{ret}}-t_{\text{top}})\right)} = 1$. So $\cosh(\frac{g}{s_\infty}(t_{\text{ret}} - t_{\text{top}})) = \sqrt{(\frac{v_0}{s_\infty})^2 + 1}$ and therefore $t_{\text{ret}} - t_{\text{top}} = \frac{s_\infty}{g}\cosh^{-1}\sqrt{(\frac{v_0}{s_\infty})^2 + 1}$.

11.64. **i.** By a formula from Section 9.9.1, $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$ and by another from Section 9.9.2 $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$. By the chain rule,

$$\frac{d}{dx}\cosh^{-1}\sqrt{x^2+1} = \frac{1}{\sqrt{(\sqrt{x^2+1})^2-1}} \cdot \tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}2x = \frac{1}{\sqrt{x^2+1-1}} \cdot \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.$$

ii. For $x > 0, x^2+1 > 1$, so that $x^2+1 > \sqrt{x^2+1}$ and hence $\frac{1}{\sqrt{x^2+1}} > \frac{1}{x^2+1}$. Therefore by (i), $\frac{d}{dx}\cosh^{-1}\sqrt{x^2+1} > \frac{d}{dx}\tan^{-1}x$ for $x > 0$ and hence $\frac{d}{dx}\big(\cosh^{-1}\sqrt{x^2+1} - \tan^{-1}x\big) > 0$ for $x > 0$. So $\cosh^{-1}\sqrt{x^2+1} - \tan^{-1}x$ is an increasing function over the interval $(0, \infty)$. Since $\cosh^{-1}\sqrt{0+1} = 0 = \tan^{-1}(0)$, it follows that $\cosh^{-1}\sqrt{x^2+1} > \tan^{-1}x$ for $x > 0$.

iii. By letting $x = \frac{v_0}{s^\infty}$ in (ii), we get $t_{\text{down}} = \frac{s_\infty}{g}\cosh^{-1}\sqrt{(\frac{v_0}{s_\infty})^2+1} > \frac{s_\infty}{g}\tan^{-1}\frac{v_0}{s_\infty} = t_{\text{up}}$.

11.65. **i.** Inserting the conclusion of Problem 11.63 into the formula $v_\downarrow(t) = -s_\infty \tanh\big(\frac{g}{s_\infty}(t - t_{\text{top}})\big)$ of Section 11.10.2, we get

$$v_\downarrow(t_{\text{ret}}) = -s_\infty \tanh\big(\tfrac{g}{s_\infty}(t_{\text{ret}} - t_{\text{top}})\big) = -s_\infty \tanh\Big(\cosh^{-1}\sqrt{(\tfrac{v_0}{s_\infty})^2+1}\Big).$$

ii. The identity $\cosh^2 x - \sinh^2 x = 1$ implies that $1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$ and hence that $\tanh^2 x = 1 - \frac{1}{\cosh^2 x}$. A look at Section 9.9.2 tells us that $\tanh x \geq 0$ when $x \geq 0$, so that $\tanh x = \sqrt{1 - \frac{1}{(\cosh x)^2}}$ and $\tanh(\cosh^{-1}x) = \sqrt{1 - \frac{1}{x^2}} = \sqrt{\frac{x^2-1}{x^2}}$ for $x > 0$.

iii. By using (i) and inserting $x = \sqrt{(\frac{v_0}{s_\infty})^2+1}$ into (ii) we can conclude that

$$v_\downarrow(t_{\text{ret}}) = -s_\infty \tanh\Big(\cosh^{-1}\sqrt{(\tfrac{v_0}{s_\infty})^2+1}\Big) = -s_\infty\sqrt{\frac{(\sqrt{(\frac{v_0}{s_\infty})^2+1})^2-1}{(\sqrt{(\frac{v_0}{s_\infty})^2+1})^2}}$$

$$= -s_\infty\sqrt{\frac{(\frac{v_0}{s_\infty})^2+1-1}{(\frac{v_0}{s_\infty})^2+1}} = -s_\infty\frac{\frac{v_0}{s_\infty}}{\sqrt{(\frac{v_0}{s_\infty})^2+1}} = \frac{-v_0}{\sqrt{(\frac{v_0}{s_\infty})^2+1}}.$$

11.66. With the block in the position described in Figure 11.33 the upward pull of kh balances the weight mg of the block, so that $kh = mg$.

