Solutions to Problems and Projects for Chapter 8

8.1. The number of bacteria in the culture is expressed by the function $y(t) = y_0e^{\mu t}$ where μ is the growth constant and y_0 is the initial number of bacteria in the culture. We are given that $y(4) = 4000, y(7) = 62{,}000$ and $y(8) = 154{,}000$. The average rates of change in the number over the time intervals $[4, 7]$ and $[7, 8]$ are $\frac{62{,}000-4000}{3} \approx 19{,}333$ and $\frac{154{,}000-62{,}000}{1} = 92{,}000$ bacteria per hour. Since $4000 = y_0e^{4\mu}$ and $62{,}000 = y_0e^{7\mu}$, it follows that $e^{3\mu} = \frac{e^{7\mu}}{e^{4\mu}} = \frac{62{,}000}{4000} = 15.5$. So $3\mu = \ln(e^{3\mu}) = \ln 15.5$. So $\mu \approx 0.9136$. Since $y_0 = \frac{4000}{e^{4\mu}} \approx \frac{4000}{e^{4(0.9136)}} \approx 103.5$. Therefore $y(t) \approx 103.5e^{0.9136t}$. Checking this equation against what we already know, we get $y(7) \approx 103.5e^{(0.9136)7} \approx 61{,}990$ and $y(8) \approx 103.5e^{(0.9136)8} \approx 154{,}570$.

8.2. **i.** We know that the equality $\mu = \frac{\ln 2}{d}$ relates the growth constant μ to the doubling time d. Since the hour is the relevant unit of time later in the problem and 15 minutes $= 0.25$ hour, $\mu \approx \frac{0.693}{0.25} = 2.772$ and hence $y(t) \approx 20{,}000e^{2.772t}$.

ii. Setting $20{,}000e^{2.772t} = 1{,}000{,}000{,}000 = 10^9$, we get $e^{2.772t} = 50{,}000$, hence $2.772t \approx \ln 50{,}000$ and therefore, $t \approx \frac{\ln 50{,}000}{2.772} \approx 3.90$ hours.

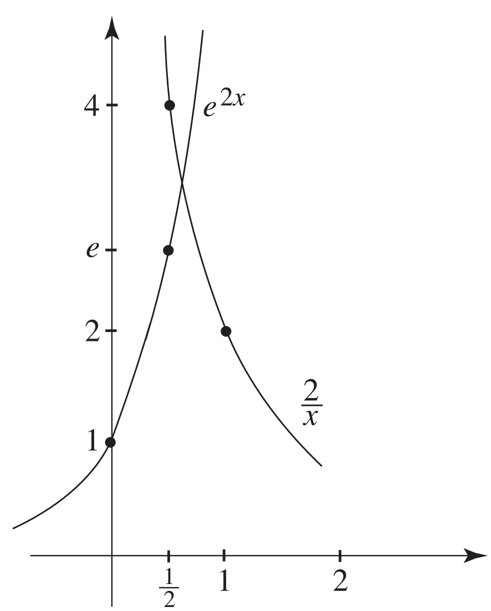
iii. Taking $t = 6$ in the formula derived in i), we get $y(6) \approx 20{,}000e^{(2.772)6} \approx 3.34 \times 10^{11}$.

8.3. **i.** Since $y_0 = 10{,}500$, $y(t) = 10{,}500e^{\mu t}$, where μ is the growth constant. Setting $10{,}500e^{\mu\cdot 2} = 23{,}000$, we get $e^{2\mu} = \frac{23{,}000}{10{,}500} \approx 2.1905$, and hence $2\mu \approx \ln 2.1905 \approx 0.7841$. So $\mu \approx 0.3921$ and hence $y(t) \approx 10{,}500e^{0.3921t}$.

ii. $y(3) \approx 10{,}500e^{(0.3921)3} \approx 34{,}000$.

iii. With $10{,}500e^{0.3921t} = 130{,}000$, we get $e^{0.3921t} = \frac{130{,}000}{10{,}500} \approx 12.3810$. Therefore $0.3921t \approx \ln 12.3810$ and hence $t \approx 6.42$ hours.

8.4. We know that $y_0 = 5000$ and that $y'(2) = 10{,}000$ cells per hour. Since $y'(t) = \mu y(t) = \mu y_0e^{\mu t}$, we get $10{,}000 = \mu 5000e^{2\mu}$. Therefore $\frac{2}{\mu} = e^{2\mu}$. So the μ that we are looking for is the x-coordinate of the point of intersection of the graphs $y = e^{2x}$ and $y = \frac{2}{x}$ as the hint suggested.



Rough sketches of the two graphs are drawn in the figure above. It follows from the figure that the x-coordinate x_0 of the point of intersection satisfies $0.5 < x_0 < 1$. After some experimenting, you will find that $e^{2(0.6)} = e^{1.2} \approx 3.32$ and $\frac{2}{0.6} \approx 3.33$, so that x_0 is close to (and a little greater than) 0.6. The graphing calculator

https://www.desmos.com/calculator

applied to $f(x) = e^{2x} - \frac{2}{x}$ tells us that this function is zero for $x_0 \approx 0.6011$, so that $\mu \approx 0.6011$ is a tight approximation of μ in the unit 1/hour.

8.5. We are given that $y_0 = 75{,}000$ and that $y'(1) = 24(150{,}000) \approx$ cells per day. Since $y'(t) = \mu y(t) = \mu y_0 e^{\mu t}$, we get $24(150{,}000) = \mu 75{,}000 e^{\mu}$, so that $\frac{48}{\mu} = e^{\mu}$. The μ we are looking for is the x-coordinate x_0 of the point of intersection of the graphs $y = e^x$ and $y = \frac{48}{x}$. Since $e^2 \approx 7.39$ and $\frac{48}{2} = 24$ and $e^3 \approx 20.09$ and $\frac{48}{3} = 16$, this x-coordinate satisfies $2 < x_0 < 3$. Since $e^3 \approx 20.09$ and $\frac{48}{3} = 16$ are close, let's try $x_0 = 2.9$. Since $e^{2.9} \approx 18.17$ and $\frac{48}{2.9} \approx 16.55$, these values are closer than before. Repeating with $x_0 = 2.8$, we get $e^{2.8} \approx 16.44$ and $\frac{48}{2.8} \approx 17.14$. Our numerical experiments have shown that x_0 satisfies $2.8 < x_0 < 2.9$. The graphing calculator already referred to tells us that $e^x - \frac{48}{x}$ has a zero very close to $x_0 = 2.8307$. So $\mu \approx 2.8307$ in the unit 1/day.

8.6. **i.** The decay constant is $\lambda = \frac{\ln 2}{h} \approx \frac{0.693}{138} \approx 0.0050$ in the unit 1/day.

ii. Since 1 milligram $= \frac{1}{1000}$ gram, Avogadro's number tells us that the sample consists of about $\frac{30}{1000} \cdot \frac{1}{210}(6.02 \times 10^{23}) \approx 2.87 \times 10^{21}$ atoms.

iii. Taking $y_0 = 2.87 \times 10^{21}$ and $\lambda \approx 0.0050$ in the expression $y(t) = y_0 e^{-\lambda t}$, we get $y(t) \approx (2.87 \times 10^{21})e^{-(0.005)t}$.

iv. Since 6 weeks have 42 days, the sample will have $y(42) \approx (2.87 \times 10^{21})e^{-(0.005)42} \approx (2.87 \times 10^{21})0.81 \approx 2.33 \times 10^{21}$ polonium-210 atoms.

8.7. **i.** The half-life that corresponds to the decay constant 0.18 1/day is $h = \frac{\ln 2}{0.18} \approx \frac{0.693}{0.18} = 3.85$ days or 92.4 hours.

ii. Setting $y_0 e^{-0.18t} = 0.9 y_0$, we get $-0.18t = \ln 0.9$ and hence that $t \approx \frac{\ln 0.9}{-0.18} \approx 0.585$ days or about 14 hours.

iii. With $y_0 e^{-0.18t} = \frac{1}{3} y_0$, we get $-0.18t = \ln 3^{-1} = -\ln 3$. So $t = \frac{\ln 3}{0.18} \approx 6.1$ days.

8.8. Let a representative measurement be taken at time $t = 0$. So $y'(0) = -3.7 \times 10^{10}$ atoms/second and using Avogadro's number $y(0) \approx \frac{1}{226}(6.2 \times 10^{23}) \approx 2.66 \times 10^{21}$ atoms. The relationship $y'(t) = -\lambda y(t)$ with $t = 0$, implies that $\lambda \approx \frac{3.7 \times 10^{10}}{2.66 \times 10^{21}} \approx 1.39 \times 10^{-11}$ in the unit 1/second. Therefore $h = \frac{\ln 2}{\lambda} \approx \frac{0.693}{1.39} \times 10^{11} \approx 4.99 \times 10^{10}$ seconds. Since 1 year has $(3600)(24)(365.25) \approx 3.156 \times 10^7$ seconds, it follows that $h \approx 1580$ years.

