Solutions to Problems and Projects for Chapter 1

Segment 1A is a reading assignment about Ptolemy's Maps. Comment about the reading in any way you wish.

Segment 1B presents Euclid's proof of the fact that for two similar triangles, the ratios of corresponding sides are equal. It remains to use Figures 1.39a and 1.39b to show that $\frac{B'C'}{BC} = \frac{A'B'}{AB}$. The argument goes as follows. Since the triangle $\Delta CAC'$ of Figure 1.39a and the triangle $\Delta CAA'$ of Figure 1.39b have the same height relative to the base CA that they share, they have the same area. Therefore the area of $\Delta B'C'A$ of Figure 1.39a is equal to the area of $\Delta B'A'C$ of Figure 1.39b. The triangles $\Delta B'C'A$ and ΔBCA of Figure 1.39a have

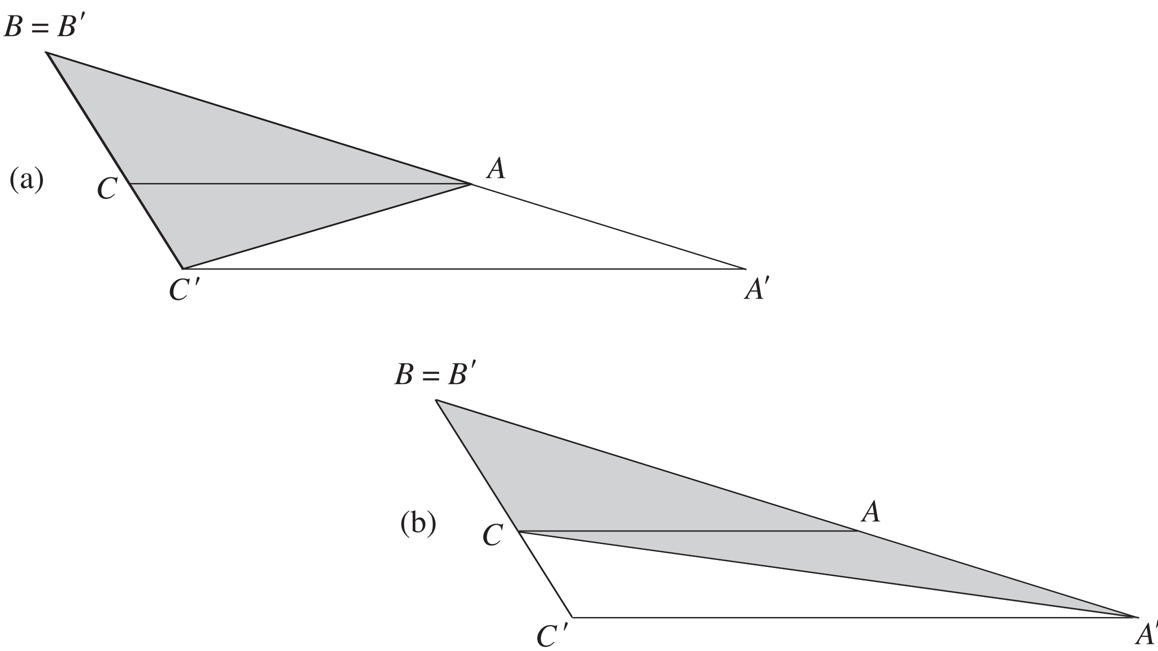


Fig. 1.39

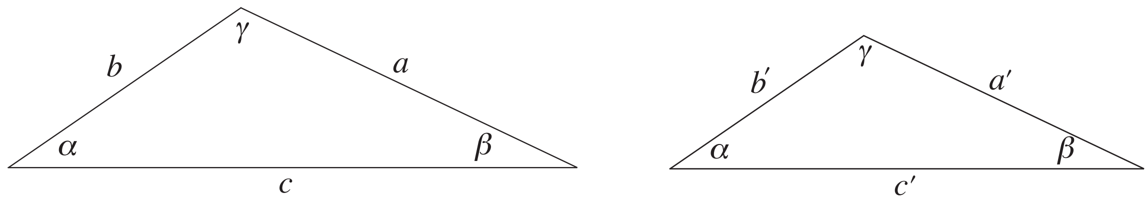
the same height h relative to the bases $B'C'$ and BC. In the same way, the triangles $\Delta B'A'C$ and ΔBAC of Figure 1.39b have the same height h' relative to the bases $B'A'$ and BA. Therefore,

$$\frac{\text{Area } \Delta B'C'A}{\text{Area } \Delta BCA} = \frac{\frac{1}{2}B'C' \cdot h}{\frac{1}{2}BC \cdot h} = \frac{B'C'}{BC} \quad \text{and} \quad \frac{\text{Area } \Delta B'A'C}{\text{Area } \Delta BAC} = \frac{\frac{1}{2}B'A' \cdot h'}{\frac{1}{2}BA \cdot h'} = \frac{B'A'}{BA}.$$

Since the areas of $\Delta B'C'A$ and $\Delta B'A'C$ are equal, we can conclude that $\frac{B'C'}{BC} = \frac{A'B'}{AB}$.