i. Suppose that the block is moved so that the position of its center is at d_0. It is released from there at time $t = 0$ with an initial velocity of v_0. The block's center is in position $y(t)$ and its velocity is $v(t) = y'(t)$ at any time $t \geq 0$ thereafter. Note that $y(0) = d_0$ and $v(0) = v_0$. If $y(t) < 0$ then the spring is stretched $h + (-y(t)) = h - y(t)$ beyond its natural length and the force exerted by the spring on the block is $k(h - y(t))$. The fact that this is positive reflects the fact that the force acts in the upward or positive direction. If $y(t) \geq 0$ then the displacement of the spring is $y(t)$ less than the h of the equilibrium position (of Figure 11.33), so that the spring is displaced by $h - y(t)$ in this case also. If $y(t) = h$ then the spring is at its natural length so that the force it exerts on the block is 0. If $y(t) > h$ then the spring is compressed by a distance of $y(t) - h$ so that the force the spring exerts on the block is downward and hence the negative quantity $k(h - y(t))$. In all cases the force exerted by the spring on the block is $k(h - y(t))$. It follows that the net force on the block is $F(t) = k(h - y(t)) - mg$. Since $y''(t)$ is the acceleration of the block,

$$my''(t) = k(h - y(t)) - mg = kh - ky(t) - mg = -ky(t),$$

so that $y''(t) = \frac{-k}{m}\,y(t)$ as asserted.

ii. Section 11.6 applies to the differential equation $y'' + \frac{k}{m}y = 0$. The two roots of the characteristic polynomial $x^2 + \frac{k}{m}$ are $\pm\sqrt{\frac{k}{m}}\,i$ so that Case 3 applies with $a = 0$ and $b = \sqrt{\frac{k}{m}}$. It follows that the function $y(t)$ has the form

$$y(t) = D_1 \cos\left(\sqrt{\tfrac{k}{m}}t\right) + D_2 \sin\left(\sqrt{\tfrac{k}{m}}t\right)$$

with D_1 and D_2 constants. Since $y(0) = d_0$, $d_0 = D_1 \cos 0 + D_2 \sin 0 = D_1$. Since

$$y'(t) = -\left(D_1 \sin\sqrt{\tfrac{k}{m}}t\right)\sqrt{\tfrac{k}{m}} + \left(D_2 \cos\sqrt{\tfrac{k}{m}}t\right)\sqrt{\tfrac{k}{m}}$$

and $v_0 = y'(0) = -\left(D_1 \sin 0\right)\sqrt{\frac{k}{m}} + \left(D_2 \cos 0\right)\sqrt{\frac{k}{m}} = D_2\sqrt{\frac{k}{m}}$, it follows that $D_2 = v_0\sqrt{\frac{m}{k}}$ and hence that $y(t) = d_0 \cos\left(\sqrt{\frac{k}{m}}t\right) + v_0\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}t\right)$.

iii. If $v_0 = 0$, then $y(t) = d_0 \cos\left(\sqrt{\frac{k}{m}}t\right)$. A look at Figure 4.24 shows that if $\sqrt{\frac{k}{m}}\,t$ varies from 0 to 2π then the block moves through exactly one complete up-and-down cycle. Dividing $0 \le \sqrt{\frac{k}{m}}\,t \le 2\pi$ through by $\sqrt{\frac{k}{m}}$ this corresponds to $0 \le t \le \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$. So the time required for one complete cycle is $2\pi\sqrt{\frac{m}{k}}$.

11.67. Since a force of 40 N is necessary to keep the spring extended 0.05 m beyond its natural length, $40 = k(0.05)$, so that the spring constant is $k = 800\,\mathrm{N/m}$. Since $m = 0.5\,\mathrm{kg}$, $\sqrt{\frac{k}{m}} = \sqrt{1600} = 40$. At the instant $t = 0$ that the block is released, $y = d_0 = 0.1\,\mathrm{m}$ and its velocity is $v_0 = -2\,\mathrm{m/s}$. Inserting this into the expression for $y(t)$ derived in Problem 11.66(ii), we get

$$y(t) = (0.1)\cos(40\,t) - \tfrac{1}{20}\sin(40\,t) \text{ meters.}$$

11.68. The study considers the responses of the front and rear suspensions under the assumption that the chassis reaches the top of its displacement at time $t = 0$ and that the position function $z(t)$ satisfies $z(0) = 0.05$ m and $z'(0) = 0$ m/s.