8.9. Given the information supplied, we'll let the minute be our unit of time and let $t = 0$ at the time of the 8:00 a.m. measurement. Using the equation $y'(t) = -\lambda y(t) = -\lambda y_0 e^{-\lambda t}$ with $t = 0$, we get $-3200 = -\lambda y_0$. Therefore $y'(t) = -3200 e^{-\lambda t}$. The 5 p.m. measurement tells us that $-900 = -3200 e^{-(9 \cdot 60)\lambda} = -3200 e^{-540\lambda}$. Therefore $-540\lambda = \ln \frac{900}{3200} \approx -1.269$, so that $\lambda \approx 2.35 \times 10^{-3}$ in the unit 1/minute. Because $h = \frac{\ln 2}{\lambda}$, we get $h \approx \frac{0.693}{2.35} \times 10^{-3}$ minutes or $h \approx (\frac{0.693}{2.35} \times 10^{-3})60 \approx 0.0177$ seconds. This corresponds most closely to the radioactive isotope boron-13.

8.10. The average cost per unit is the function $\frac{C(x)}{x}$ of the production level x. If this is equal to a constant, say, A, then $C(x) = Ax$. It follows that the marginal cost $C'(x)$ is also equal to A.

8.11. **i.** With a marginal cost of $C'(x) = 0.000012x^2 - 0.002x + 2800$, the cost function $C(x) = 0.000004x^3 - 0.001x^2 + 2800x + C_0$, where C_0 is the fixed cost. Since $C(10{,}000) =$ 39,476,000 dollars (the value $C(10{,}000) = 39{,}476$ given in the problem is in error), we get that

$$39{,}476{,}000 = 0.000004(10{,}000)^3 - 0.001(10{,}000)^2 + 2800(10{,}000) + C_0,$$

so that $C_0 = 7{,}576{,}000$ dollars.

ii. Since the revenue from the sale of x units is $R(x) = px$ with $p = 6652$ dollars, the profit from these sales is $P(x) = R(x) - C(x) = 6652x - C(x)$ dollars.

iii. At the production level x that provides the maximal profit, $P'(x) = R'(x) - C'(x) = 0$. Solving $P'(x) = 6652 - C'(x) = -0.000012x^2 + 0.002x + 3852 = 0$ with the quadratic formula, gives us $x = \frac{-0.002 \pm \sqrt{(0.002)^2 + (0.000048)(3852)}}{-0.000024} = \frac{-0.002 \pm 0.43}{-0.000024}$. Since x is positive, -0.43 is the only option and we get $x = \frac{0.432}{0.000024} = 18{,}000$ units as the production level that yields the maximum profit. (Considering $P(x)$ as an abstract function and observing that $P'(0) = 3852 > 0$ and $P'(x) < 0$ for x very large, tells us that P has a maximum at $x = 18{,}000$.) The average cost per unit at a production level of $x = 18{,}000$ units is $\frac{C(18{,}000)}{18{,}000} = \frac{0.000004(18{,}000)^3 - 0.001(18{,}000)^2 + 2800(18{,}000) + 7{,}576{,}000}{18{,}000} = \frac{80{,}980{,}000}{18{,}000} = 4499$ dollars.

iv. Since $P(18{,}000) = 6652(18{,}000) - C(18{,}000) = 119{,}736{,}000 - 80{,}980{,}000 = 38{,}756{,}000$, the maximum profit is 38,756,000 dollars.

8.12. Note that $CF = a = 8$ and $BC = c = 10$. Therefore $EC = x = \frac{a}{4c}\big(a + \sqrt{a^2 + 8c^2}\big) = \frac{8}{4 \cdot 10}\big(8 + \sqrt{8^2 + 8(10^2)}\big) \approx 7.48$. So $EF^2 = 8^2 - x^2 \approx 8^2 - 7.48^2 \approx 8.05$ and $EF \approx 2.84$. Similarly, $BF^2 = BE^2 + EF^2 \approx (10 - x)^2 + 8.05 \approx (10 - 7.48)^2 + 8.05 \approx 14.40$, so that $BF \approx 3.79$. It follows that $\tan\theta_1 = \frac{EF}{BE} \approx \frac{2.84}{10 - 7.48} \approx 1.13$, so that $\theta_1 = \tan^{-1} 1.13 \approx 48.49°$. In the same way, $\tan\theta_2 = \frac{EF}{x} \approx \frac{2.84}{7.48} \approx 0.38$, and therefore, $\theta_2 = \tan^{-1} 0.38 \approx 20.81°$. (The inverse trig functions will be discussed in Section 9.9.1. For now, regard $\tan^{-1}$ as the button on a calculator that tells us for a given number $\tan\theta$ what the corresponding angle θ is.)

8.13. Since $W = 150$ pounds, we know that $T_1 = W = 150$ pounds. Since $T_2 \cos\theta_2 = T_1 \cos\theta_1$, $T_2 = 150\,\frac{\cos\theta_1}{\cos\theta_2} \approx 150\,\frac{\cos 48.49°}{\cos 20.81°} \approx 106.35$ pounds.

8.14. Recall that $a < c$. So if ΔBCF is isosceles, then either $BF = c$ or $BF = a$. Suppose that $BF = c$. It follows that $c^2 = BE^2 + EF^2 = (c - x)^2 + (a^2 - x^2)$. So $c^2 = c^2 - 2cx + x^2 + a^2 - x^2$, hence $2cx = a^2$, and therefore $x = \frac{a^2}{2c}$. Since we know that $x = \frac{a}{4c}\big(a + \sqrt{a^2 + 8c^2}\big) = \frac{a^2}{4c} + \frac{a}{4c}\sqrt{a^2 + 8c^2}$, it follows that $\frac{a}{4c}\sqrt{a^2 + 8c^2} = \frac{a^2}{4c}$. Therefore, $\sqrt{a^2 + 8c^2} = a$, but this is clearly impossible.

So if ΔBCF is isosceles, then $BF = a$. So the angles at B and C in Figure 8.3 are equal. This means that the triangles ΔBEF and ΔCEF are congruent. Hence $c - x = BE = EC = x$, and therefore $x = \frac{c}{2}$. Since $x = \frac{a}{4c}(a + \sqrt{a^2 + 8c^2})$, it follows that $\frac{c}{2} = \frac{a}{4c}(a + \sqrt{a^2 + 8c^2})$. By easy algebra, $2c^2 - a^2 = a\sqrt{a^2 + 8c^2}$ and $\frac{2c^2}{a} - a = \sqrt{a^2 + 8c^2}$. By squaring both sides, $\frac{4c^4}{a^2} - 4c^2 + a^2 = a^2 + 8c^2$. So $\frac{4c^4}{a^2} = 12c^2$, hence $\frac{c^2}{a^2} = 3$, so that finally $a = \frac{c}{\sqrt{3}}$. A second argument goes as follows. Since $BF = a$, the angles θ_1 and θ_2 are equal. It follows from Section 8.2.2 that $2\tan\theta_1 = \tan\theta_1 + \tan\theta_2 = \frac{1}{\cos\theta_1}$. So $2\,\frac{\sin\theta_1}{\cos\theta_1} = \frac{1}{\cos\theta_1}$ and therefore, $2\sin\theta_1 = 1$. So $\sin\theta_1 = \frac{1}{2}$ and hence $\theta_2 = \theta_1 = 30°$. Applying the law of sines to the triangle ΔBCF, shows that $\frac{\sin 120°}{c} = \frac{\sin 30°}{a}$ and hence that $a = \frac{c}{\sqrt{3}}$.

8.15. The assumption that the angle at F is a right angle and the Pythagorean theorem together imply that $BF^2 + a^2 = c^2$ and hence that $(c-x)^2 + (a^2 - x^2) + a^2 = c^2$. So $c^2 - 2cx + 2a^2 = c^2$. So $2cx = 2a^2$ and $x = \frac{a^2}{c}$. Therefore, $\frac{a^2}{c} = \frac{a}{4c}(a + \sqrt{a^2 + 8c^2})$. It follows that $4a = a + \sqrt{a^2 + 8c^2}$, hence $9a^2 = a^2 + 8c^2$, and therefore $a = c$. But this was ruled out early in the analysis of L'Hospital's pulley problem.