i. For the front wheels, $m = 118.0\,\mathrm{kg}$, $k = 525{,}000\,\mathrm{N/m}$, and $d = 2\zeta\sqrt{mk} = 11{,}019\,\frac{\text{N-s}}{\text{m}}$. For the rear wheels, $m = 186.0\,\mathrm{kg}$, $k = 612{,}000\,\mathrm{N/m}$, and $d = 2\zeta\sqrt{mk} = 19{,}205\,\frac{\text{N-s}}{\text{m}}$.

ii. In either case, Section 11.11.1 tells us that the position function $z(t)$ is given by

$$z(t) = e^{at}(D_1 \cos bt + D_2 \sin bt)$$

where $a = -\frac{d}{2m}$, $b = \frac{\sqrt{4mk-d^2}}{2m}$, $D_1 = z(0)$, and $D_2 = \frac{1}{b}(z'(0) - az(0))$.

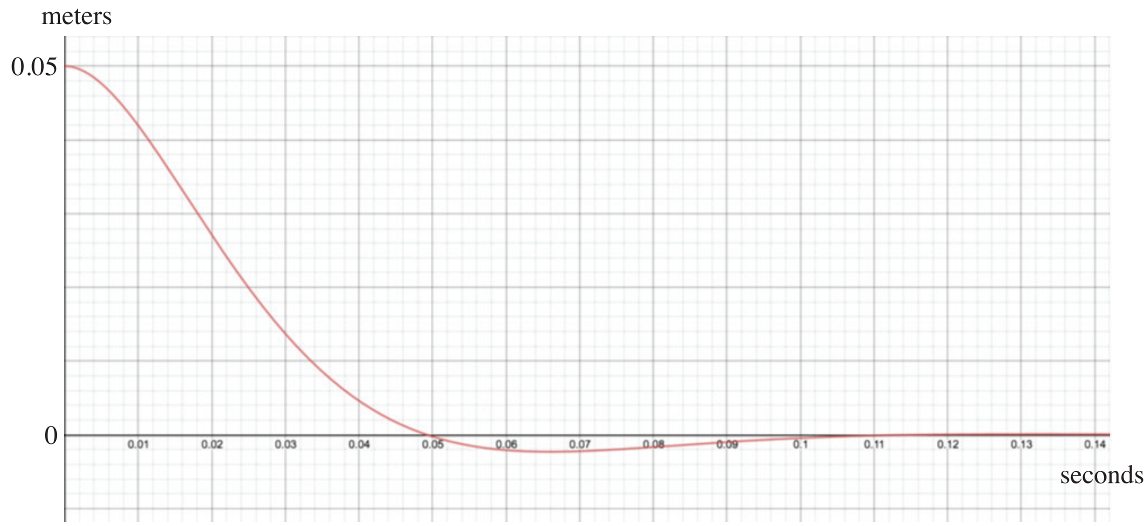
For each front wheel, $a \approx -46.69, b \approx 47.64, D_1 = 0.05$ and $D_2 \approx 0.049$ so that the particular solution is

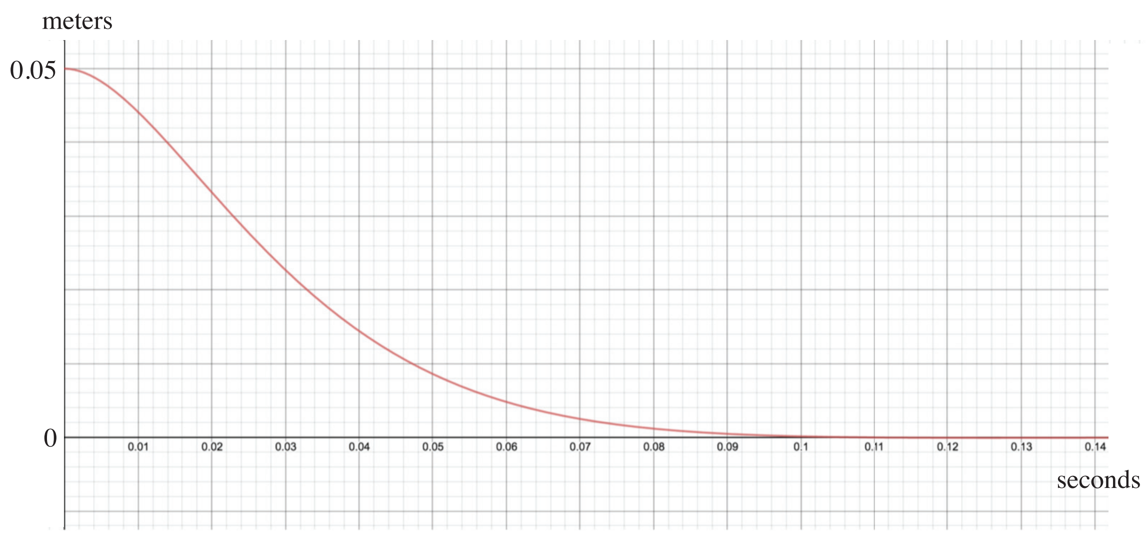
$$z(t) = e^{-46.69t}(0.05 \cos 47.64t + 0.049 \sin 47.64t).$$