8.16. The pull T_1 of cable CA at C has horizontal component $T_1\cos\alpha$ and vertical component $T_1\sin\alpha$. Similarly, the pull T_2 of cable CB at C has horizontal component $T_2\cos\beta$ and vertical component $T_2\sin\beta$. The assumption that this weight-cable system is in equilibrium tells us that the horizontal forces at C and the vertical forces at C are in balance. Therefore

$$T_1\cos\alpha = T_2\cos\beta \quad \text{and} \quad T_1\sin\alpha + T_2\sin\beta = W.$$

Since $T_2 = T_1\,\frac{\cos\alpha}{\cos\beta}$, we get $T_1\sin\alpha + T_1\,\frac{\cos\alpha}{\cos\beta}\sin\beta = W$. It follows that $T_1\big(\sin\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta\big) = W$. Hence T_1 and T_2 can be expressed in terms of α, β, and W as follows

$$T_1 = \frac{W}{\big(\sin\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta\big)} \text{ and } T_2 = \frac{\cos\alpha}{\cos\beta}\,\frac{W}{\big(\sin\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta\big)}.$$

8.17. Since $\sin\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\beta}$ and $\cos\beta\big(\sin\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta\big) = \cos\beta\sin\alpha + \cos\alpha\sin\beta$, we get by inserting the law of sines and a little algebra into the conclusion of Problem 8.16, that

$$T_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)} \quad \text{and} \quad T_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)}$$

as required. If the weight W is attached to a pulley wheel at C that is free to move along the cable, then the tensions T_1 and T_2 in the cable segments AC and CB are free to adjust until they are equal. Once $T_1 = T_2$ is achieved, the equality $T_1\cos\alpha = T_1\cos\beta$ tells us that $\cos\alpha = \cos\beta$. The graph of the cosine function (see Figure 4.24) and fact that the angles α and β are both between $0°$ and $90°$ imply that $\alpha = \beta$. So the triangle ΔABC is isosceles and $AC = CB$.

8.18. With $W = 160$ pounds, $\alpha = 10°$ and $\beta = 5°$,

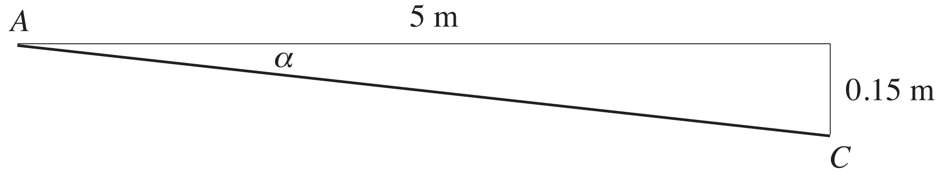
$$T_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)} = \frac{160\cos 5°}{\sin 15°} \approx 615.84 \quad \text{and} \quad T_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)} = \frac{160\cos 10°}{\sin 15°} \approx 608.80$$

both in pounds. With $W = 200$ pounds, $\alpha = 4°$ and $\beta = 2°$,

$$T_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)} = \frac{200\cos 2°}{\sin 6°} \approx 1912.19 \quad \text{and} \quad T_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)} = \frac{200\cos 4°}{\sin 6°} \approx 1908.69$$

in pounds. The message is that a modest weight W, when suspended as Figure 8.46 illustrates, can generate great tensions in the cable that carries it, if the angles α and β that the cable makes with the horizontal are small.

8.19. The figure below highlights the relevant geometry of the segment AC of the cable in the situation the problem describes. Observe that $\tan\alpha = \frac{0.15}{5}$, so that $\alpha = \tan^{-1}\left(\frac{0.15}{5}\right) \approx 1.72°$.



With β the analogous angle at the point B, $\tan\beta = \frac{0.15}{15}$, so that $\beta = \tan^{-1}\left(\frac{0.15}{15}\right) \approx 0.57°$. It follows from one of the formulas of Problem 8.17 that the ratio of the tension T in the cable segment AC to the weight W of the clown is

$$\frac{T}{W} = \frac{\cos\beta}{\sin(\alpha+\beta)} \approx \frac{\cos 0.57°}{\sin 2.29°} \approx 25.$$

8.20. Let T be the tension in the string. Consider the right triangle formed by the string, the vertical wall, and the horizontal segment from the center of the sphere to the wall. Using it, we see that the horizontal and vertical components of T are $T\sin\alpha$ and $T\cos\alpha$, respectively.

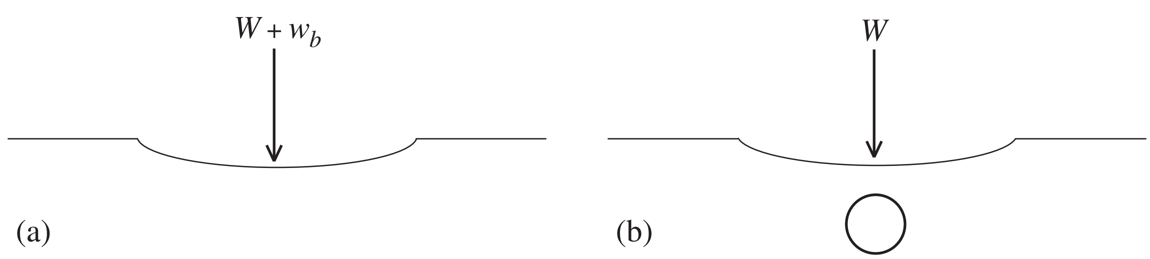
i. The vertical component $T\cos\alpha$ of the tension counterbalances the weight of the sphere. Since the weight of the sphere is $(9.8)(12) = 117.6$ newtons, $T\cos\alpha = 117.6$ and hence $T = \frac{117.6}{\cos\alpha}$ newtons.

ii. This force balances the horizontal component $T\sin\alpha$ of the tension T. So is equal to $F = T\sin\alpha = \frac{117.6}{\cos\alpha}(\sin\alpha) = 117.6\tan\alpha$ newtons.

iii. $T = \frac{117.6}{\cos 15°} \approx 121.75\,\text{N}$ and $F = 117.6\tan 15° \approx 31.51\,\text{N}$.

The fact that the diameter of the sphere is 20 cm plays no role in the solution.

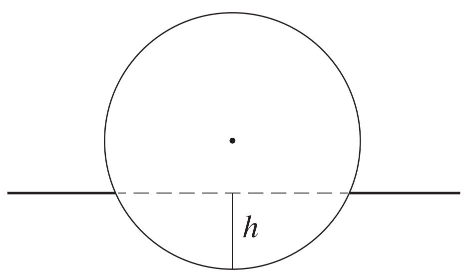
8.21. The figures below show cross sections of the surface of the water and the submerged part of the boat. The weight of the boat is denoted by W. The weight and volume of the bowling ball are denoted by w_b and v_b, respectively. In figure (a) the bowling ball is on board and in figure (b) it is shown overboard and under water. Let V_1 be the volume added to the volume

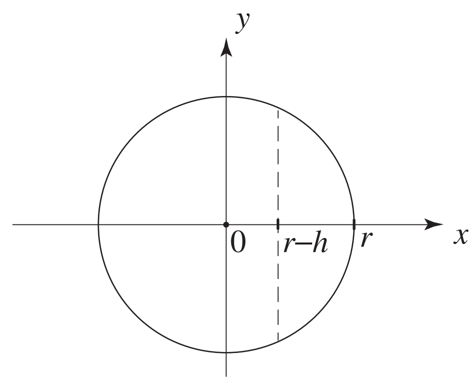


of water in the lake in situation (a) and let V_2 be the volume added to the lake in situation (b). In each case, the added volume is equal to the volume of water displaced. We'll use the units pounds and cubic feet. Archimedes's law of hydrostatics tells us that the buoyant force on the boat (that balances its full weight) is equal to the weight of the water that the boat displaces.

Applied in case (a) this tells us that $V_1 \cdot 62.5 = W + w_b$. So $V_1 = \frac{W+w_b}{62.5}$. In case (b) it tells us that $V_2 = \frac{W}{62.5} + v_b$. Because the ball is sinking, the buoyant force of $62.5v_b$ pounds on the bowling ball is less than its weight w_b. So $v_b < \frac{w_b}{62.5}$. Therefore $V_1 = \frac{W}{62.5} + \frac{w_b}{62.5} > \frac{W}{62.5} + v_b = V_2$. It follows that more water is displaced in situation (a) than is situation (b) so that the water level of the lake will drop after the ball is thrown overboard. So Marilyn had it right.

8.22. By Archimedes's law, the weight of the basketball is equal to the weight of the water that it displaces. To compute the weight of the water that is displaced, we need to compute the volume of the water that is displaced. Let h be the vertical distance from the bottom of the submersed part of the ball to the water surface. Notice that this volume is obtained by rotating one revolution about the x axis the part of the circle of radius r centered at the origin





that lies over the interval $[r-h, r]$. See the two figures above. As in previous situations, the units are pounds and feet. By the volume formula of Section 5.9 this is equal to

$$\int_{r-h}^{r} \pi f(x)^2 dx$$

with $f(x) = \sqrt{r^2 - x^2}$, the function whose graph is the top half of the circle. Observe that

$$\begin{aligned}
\int_{r-h}^{r} \pi(r^2 - x^2)\, dx &= \pi \int_{r-h}^{r} (r^2 - x^2)\, dx = \pi(r^2 x - \tfrac{x^3}{3}) \Big|_{r-h}^{r} \\
&= \pi\big(r^3 - \tfrac{r^3}{3} - (r^2(r-h) - \tfrac{(r-h)^3}{3})\big) \\
&= \pi\big(- \tfrac{r^3}{3} + r^2 h + \tfrac{1}{3}(r^3 - 3r^2 h + 3rh^2 - h^3)\big) \\
&= \pi\big(r^2 h - r^2 h + rh^2 - \tfrac{1}{3}h^3\big) = \pi\big(rh^2 - \tfrac{1}{3}h^3\big).
\end{aligned}$$

It follows that the weight of the volume of water that the ball displaces is

$$62.5\pi\big(rh^2 - \tfrac{1}{3}h^3\big) \text{ pounds.}$$

Since this is equal to the weight of the basketball, we get that $62.5(\pi rh^2 - \frac{\pi}{3}h^3) = 1.3$ or, equivalently, that $\pi rh^2 - \frac{\pi}{3}h^3 = 0.0208$. Inserting $r = 0.39$ and rounding off to three decimal accuracy gives us

$$1.047h^3 - 1.225h^2 + 0.021 = 0.$$

We now let $f(x) = 1.047x^3 - 1.225x^2 + 0.021$ and apply Newton's method (segment 7O of the Problems and Projects section of Chapter 7) to find the h with $f(h) = 0$ that we need.