For each rear wheel, $a \approx -51.63, b \approx 25.00, D_1 = 0.05$ and $D_2 \approx 0.103$ and the particular solution is

$$z(t) = e^{-51.63t}(0.05 \cos 25.00t + 0.103 \sin 25.00t).$$

iii. The results of https://www.desmos.com/calculator are provided below with the response of the front suspension at the top and that of the rear suspension at the bottom.





iv. Compared to the suspension of the family sedan and the stock car studied in Section 11.11.1, the suspensions of the race car—both front and rear—are hard. The rear suspension with its large damping ratio of $\zeta = 0.90$ is so hard that it almost seems to lead to a critically damped response. The graph of this response reflects this. However, a careful analysis of the function $z(t)$ shows that its graph dips below the time axis from about 0.108 seconds to about 0.126 seconds before rising above it again.

11.69. We saw that with $b = \frac{c}{T_0}$ and (d, s) on the graph, that $s = \frac{1}{b}(\cosh bd - 1)$ and hence that $e^{bd} + e^{-bd} = 2(1 + bs)$. Multiplying this last equation through by e^{bd} we get

$$e^{bd} \cdot e^{bd} + e^{bd} \cdot e^{-bd} = 2e^{bd}(1 + bs).$$

Since $e^{bd} \cdot e^{bd} + e^{bd} \cdot e^{-bd} = e^{bd+bd} + e^{bd-bd} = e^{2bd} + 1 = (e^{bd})^2 + 1$, we get

$$(e^{bd})^2 - 2(1+bs)e^{bd} + 1 = 0.$$

The quadratic formula applied to $X^2 - 2(1+bs)X + 1 = 0$ tells us that

$$e^{bd} = \tfrac{2(1+bs)\pm\sqrt{4(1+bs)^2-4\cdot 1\cdot 1}}{2} = 1 + bs \pm \sqrt{(bs)^2 + 2bs}.$$

i. Since $bd > 0$, $e^{bd} > 1$. So $bs \pm \sqrt{(bs)^2 + 2bs} > 0$. Since $\sqrt{(bs)^2 + 2bs} > \sqrt{(bs)^2} = bs$ only the $+$ option can arise. Therefore $e^{bd} = 1 + bs + \sqrt{(bs)^2 + 2bs}$. Letting $x = bs$, we get $e^{\frac{d}{s}x} = 1 + x + \sqrt{x^2 + 2x}$.

ii. It follows directly from (i) that $x = bs = \frac{cs}{T_0}$ is the x-coordinate of a point of intersection of the curves $y = e^{\frac{d}{s}x}$ and $y = 1 + x + \sqrt{x^2 + 2x}$.

11.70. We'll start the analysis of the function $g(x) = 1 + x + \sqrt{x^2 + 2x}$ for $x \geq 0$ with its derivative.

i. Since $g(x) = 1 + x + (x^2 + 2x)^{\frac{1}{2}}$ we see that

$$g'(x) = 1 + \tfrac{1}{2}(x^2 + 2x)^{-\frac{1}{2}}(2x + 2) = 1 + \tfrac{x+1}{(x^2+2x)^{\frac{1}{2}}} = 1 + \tfrac{x+1}{\sqrt{x^2+2x}} = 1 + \sqrt{\tfrac{x^2+2x+1}{x^2+2x}}.$$