Notice that $f'(x) = 3.141x^2 - 2.450x$. Let's start with the guess $c_1 = 0.39$, the radius of the circle. With this c_1 we get

$$c_2 \;=\; 0.39 - \tfrac{f(0.39)}{f'(0.39)} \;=\; 0.39 - \tfrac{0.103216}{0.477754} \;=\; 0.173956.$$

Next,

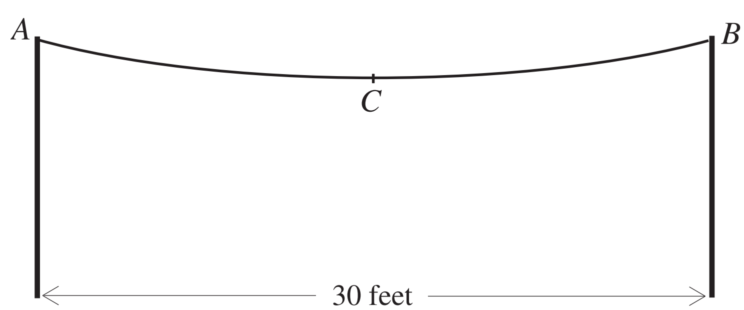
$$\begin{aligned} c_3 &= 0.173956 - \tfrac{f(0.173956)}{f'(0.173956)} = 0.173956 - \tfrac{0.010558}{0.331143} = 0.142072, \text{ and} \\ c_4 &= 0.142072 - \tfrac{f(0.142072)}{f'(0.142072)} = 0.142072 - \tfrac{0.000724}{0.284678} = 0.139530. \end{aligned}$$

Plugging $x = 0.139530$ into $f(x)$ gives us $f(0.139530) = 0.000005 \approx 0$. Rounding off to the two significant figures that parallels the data gives $h = 0.14$.

If you have any energy left, you can check that the convergence of Newton's method in this example follows the second case of Problem 7.91i.

The theory of the suspension bridge relies on the crucial assumption that the vertical load at any point on the bridge is constant over the length of the bridge. In the theory of Section 8.3, w is taken to be the maximum weight (dead load plus maximum live load) per foot that the bridge needs to support distributed over the number of cables. This is the w relevant for determining the greatest loads, tensions, and compressions that the structure is subject to. However, the analysis of the bridge applies to any w so that w could be the weight per foot (distributed over the number of cables) in the situation of zero live load, maximum live load, or anything in between. The theory applies to any situation of a cable under constant vertical load (under the assumption that the cable is completely flexible—there is no resistance to bending it—and inelastic—it does not get longer when stretched).

8.23. The figure below shows the clothesline that has been described. The birds weigh a total of $50 \cdot \frac{14}{16} = 43.75$ pounds. So the vertical load on the clothesline is $w = \frac{43.75}{30} \approx 1.458$ pounds per foot. Taking $d = 15$ feet to be half the distance between the two supporting poles and $s = 0.5$ feet, we get by applying two formulas from Section 8.3, that the maximum and minimum



tensions in the line are

$$T_d = wd\sqrt{\left(\tfrac{d}{2s}\right)^2 + 1} \approx (1.458)(15)\sqrt{(\tfrac{15}{1})^2 + 1} \approx 328.78 \text{ pounds and}$$

$$T_0 = \tfrac{1}{2}\tfrac{wd^2}{s} \approx \tfrac{1}{2}\tfrac{(1.458)(15^2)}{\frac{1}{2}} \approx 328.10 \text{ pounds.}$$

8.24. Our peace corps volunteer begins by thinking about the dead and live loads that the bridge would have to support. He knows that 60 boards are needed for the footpath so that their total weight would be 600 pounds. He also knows that the two heavy ropes that will carry the bridge need to be secured several feet beyond the edge of the gorge so that they would each be about 110 feet long. Knowing that the heavy rope that he used for his fitness workouts back in the U.S. weighs close to 1 pound per foot, he estimated that the ropes would add about 220 pounds to the dead load of the bridge. Adding this to the weight of the 60 boards and an estimated $4(60) = 240$ pounds for the vertical cords that will hold them, he approximates the dead load of the bridge to be $220+10(60)+4(60) = 1060$ pounds. Given what he knows about the carrying capacity that the bridge is to have, the peace corps volunteer gets the estimate of $w = \frac{1060+3900}{2\cdot 90} \approx \frac{4960}{180} \approx 28$ pounds/foot per heavy rope for the dead plus maximum live load of the bridge. He estimates the sag of the span to be 10 feet. Knowing that half the span is $d = \frac{90}{2} = 45$ feet, he can apply the formula $T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2+1}$ to derive the estimate $T_d = (28)(45)\sqrt{(\frac{45}{20})^2+1} \approx 3100$ pounds for the maximum tension that the heavy ropes would be subjected to. Stipulating that these ropes would have a circular cross-section of radius about 2 inches, our peace corps volunteer—with safety factor considerations in mind—recommended that the ultimate strength of these ropes should exceed $\frac{3100}{4\pi} \approx 250$ pounds per square inch by a factor of about two.

8.25. The information implies that $w = \frac{8680+4000}{2} = \frac{12{,}680}{2} = 6340$, $d = 1400$, and $s = 280$ with respect to the units pounds and feet. The maximum tension on a main cable is $T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2+1} \approx (6340)(1400)\sqrt{\left(\frac{1400}{560}\right)^2+1} \approx 23{,}900{,}000$ pounds. Since $\tan\alpha = \frac{2s}{d} = \frac{560}{1400} = 0.4$, we get $\alpha = \tan^{-1} 0.4 \approx 21.80°$. The compression by one cable over the center span on each tower is $T_d \sin\alpha \approx (23{,}900{,}000)\sin 21.80° \approx 8{,}875{,}000$ pounds.

8.26. We know that the dead load per cable is $\frac{20{,}170}{2} = 10{,}085$ pounds per foot and that the dead plus live load per cable is $\frac{20{,}170+4000}{2} = \frac{20{,}170+4000}{2} = \frac{24{,}170}{2} = 12{,}085$ pounds per foot. From the given, $d = 2100$, and $s = 470$. So under dead load only,

$$T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2+1} \approx (10{,}085)(2100)\sqrt{\left(\frac{2100}{940}\right)^2+1} \approx 51{,}800{,}000 \text{ pounds.}$$

In the same way, for the dead plus live loads

$$T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2+1} \approx (12{,}085)(2100)\sqrt{\left(\frac{2100}{940}\right)^2+1} \approx 62{,}100{,}000 \text{ pounds.}$$

The 54,000,000 and 64,100,000 pounds for the tensions listed in the statement of the problem refer to the values that predate the structural work on the Golden Gate that lightened its deck.

8.27. Notice that $w = \frac{37{,}000+4800}{4} = 10{,}450, d = 2130$, and $s = 385$. Thus, $T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2+1} \approx 6.55 \times 10^7$ pounds. The angle α satisfies $\tan\alpha = \frac{2s}{d} \approx 0.36$, so that $\alpha \approx \tan^{-1} 0.36 \approx 19.88°$. The compression produced by all four cables is $4\,T_d \sin\alpha \approx 8.91 \times 10^7$ pounds.

8.28. Since $d = \frac{1990}{2} \approx 995$ m and $s \approx 201$ m, we get that $\tan\alpha = \frac{2s}{d} \approx \frac{402}{995} \approx 0.40$, and hence $\alpha \approx \tan^{-1} 0.40 \approx 22.0°$. The information given implies that the maximal tension T_d satisfies $4T_d \sin\alpha \approx 980{,}000{,}000$ newtons, and hence that

$$T_d \approx \frac{980{,}000{,}000}{4\sin 22^\circ} \approx 654{,}000{,}000 \text{ newtons}.$$

The formula $T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2 + 1}$ implies that $w \approx 245{,}000$ N/m.

8.29. Since $\theta(t) = \frac{t^3}{125} + \frac{t}{5}$, it follows that the angular velocity and angular acceleration at any time t are $\omega(t) = \theta'(t) = \frac{3}{125}t^2 + \frac{1}{5}$ and $\alpha(t) = \theta''(t) = \frac{6}{125}t$, respectively. Note that $\theta(10) = \frac{1000}{125} + 2 = 10$ radians and that the average angular velocity from time $t = 0$ to time $t = 10$ is $\frac{\theta(10)-\theta(0)}{10} = \frac{1}{10}((\frac{1000}{125} + 2) - 0) = \frac{10}{10} = 1$ radian per second. At the instant $t = 10$, the angular velocity is $\omega(10) = \frac{300}{125} + \frac{1}{5} = 2.6$ radians per second. The average acceleration from time $t = 0$ to time $t = 10$ is $\frac{\omega(10)-\omega(0)}{10} = \frac{1}{10}((\frac{300}{125} + \frac{1}{5}) - \frac{1}{5}) = \frac{30}{125} = 0.24$ radians per second^2. Finally, the angular acceleration at $t = 10$ is $\alpha(t) = \frac{60}{125} = 0.48$ radians per second^2.

8.30. **i.** Refer to Figure 8.50b. The circle on which the graph of the function $y = f_1(x)$ lies has radius r and center $(-\frac{r}{2}, 0)$. A look at the graph tells us the following. Since the segment $[-\frac{r}{2}, \frac{r}{2}]$ on the x-axis is a radius of the circle, the tangent to the circle at the point $(\frac{r}{2}, 0)$ is vertical. In addition, the slopes of the tangents to the graph of $y = f_1(x)$ are all negative and the slope at the point $(0, f_1(0))$ is the smallest of these negative numbers. The equation of the circle is $(x + \frac{r}{2})^2 + y^2 = r^2$ and hence $y^2 = r^2 - (x^2 + rx + (\frac{r}{2})^2)$. Solving for y with $y \geq 0$, we get $y = \sqrt{\frac{3}{4}r^2 - x^2 - rx}$. So $f_1(x) = (\frac{3}{4}r^2 - x^2 - rx)^{\frac{1}{2}}$ with $0 \leq x \leq \frac{r}{2}$. It follows that $f_1'(x) = \frac{1}{2}(\frac{3}{4}r^2 - x^2 - rx)^{-\frac{1}{2}}(-2x - r) = \frac{-x-\frac{r}{2}}{(\frac{3}{4}r^2-x^2-rx)^{\frac{1}{2}}}$. Taking $x = 0$ tells us that the slope of the tangent of the graph of $y = f_1(x)$ at the point $(0, f_1(0)) = (0, \frac{\sqrt{3}}{2}r)$ is $\frac{-\frac{r}{2}}{\frac{\sqrt{3}}{2}r} = \frac{-1}{\sqrt{3}}$. The assertion about the bounds on the derivative $f_1'(x)$ has now been verified.

Let's consider the function $y = f_2(x)$ with domain $[\frac{-r}{2}, \frac{r}{2}]$ next. Its graph lies on the circle of radius r and center the point $(0, f_1(0)) = (0, \frac{\sqrt{3}}{2}r)$. A look at Figure 8.50b tells us that the graph of $y = f_2(x)$ has its smallest (negative) slope at $x = \frac{-r}{2}$ and its largest slope at $x = \frac{r}{2}$. The equation of the circle is $x^2 + (y - \frac{\sqrt{3}}{2}r)^2 = r^2$. So $(y - \frac{\sqrt{3}}{2}r)^2 = r^2 - x^2$ and hence $y - \frac{\sqrt{3}}{2}r = \pm\sqrt{r^2 - x^2}$. The fact that $y \leq 0$ for all y-coordinates of $y = f_2(x)$, tells us that $y = \frac{\sqrt{3}}{2}r - \sqrt{r^2 - x^2}$. So $f_2(x) = \frac{\sqrt{3}}{2}r - (r^2 - x^2)^{\frac{1}{2}}$ with $\frac{-r}{2} \leq x \leq \frac{r}{2}$. Using the chain rule, we get $f_2'(x) = -\frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x) = x(r^2 - x^2)^{-\frac{1}{2}}$. So the slope of the graph at the point $(\frac{-r}{2}, 0)$ is $\frac{-r}{2}(r^2 - \frac{r^2}{4})^{-\frac{1}{2}} = \frac{\frac{-r}{2}}{(\frac{3r^2}{4})^{\frac{1}{2}}} = \frac{-1}{\sqrt{3}}$ and the slope of the graph at the point $(\frac{r}{2}, 0)$ is $\frac{1}{\sqrt{3}}$. This verifies the asserted bounds on $f_2'(x)$. The remaining case of the function $f_3(x)$ is very similar to that of $f_1(x)$.

ii. Observe that the slopes of the tangents to the graph of $f_1(x)$ decrease over its domain $[0, \frac{r}{2}]$ because they get more and more negative for increasing x. (This conclusion is confirmed by the fact—easily established—that $f_1''(x)$ is negative.) The slopes of the tangents to the graph of $f_2(x)$ increase over $[\frac{-r}{2}, \frac{r}{2}]$ because for increasing x the slopes become less negative and (after $x = 0$) more and more positive. (The fact that $f_2''(x)$ is positive confirms this.) For the graph of $f_3(x)$ over $[\frac{-r}{2}, 0]$, the slopes are positive but decreasing for increasing x. These observations in combination with the inequalities of

(i) tell us that no two tangent lines to the curve of Figure 8.50b have the same slope. Therefore no matter where the point P is chosen, the parallel line L' cannot be tangent to the curve but must go through one of the vertices. So the point P' is a vertex. Now turn to Figure 8.51 and observe that since L' touches the curve at only one point, this vertex cannot be an endpoint of the arc on which P lies. It follows that P' is the center of the circle of the arc on which P lies.

iii. It follows from part (ii) that $P'P$ is a radius of the circle on which P lies. Since L is tangent to the circle at P, this radius is perpendicular to L. Note also that the distance between P and P' is r.

8.31. Since $A'B = A'C$, $BA = CA$, and the angle $\angle A'BA$ is equal to the angle at C, it follows that the triangles $\Delta AA'B$ and $\Delta AA'C$ are congruent. It follows that $\angle BA'A = \angle CA'A$ so that both angles must be right angles, and also that $\angle BAA' = \angle CAA'$ so that AA' bisects $\angle BAC$.

8.32. Let's denote by O the center of the figure depicted as rolling on the horizontal plane of Figure 8.52. Considering Figures 8.52 and 8.53a together tells us that the distance from O to this plane is least when the triangle within the rolling figure is positioned as in Figure 8.53a and greatest when this triangle is obtained from Figure 8.53a by a 180° rotation. To compute the greatest distance from O to the horizontal plane, we need to compute the distance CM of Figure 8.53a. Note that $CM = AM = \frac{2}{3}AA'$. (See the information provided for Problem 8.31.) By the Pythagorean theorem, $(AA')^2 + (A'B)^2 = (AA')^2 + 1 = (AB)^2 = 2^2$ and hence that $AA' = \sqrt{3}$. So $CM = \frac{2}{3}\sqrt{3}$. When O is at its lowest point, its distance above the horizontal plane is $2 - \frac{2}{3}\sqrt{3}$.

The next several problems rely on results and formulas developed in Sections 8.4.1, 8.4.2, and 8.4.3.

8.33. **i.** The magnitude of the force that drives the ice cube is $mg\sin\beta = 0.25(9.81)\sin 15° \approx 0.625$ N and that driving the ball is $\frac{5}{7}mg\sin\beta \approx \frac{5}{7}(0.625) \approx 0.453$ N.

ii. The acceleration of the ice cube is $g\sin\beta = (9.81)\sin 15° \approx 2.539$ m/s^2 and that of the ball is $\frac{5}{7}g\sin\beta = \frac{5}{7}(2.539) \approx 1.814$ m/s^2.

iii. We'll assume that the ball starts from rest at the top of the 5 m long plane. With h the plane's height, $\sin\beta = \sin 15° = \frac{h}{5}$, so that $h = 5\sin 15° \approx 1.294$ m. The ball's velocity at the bottom of the plane is $v(t_b) = \sqrt{\frac{10}{7}gh} \approx \sqrt{\frac{10}{7}(9.81)(1.294)} \approx 4.258$ m/s. Since $r = 0.06$ meters is the radius of the ball, its angular velocity at the bottom of the plane is $w = \frac{v}{r} \approx \frac{4.258}{0.06} \approx 70.967$ radians/second. Since 1 revolution corresponds to 2π radians, this is equivalent to $\frac{70.967}{2\pi} \approx 11.295$ revolutions per second.

iv. The time at which the rolling ball reaches the bottom of the plane is $t_b = \sqrt{\frac{14h}{5g}\frac{1}{\sin\beta}} \approx \sqrt{\frac{14(1.294)}{5(9.81)}\frac{1}{\sin 15°}} \approx 2.348$ seconds. By using (ii), we get that after time t the velocity of the ice cube (also assumed to start from rest at the top of the plane) is $v(t) = (g\sin\beta)t = (9.81)(\sin 15°)t$ m/s and, in turn, that its distance from the top of the plane is $p(t) = \frac{1}{2}(g \sin\beta)t^2 = \frac{1}{2}(9.81)(\sin 15°)t^2$ meters. Setting $\frac{1}{2}(9.81)(\sin 15°)t^2 = 5$ and solving for t, we get $t = \sqrt{\frac{2\cdot 5}{9.81 \sin 15°}} \approx 1.985$ seconds for the time it takes for the ice cube to reach the bottom.

v. At the instant $t \approx 1.985$ seconds at which the ice cube reaches the bottom, the ball is $p(1.985) = \big(\frac{5g}{14}\sin\beta\big)1.985^2 \approx 3.573$ meters down the plane. So the ice cube wins by about 1.4 meters.