Clearly $x^2 + 2x + 1 > x^2 + 2x$ and hence $\sqrt{x^2 + 2x + 1} > \sqrt{x^2 + 2x}$. So $1 + \sqrt{\frac{x^2+2x+1}{x^2+2x}} > 2$. It follows that the graph of $y = g(x)$ is increasing over $[0, \infty)$. Notice that $\lim\limits_{x\to 0^+} g'(x) = +\infty$ and hence that the graph has a vertical tangent at the point $(0, 1)$.

ii. Starting with $g'(x) = 1 + \frac{x+1}{(x^2+2x)^{\frac{1}{2}}}$ and using the quotient rule, we get

$$g''(x) = \tfrac{1\cdot(x^2+2x)^{\frac{1}{2}} - (x+1)\cdot\frac{1}{2}(x^2+2x)^{-\frac{1}{2}}(2x+2)}{(x^2+2x)} = \tfrac{(x^2+2x)^{\frac{1}{2}}}{(x^2+2x)^{\frac{1}{2}}}\left[\tfrac{(x^2+2x)^{\frac{1}{2}} - (x+1)(x^2+2x)^{-\frac{1}{2}}(x+1)}{(x^2+2x)}\right]$$

$$= \tfrac{(x^2+2x)-(x+1)(x+1)}{(x^2+2x)^{\frac{3}{2}}} = \tfrac{(x^2+2x)-(x^2+2x+1)}{(x^2+2x)^{\frac{3}{2}}} = \tfrac{-1}{(x^2+2x)^{\frac{3}{2}}}.$$

Since this is negative for all $x > 0$ the graph of $y = g(x)$ is concave down.

iii. Since $g(x) = 1 + x + \sqrt{x^2 + 2x}$, we get that

$$2x + 2 - g(x) = 2x + 2 - (1 + x + \sqrt{x^2 + 2x}) = 1 + x - \sqrt{x^2 + 2x}$$

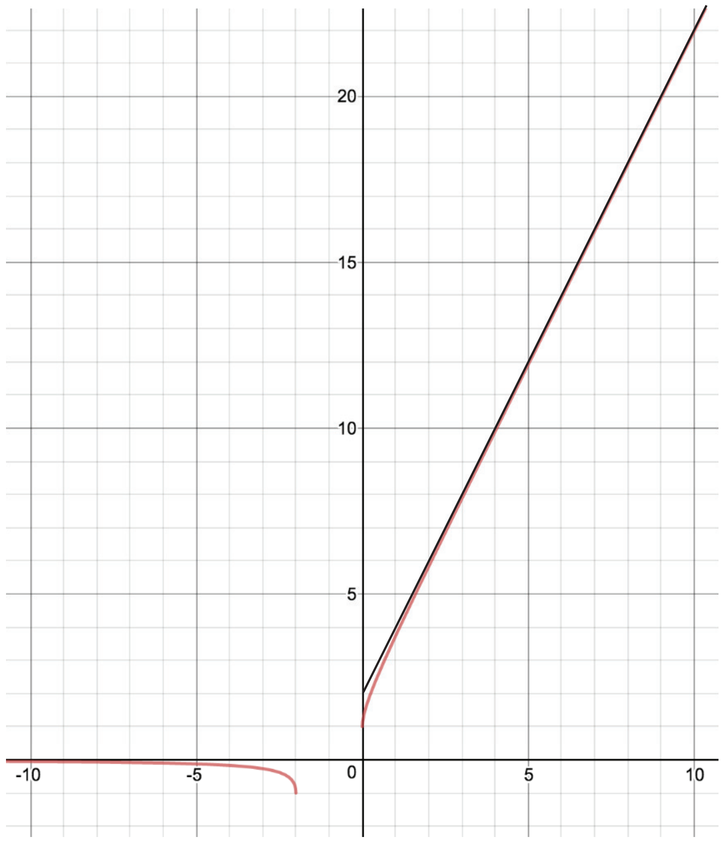
By rationalizing,

$$1 + x - \sqrt{x^2 + 2x} = (1 + x - \sqrt{x^2 + 2x}) \cdot \tfrac{1+x+\sqrt{x^2+2x}}{1+x+\sqrt{x^2+2x}} = \tfrac{(1+x)^2-(x^2+2x)}{1+x+\sqrt{x^2+2x}} = \tfrac{1}{1+x+\sqrt{x^2+2x}}.$$

Since this last term is positive for all $x \geq 0$, it follows that $2x + 2 - g(x) > 0$ and hence that $2x + 2 > g(x)$ for all $x \geq 0$. The fact that $\lim\limits_{x\to+\infty} \frac{1}{1+x+\sqrt{x^2+2x}} = 0$ tells us that $\lim\limits_{x\to+\infty} \big((2x - 2) - g(x)\big) = 0$ so that that the line $y = 2x + 2$ is an asymptote of the graph of $y = g(x)$.

iv. The graph of $y = g(x)$ depicted on the next page was provided by the graphing calculator https://www.desmos.com/calculator. The slanting black line is the asymptote $y = 2x+2$.