8.34. As before, let the ball start at the top of the inclined plane at time $t = 0$. Consider the rolling ball at any time t thereafter and refer to Section 8.4.3.

i. Since $v(t) = \big(\frac{5g}{7}\sin\beta\big)t$ and $\sqrt{p(t)} = \sqrt{\frac{5g}{14}\sin\beta}\, t$, we get $\frac{v(t)}{\sqrt{p(t)}} = \frac{\frac{5g}{7}\sin\beta}{\sqrt{\frac{5g}{14}\sin\beta}}$ so that $v(t)$ and $\sqrt{p(t)}$ are proportional with $\frac{\frac{5g}{7}\sin\beta}{\sqrt{\frac{5g}{14}\sin\beta}} = \frac{5g}{7}\sqrt{\frac{14}{5g}}\sqrt{\sin\beta} = \frac{\sqrt{10g}}{\sqrt{7}}\sqrt{\sin\beta}$ the constant of proportionality.

ii. Combining formulas, we get that $\frac{p(t)-p(0)}{t} = \frac{p(t)}{t} = \big(\frac{5g}{14}\sin\beta\big)t = \frac{1}{2}v(t)$.

iii. Let $h(t)$ be the amount of the vertical drop of the ball during its motion from $t = 0$ to t. Note that $\sin\beta = \frac{h(t)}{p(t)}$ and $p(t) = (\frac{5g}{14}\sin\beta)t^2$, so that $h(t) = p(t)\sin\beta = (\frac{5g}{14}\sin^2\beta)t^2$. So $\sqrt{h(t)} = \sqrt{\frac{5g}{14}}(\sin\beta)t$ and hence $(\sin\beta)t = \sqrt{\frac{14}{5g}}\sqrt{h(t)}$. It follows that

$$v(t) = (\tfrac{5g}{7}\sin\beta)t = \tfrac{5g}{7}\sqrt{\tfrac{14}{5g}}\sqrt{h(t)} = \sqrt{\tfrac{5^2g^2\cdot 14}{7^2\cdot 5g}}\sqrt{h(t)} = \sqrt{\tfrac{10g}{7}}\sqrt{h(t)}.$$

So $v(t)$ depends only on $h(t)$ (and the gravitational constant g).

iv. Up to now it was assumed that the ball starts from rest. We'll now assume that it starts from the top of the plane with an initial velocity of v_0 and show that its velocity at the bottom of the plane depends on v_0 together with its vertical distance of fall. Starting with the acceleration $a = \frac{5g}{7}\sin\beta$ of the ball, we see that its velocity is $v(t) = (\frac{5g}{7}\sin\beta)t + v_0$ and its position relative to the top of the plane is $p(t) = (\frac{5g}{14}\sin\beta)t^2 + v_0 t$. Since $\sin\beta = \frac{h(t)}{p(t)}$, it follows that $h(t) = p(t)(\sin\beta)$. So $h(t) = \frac{5g}{14}(\sin^2\beta)t^2 + v_0(\sin\beta)t$ and hence $\frac{5g}{14}(\sin^2\beta)t^2 + v_0(\sin\beta)t - h(t) = 0$. By an application of the quadratic formula we get

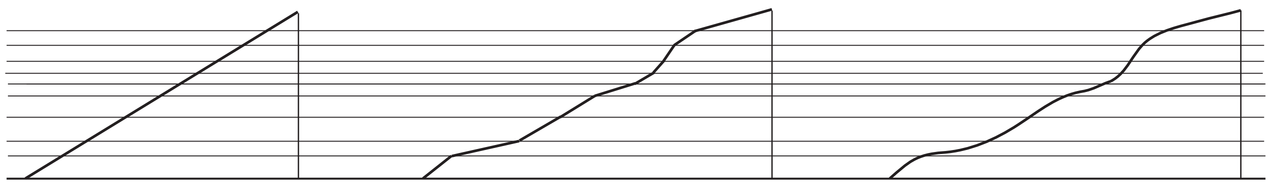
$$(\sin\beta)t = \frac{-v_0 \pm \sqrt{v_0^2 + 4\cdot\frac{5g}{14}\cdot h(t)}}{\frac{5g}{7}}.$$

Since $\sin\beta$ and t are both positive, the $+$ option applies and it follows that $(\frac{5g}{7}\sin\beta)t = \sqrt{v_0^2 + \frac{10g}{7}h(t)} - v_0$. After inserting this into the earlier formula for $v(t)$ we see that

$$v(t) = (\tfrac{5g}{7}\sin\beta)t + v_0 = \Big(\sqrt{v_0^2 + \tfrac{10g}{7}h(t)} - v_0\Big) + v_0 = \sqrt{v_0^2 + \tfrac{10g}{7}h(t)}.$$

The assertion made in part (iv) of the problem is a direct consequence of this formula.

8.35. Consider the ramps of Figures (a), (b), and (c). Figure (a) is the ramp on the left of Figure 8.55 and Figure (c) is the curving ramp on the right of Figure 8.55. The ramp (b) in the middle is composed of a sequence of straight, slanting segments that are determined by parallel lines chosen in such a way that the sequence of segments closely approximates the curving ramp (c). Given this close approximation, it follows that a ball rolling down from rest on ramp (b) will develop approximately the same speed at the bottom as the ball rolling down from rest on the curving ramp (c). By dividing the curve (c) more and more finely, the approximation of ramp (c) by ramp (b)—and hence the approximation of the two speeds at the bottom of the two ramps—can be made as tight as one would like. Focusing on ramp (b), notice that each small slanting segment and the two parallel lines that define it, determine a small inclined plane on ramp (a). Since the parallel lines are horizontal, the heights of corresponding inclined planes



on (b) and (a) are the same. By applying part (iv) of Problem 8.34 to each pair of corresponding small inclined planes from the top down, it follows that two balls starting from rest at the top of ramps (b) and (a) will reach the respective bottoms with the same final speed. We can now conclude that that if two balls start from the top of the ramps of Figures (a) and (c) from rest, then they will reach the bottom of these ramps with the same final speed.

8.36. Regard the mirror to be an inclined plane as depicted in Figure 8.56 and let β be the angle of inclination. Label the axis pointing across the rising mirror as x-axis and the one pointing up in the direction of the rise as y-axis. As was done in Section 6.7 we'll conceive of the motion of the ball to occur in components along the x- and y-axes separately. Let the motion of the ball start at time $t = 0$ and let the position of the ball be $(x(t), y(t))$ at any time $t \geq 0$ into its motion. We know that the acceleration of the ball in the x-direction is $x''(t) = 0$ and Section 8.4.3 tells us that the acceleration due to gravity in the y-direction is $y''(t) = -\frac{5}{7}g\sin\beta$. As in Figure 6.22, let φ_0 be the angle between the initial direction of the motion and the x-axis and let y_0 be the y-coordinate of starting point of the ball. (Notice that for the path of the ball drawn in Figure 8.56, φ_0 is close to 90° and y_0 is close to 0.) Arguing exactly as in the development of equation (6c) in Section 6.7, we get that

$$y(t) = \frac{-\frac{5}{7}g\sin\beta}{2v_0^2\cos^2\varphi_0}\,x(t)^2 + (\tan\varphi_0)x(t) + y_0.$$

So the path of the ball is indeed parabolic.

8.37. Let the point of intersection P have coordinates (x_0, y_0) and notice that $x_0^2 + (y_0 + r)^2 = r^2$ and $y_0 = mx_0$. Therefore $x_0^2 + (mx_0 + r)^2 = r^2$ and $(1 + m^2)x_0^2 + 2mrx_0 = 0$. It follows that $x_0 = \frac{-2mr}{1+m^2}$ and $y_0 = \frac{-2m^2r}{1+m^2}$. The distance between P and O is

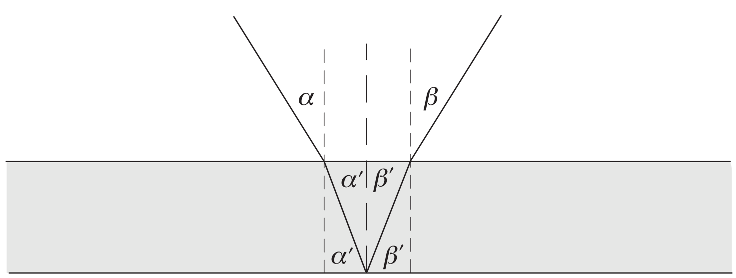
$$\sqrt{\left(\tfrac{-2mr}{1+m^2} - 0\right)^2 + \left(\tfrac{-2m^2r}{1+m^2} - 0\right)^2} = \sqrt{\tfrac{4m^2r^2}{(1+m^2)^2} + \tfrac{4m^4r^2}{(1+m^2)^2}} = \sqrt{\tfrac{4m^2r^2}{(1+m^2)^2}(1 + m^2)} = \tfrac{2mr}{\sqrt{1+m^2}}.$$

8.38. Note that the height of the inclined plane depicted in Figure 8.57b is h. By Section 8.4.3 the ball reaches the bottom of the inclined plane at time $t = \sqrt{\frac{14h}{5g}\frac{1}{\sin\beta}}$. Insert a coordinate system into Figure 8.57b, let $P = (x_0, y_0)$ and let $y = mx$ be the equation of the line on which the segment OP lies. By using conclusions of Problem 8.37, we get $h = -y_0 = \frac{2m^2r}{1+m^2}$ and $\sin\beta = \frac{h}{OP} = \frac{2m^2r}{1+m^2} \cdot \frac{\sqrt{1+m^2}}{2mr} = \frac{m}{\sqrt{1+m^2}}$. Therefore $t = \sqrt{\frac{14}{5g}\frac{2m^2r}{1+m^2} \cdot \frac{\sqrt{1+m^2}}{m}} = \sqrt{\frac{14 \cdot 2r}{5g}} = 2\sqrt{\frac{7r}{5g}}$.

8.39. Place an xy-coordinate system into the vertical plane of Figure 8.58 in such a way that the x-axis is horizontal and the given point O is the origin. Consider any inclined plane in this vertical plane that slants downward from the fixed point O. Let a ball roll down it from O starting at time $t = 0$. At a time $t > 0$ later let the position of the ball—more precisely, the point of contact of the ball with the inclined plane—be P. The intersection of the perpendicular bisector of the segment OP and the y-axis determine the center of a circle on which both O and P lie. (Make use of Problem 1.9.) Let r be the radius of this circle and note that the point $(0, -r)$ on the negative y-axis is its center. By the conclusion of Problem 8.38, the elapsed time t and the radius r are related by the equality $t = 2\sqrt{\frac{7r}{5g}}$. So $t^2 = 4\frac{7r}{5g}$, and hence $r = \frac{5g}{28}t^2$. So the radius r of the circle that the position P of the ball determines depends only on t (and the gravitational constant g). Let a second ball roll down some other inclined plane simultaneously with the first starting at O at time $t = 0$. Observe it again in position P' after the same time $t > 0$ as the first. The equality $r = \frac{5g}{28}t^2$ tells us that the circle that the perpendicular bisector of OP' determines has the same radius and hence the same center as the first. So it is the same circle as the first. It follows that all the balls that are released at the same time from the point O lie at any time t later on the circle of radius $r = \frac{5g}{28}t^2$ and center $(0, -r)$.

8.40. By Table 8.1 the indices of refraction of air and crown glass are $n_A = 1.00029$ and $n_B = 1.52$, respectively. By Snell's law and Figure 8.30, we get that $n_A \sin 30° = n_B \sin\beta$ where $\alpha = 30°$ is the angle of incidence and β is the angle of refraction. Since $\sin 30° = \frac{1}{2}$, it follows that $\sin\beta = \frac{1}{2}(\frac{1.00029}{1.52}) \approx 0.33$. So $\beta \approx \sin^{-1}(0.33) \approx 19.26°$.

8.41. The figure below is a blown-up version of Figure 8.59. The two lower angles of reflection α' and β' are the angles of reflection from the reflective silver coating. Therefore $\alpha' = \beta'$. The equal upper pair of angles $\alpha' = \beta'$ are the angles of refraction corresponding to the angles of incidence α and β, respectively. It follows from Snell's law that $n_A \sin\alpha = n_B \sin\alpha'$ where n_A



and n_B are the indices of refraction of air and the glass of the plate, respectively. By Snell's law once more, $n_A \sin\beta = n_B \sin\beta'$. Since $\alpha' = \beta'$, we see that $n_A \sin\alpha = n_A \sin\beta$. So $\sin\alpha = \sin\beta$. Since the angles α and β are both acute, $\alpha = \beta$.

8.42. The first term of the aspheric lens equation $x = \frac{y^2}{R\left[1+\sqrt{1-(1-k)\frac{y^2}{R^2}}\right]}$ is put into the form $y^2 = 2Rx - (1-k)x^2$ as follows. Since

$$y^2 = xR\big[1 + \sqrt{1-(1-k)\tfrac{y^2}{R^2}}\big] = xR + xR\sqrt{1-(1-k)\tfrac{y^2}{R^2}},$$

we get $y^2 - xR = xR\sqrt{1-(1-k)\frac{y^2}{R^2}}$. After squaring both sides,

$$y^4 - 2xRy^2 + x^2R^2 = x^2R^2\big(1-(1-k)\tfrac{y^2}{R^2}\big) = x^2R^2 - x^2R^2(1-k)\tfrac{y^2}{R^2}.$$

So $y^4 - 2xRy^2 = -x^2R^2(1-k)\frac{y^2}{R^2}$ and hence $y^2 - 2xR = -x^2(1-k)$.

- **i.** Suppose that $k = 0$. So the equation reduces to $x^2 - 2xR + y^2 = 0$. By completing the square, we get $(x-R)^2 + y^2 = R^2$ and this is the required circle.
- **ii.** Suppose that $k \neq 0$ and that (x, y) satisfies both $y^2 = 2Rx-(1-k)x^2$ and $(x-R)^2+y^2 = R^2$. So $R^2 - (x-R)^2 = 2Rx - (1-k)x^2$. After multiplying things out,

  $$R^2 - x^2 + 2Rx - R^2 = 2Rx - x^2 + kx^2,$$

  so that $kx^2 = 0$. It follows that $x = 0$ and since $y^2 - 2xR = -x^2(1-k)$ that $y = 0$.
- **iii.** Take $k > 0$. To show that the graph of $y^2 = 2Rx - (1-k)x^2$ lies completely outside the circle $(x-R)^2 + y^2 = R^2$ except for the point $(0, 0)$, we'll take any point (x_0, y_0) satisfying $y_0^2 = 2Rx_0 - (1-k)x_0^2$ and compute the distance from (x_0, y_0) to the center $(R, 0)$ of the circle. This distance is

  $$\sqrt{(x_0-R)^2 + y_0^2} = \sqrt{(x_0-R)^2 + 2Rx_0 - (1-k)x_0^2} = \sqrt{R^2 + kx_0^2}.$$

  Since $k > 0$, the distance $\sqrt{R^2 + kx_0^2}$ is greater than R unless $x_0 = 0$. It follows that point (x_0, y_0) lies outside the circle, unless $x_0 = 0$ and hence $y_0 = 0$. Notice that the point $(0, 0)$ lies on the circle.
- **iv.** If $k < 0$, then the same argument shows again that the distance between the point (x_0, y_0) and the center $(R, 0)$ of the circle is $\sqrt{R^2 + kx_0^2}$. Since $k < 0$, the distance $\sqrt{R^2 + kx_0^2}$ is now less than R, so that the graph of $y^2 = 2Rx - (1-k)x^2$ lies completely inside the circle $(x-R)^2 + y^2 = R^2$ except for the point $(0, 0)$. It lies on the circle.