11.71. The function $g(x) = 1 + x + \sqrt{x^2 + 2x}$ is also defined for $x \leq -2$ (but not for $-2 < x < 0$) as the graph shows. The strategies of Problem 11.70 can be used to confirm that the graph is



$g(x) = 1 + x + \sqrt{x^2 + 2x}$

decreasing over $(-\infty, -2]$, that it is concave down, that it lies below the x-axis, and that the x-axis is a horizontal asymptote. Since none of these conclusions are used in the discussion that follows we'll skip the details.

The final three problems deal with a chain that weighs $c = 1.47\,\mathrm{N/m}$. The two supporting posts of the chain have the same height and are a distance $2d = 20\,\mathrm{m}$ apart. The solutions rely on Table 11.4 as well as the formulas $L = L(d) = \frac{T_0}{c}\sinh(\frac{cd}{T_0})$ and $T(d) = \sqrt{T_0^2 + c^2L^2}$ for the length of the chain and the maximal tension it is subject to.

11.72. **i.** Suppose that the sag in the chain is $s = 10\,\mathrm{m}$. Since $\frac{d}{s} = 1$, Table 11.4 tells us that $\frac{cs}{T_0} \approx 1.616$. So $T_0 \approx \frac{1}{1.616}cs \approx \frac{(1.47)(10)}{1.616} \approx 9.10\,\mathrm{N}$.

ii. The length of the chain is $2L = \frac{2T_0}{c}\sinh(\frac{cd}{T_0}) \approx \frac{2(9.10)}{1.47}\sinh(\frac{1.47\cdot 10}{9.10}) \approx 29.91$ m. The weight of the chain is approximately $(29.91)(1.47) \approx 43.96\,\mathrm{N}$ and the maximal tension that the chain is under is $T(d) = \sqrt{T_0^2 + c^2L^2} \approx \sqrt{9.10^2 + (1.47)^2(\frac{29.91}{2})^2} \approx 23.79\,\mathrm{N}$.

11.73. **i.** Let's shorten the chain so that its sag is $s = 1$ m. Now $\frac{d}{s} = 10$ and Table 11.4 tells us that $\frac{cs}{T_0} \approx 0.0199$. So $T_0 \approx \frac{1}{0.0199}cs \approx \frac{(1.47)(1)}{0.0199} \approx 73.87\,\mathrm{N}$.

ii The length of the chain is $2L = \frac{2T_0}{c}\sinh(\frac{cd}{T_0}) \approx \frac{2(73.87)}{1.47}\sinh(\frac{1.47\cdot 10}{73.87}) \approx 20.13$ m. Hence the weight of the chain is approximately $(20.13)(1.47) \approx 29\,\mathrm{N}$ and the maximal tension that the chain is under is $T(d) = \sqrt{T_0^2 + c^2L^2} \approx \sqrt{73.87^2 + (1.47)^2(\frac{20.13}{2})^2} \approx 75.34\,\mathrm{N}$.

11.74. **i.** Let's shorten the chain some more so that its sag is $s = 0.1$ m. Since $\frac{d}{s} = 100$ the table tells us that $\frac{cs}{T_0} \approx 0.0002$. It follows that $T_0 \approx \frac{1}{0.0002}cs \approx \frac{(1.47)(0.1)}{0.0002} \approx 735\,\text{N}$.

ii. The length of the chain is $2L = \frac{2T_0}{c}\sinh(\frac{cd}{T_0}) \approx \frac{2(735)}{1.47}\sinh(\frac{1.47\cdot 10}{735}) \approx 20.00\,\text{m}$. So the weight of the chain is approximately $(20.00)(1.47) \approx 29.40$ N, and the maximal tension that the chain is under is $T(d) = \sqrt{T_0^2 + c^2L^2} \approx \sqrt{735^2 + (1.47)^2(\frac{20.00}{2})^2} \approx 735.15\,\text{N}$.