8.43. This problem deals with the equation $y^2 = 2Rx - (1-k)x^2$ and its graph. This is a parabola, an ellipse, or a hyperbola as follows:

- **i.** In the case $k = 1$, the graph of $y^2 = 2Rx$ is the parabola with focal point $(\frac{R}{2}, 0)$ and directrix $x = -\frac{R}{2}$. To see this, derive the equation of the parabola with focus $(\frac{R}{2}, 0)$ and directrix $x = -\frac{R}{2}$ from first principles. This is illustrated in Section 4.3. Confirm that the resulting equation is $y^2 = 2Rx$.
- **ii.** Suppose that $k \neq 1$. Write the equation $(1-k)x^2 - 2Rx + y^2 = 0$. After dividing through by $1-k$, $x^2 - \frac{2R}{1-k}x + \frac{y^2}{1-k} = 0$. By completing the square,

  $$x^2 - \tfrac{2R}{1-k}x + (\tfrac{R}{1-k})^2 + \tfrac{y^2}{1-k} = (\tfrac{R}{1-k})^2 \text{ and hence } (x - \tfrac{R}{1-k})^2 + \tfrac{y^2}{1-k} = (\tfrac{R}{1-k})^2.$$

After dividing through by $(\frac{R}{1-k})^2$, we get

$$\frac{(x-\frac{R}{1-k})^2}{\frac{R^2}{(1-k)^2}} + \frac{y^2}{\frac{R^2}{1-k}} = 1.$$

iii. For $k < 0$, the graph is the ellipse with focal points $\left(\frac{R}{1-k}, \pm\frac{\sqrt{-k}R}{1-k}\right)$ and eccentricity $\varepsilon = \sqrt{\frac{-k}{1-k}}$. This can be seen as follows. Since $k < 0$, it follows that $1-k > 0$. So we can write $1-k$ as $(\sqrt{1-k}\,)^2$. Therefore,

$$\frac{(x-\frac{R}{1-k})^2}{\frac{R^2}{(1-k)^2}} + \frac{y^2}{\frac{R^2}{1-k}} = 1$$

becomes the ellipse

$$\frac{(x-\frac{R}{1-k})^2}{\left(\frac{R}{1-k}\right)^2} + \frac{y^2}{\left(\frac{R}{\sqrt{1-k}}\right)^2} = 1.$$

Now let $b = \frac{R}{1-k}$ and $a = \frac{R}{\sqrt{1-k}}$. Since $1-k > 1$, we have that $1-k > \sqrt{1-k}$, so that $a > b$. So a is the semimajor axis and b is the semiminor axis of the ellipse. Observe that this ellipse is obtained from the standard "vertical" ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ of Problem 4.67 and Figure 4.36 by shifting it $b = \frac{R}{1-k}$ units to the right. (See Section 4.2 for a discussion of shifts.) The focal points of $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ are the points $(0, \pm c)$ on the y axis, where $c = \sqrt{a^2-b^2} = \sqrt{\frac{R^2}{1-k} - \frac{R^2}{(1-k)^2}} = \sqrt{\frac{R^2(1-k)-R^2}{(1-k)^2}} = \frac{R\sqrt{-k}}{1-k}$. So the focal points of this standard vertical ellipse are $\left(0, \pm\frac{R\sqrt{-k}}{1-k}\right)$. Shifting things $b = \frac{R}{1-k}$ units to the right tells us that the focal points of the shifted ellipse we are considering are $\left(\frac{R}{1-k}, \pm\frac{R\sqrt{-k}}{1-k}\right)$. The eccentricity is $\varepsilon = \frac{c}{a} = \frac{\frac{R\sqrt{-k}}{1-k}}{\frac{R}{\sqrt{1-k}}} = \frac{R\sqrt{-k}}{1-k} \cdot \frac{\sqrt{1-k}}{R} = \sqrt{\frac{-k}{1-k}}$.

iv. For $k = 0$, the equation reduces to $y^2 + x^2 - 2Rx = 0$. So $(x-R)^2 - R^2 + y^2 = 0$, and hence $(x-R)^2 + y^2 = R^2$. So the graph is the circle of radius R and center $(R, 0)$.

v. For $0 < k < 1$, the graph is the ellipse with focal points $\left(\frac{R}{1\pm\sqrt{k}}, 0\right)$ and eccentricity $\varepsilon = \sqrt{k}$. To see this, go back to the equality derived in (ii). Since $0 < k < 1$, we get $-1 < -k < 0$ (by multiplying through by -1) and hence by adding 1, $0 < 1-k < 1$. Therefore as in the solution of (iii),

$$\frac{(x-\frac{R}{1-k})^2}{\left(\frac{R}{1-k}\right)^2} + \frac{y^2}{\left(\frac{R}{\sqrt{1-k}}\right)^2} = 1,$$

again an ellipse. Since $1-k < 1$, we see that $\sqrt{1-k} > 1-k$. Now let $a = \frac{R}{1-k}$ and $b = \frac{R}{\sqrt{1-k}}$. Notice that $a > b$, so that a is the semimajor axis and b the semiminor axis. This ellipse is obtained by taking the standard "horizontal" ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of Section 4.4 and shifting it $a = \frac{R}{1-k}$ units to the right. (See Section 4.2 for a discussion of shifts.) The focal points of the standard ellipse are the points $(\pm c, 0)$ where $c = \sqrt{a^2-b^2}$.

It follows that the focal points of the shifted ellipse are $(a \pm c, 0)$. Since $c = \sqrt{a^2 - b^2} = \sqrt{\frac{R^2}{(1-k)^2} - \frac{R^2}{1-k}} = \sqrt{\frac{R^2 - R^2(1-k)}{(1-k)^2}} = \sqrt{\frac{R^2 k}{(1-k)^2}} = \frac{R}{(1-k)}\sqrt{k}$, the focal points of the shifted ellipse are

$$(a \pm c, 0) = \left(\tfrac{R}{1-k} \pm \tfrac{R\sqrt{k}}{1-k}, 0\right) = \left(\tfrac{R(1 \pm \sqrt{k})}{1-k}, 0\right).$$

Finally, rewrite $1-k$ as $1 - (\sqrt{k})^2 = (1+\sqrt{k})(1-\sqrt{k})$ to see that the focal points are $\left(\frac{R}{1 \pm \sqrt{k}}, 0\right)$. That $\varepsilon = \frac{c}{a} = \sqrt{k}$ is easy to see.

vi. For $k > 1$, the graph is the hyperbola with focal points $\left(\frac{R}{1 \pm \sqrt{k}}, 0\right)$ and eccentricity $\varepsilon = \sqrt{k}$. To see this, go back to the equality derived in (ii) once more. Since $k > 1$, notice that $k - 1 > 0$ and the equality of (ii) can be written as

$$\frac{(x + \frac{R}{k-1})^2}{\frac{R^2}{(k-1)^2}} - \frac{y^2}{\frac{R^2}{k-1}} = 1$$

and, since $k - 1 = (\sqrt{k-1}\,)^2$, as

$$\frac{(x + \frac{R}{k-1})^2}{\frac{R^2}{(k-1)^2}} - \frac{y^2}{\frac{R^2}{(\sqrt{k-1})^2}} = 1.$$

Now let $a = \frac{R}{k-1}$ and $b = \frac{R}{\sqrt{k-1}}$ and observe that what we are dealing with is the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ discussed in Section 4.5, shifted $a = \frac{R}{k-1}$ units to the left. This standard hyperbola has focal points $(\pm c, 0)$, where $c = \sqrt{a^2 + b^2} = \sqrt{\frac{R^2}{(k-1)^2} + \frac{R^2}{k-1}} = \sqrt{\frac{R^2 + R^2(k-1)}{(k-1)^2}} = \frac{R\sqrt{k}}{k-1}$. So the shifted hyperbola has focal points

$$(-a \pm c, 0) = \left(-\tfrac{R}{k-1} \pm \tfrac{R\sqrt{k}}{k-1}, 0\right) = \left(\tfrac{-R \pm R\sqrt{k}}{k-1}, 0\right) = \left(\tfrac{R(-1 \pm \sqrt{k})}{\sqrt{k}^2 - 1}, 0\right) = \left(\tfrac{R(\pm\sqrt{k} - 1)}{(\sqrt{k}+1)(\sqrt{k}-1)}, 0\right).$$

With the $+$ in place, this is the point $\left(\frac{R}{1+\sqrt{k}}, 0\right)$ and with the $-$ in place it is the point $\left(\frac{-R}{\sqrt{k}-1}, 0\right) = \left(\frac{R}{1-\sqrt{k}}, 0\right)$. So the focal points are $\left(\frac{R}{1 \pm \sqrt{k}}, 0\right)$. The eccentricity of the hyperbola is $\varepsilon = \frac{c}{a} = \sqrt{k}$.

In reference to the particular lens depicted in Figure 8.60 some cases of Problems 8.42 and 8.43 are relevant but others are not. The circle $(x - R)^2 + y^2 = R^2$ is relevant as a kind of a base curve for the lens. Cases (iii) and (iv) of Problem 8.42 tell us that a graph with $k > 0$ is a relevant outer boundary of the lens but that a graph with $k < 0$ is not (as this curve falls inside the basic circle). Turning to the more specific Problem 8.43, we see that case (iii) is not relevant (because $k < 0$), but that cases (i), (v), and (vi) are relevant (because $k > 0$). With regard to case (v), the shift of the hyperbola to the left by a units, puts the leftmost point of its right branch at the origin as called for by Figure 8.60. But the left branch is irrelevant as it ends up to the left of the y-axis.

8.44. The formula $\rho = 58.37 \tan z_{\text{app}}$ seconds for the difference $\rho = z_{\text{true}} - z_{\text{app}}$ and a calculator tells us that for the angles z_{app} equal to $20°, 40°, 60°$, and $80°$ respectively, the values for ρ are $21.24, 48.98, 101.10$, and 331.03 seconds. The table tells us for z_{app} equal to $20°, 40°, 60°$, and $80°$, that the corresponding values for ρ are $21, 49, 101$, and 319 seconds.