1.1. The figure below depicts two similar triangles. The corresponding equal angles are labelled accordingly and the corresponding sides are as well. Applying the law of sines to each of the triangles, we get

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \quad \text{and} \quad \frac{\sin\alpha}{a'} = \frac{\sin\beta}{b'} = \frac{\sin\gamma}{c'}.$$



Inverting the first two equalities and multiplying then into the second two, give us

$$\frac{a}{\sin\alpha}\cdot\frac{\sin\alpha}{a'}=\frac{b}{\sin\beta}\cdot\frac{\sin\beta}{b'}=\frac{c}{\sin\gamma}\cdot\frac{\sin\gamma}{c'}.$$

Therefore $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

1.2. The triangles ΔPAC and ΔABC are right triangles and $\angle CAP=\angle CAB$. So two of the angles of these two triangles match and therefore all three. So ΔPAC and ΔABC are similar. The same reasoning tells us that ΔPBC and ΔABC are similar. It follows that $\frac{c}{b}=\frac{b}{c_1}$ and $\frac{a}{c_2}=\frac{c}{a}$. Therefore, $a^2+b^2=cc_2+cc_1=c^2$.

1.3. The inner square is a $(a-b)$ by $(a-b)$ square. So $(a-b)^2+4\times\frac{1}{2}ab=c^2$. Hence $a^2-2ab+b^2+2ab=c^2$.

1.4. After they are slid into position in Figure 1.43, the two small squares cover the region A twice and the two regions B not at all. In terms of areas this means that $A=B+B$. Let r and s be the side lengths of the squares A and B respectively. Figure 1.43 tells us that $r+s=i$ and $r+2s=j$. So $s=j-i$ and $r=i-s=2i-j$. Therefore $A=B+B$ with both A and B of integer side lengths. Since the square A is smaller than the earlier smallest one, we have reached a contradiction. So the assumption that $\sqrt{2}$ is rational must have been wrong.

1.5. Since an angle of π radians has $180°$, 1 radian corresponds to $\frac{180}{\pi}$ degrees and 1 degree corresponds to $\frac{\pi}{180}$ radians. The other answers are $\frac{78.5}{180}\pi$ radians and $\frac{1.238\cdot 180}{\pi}$ degrees respectively.

1.6. Since θ in radians is equal to $\frac{\text{arc }AB}{3}$ and $57.3°=(57.3)\cdot\frac{\pi}{180}$ radians, we see that $\frac{\text{arc }AB}{3}=(57.3)\cdot\frac{\pi}{180}$. So arc AB is $(57.3)\cdot\frac{3\pi}{180}\approx 3.00$ units long.

1.7. In radians, the angle θ is equal to $\frac{\text{arc }AB}{2}=\frac{1\frac{1}{2}}{2}=\frac{3}{4}$. Since 1 radian is equal to $\frac{180}{\pi}$ degrees, $\theta=\frac{\frac{3}{4}\cdot 180}{\pi}=\frac{3\cdot 45}{\pi}=\frac{135}{\pi}\approx 42.97$ degrees.

1.8. The radian measure of the angle is $\frac{4}{5}$ radians or $0.8\cdot\frac{180}{\pi}=\frac{144}{\pi}\approx 45.84$ degrees. If the angle is $21°$ then it has $21\cdot\frac{\pi}{180}=\frac{21\pi}{180}=\frac{7\pi}{60}$ radians. If r is the radius of the circle, then $\frac{4}{r}=\frac{7\pi}{60}$, so that $r=\frac{240}{7\pi}\approx 10.91$ centimeters.

1.9. Let's start by pointing out that even though the center O of the circle is shown to lie on the constructed segment DE, this is not assumed. This is precisely what needs to be established. Since OB and OB' are both radii of the circle, the triangle $\Delta BOB'$ is isosceles. So $\angle OBC = \angle OB'C$ as asserted in the hint. Since C is the midpoint of the segment BB', the triangles ΔOBC and $\Delta OB'C$ are congruent. Therefore $\angle BCO = \angle B'CO$ as asserted in the hint. Since $\angle BCO + \angle B'CO = 180°$, this means that both $\angle BCO$ and $\angle B'CO$ are right angles. It follows that O lies on the perpendicular bisector DE.

1.10. Let B and B' be the two points. By Problem 1.9, the center O of any circle that has both B and B' on it lies on the perpendicular bisector of the segment BB'. Conversely, if O is any point on the perpendicular bisector of BB', then (refer to Figure 1.45) $BC = B'C$ and $\angle BCO$ and $\angle B'CO$ are both right angles. So the triangles ΔOCB and ΔOCB are congruent and therefore $OB = OB'$. So the circle with radius OB has the point B' on it. It follows that the center of any circle that has both B and B' on it lies on the perpendicular bisector of the segment BB'.

1.11. Let B, B', and B'' be the three points. By the previous problem, the center O of the circle on which the three points lie is the point of intersection of the perpendicular bisectors of the segments BB' and BB''. Since the radius of the circle is OB, there is exactly one such circle.

1.12. Assume that the cross section of the Earth depicted in Figure 1.46b is a circle with center the point that the figure singles out (this is not quite true because Earth is flattened at the poles) and let r be its radius. If d is the distance from the ship to the equator (along the arc), then $\frac{d}{r}$ is the radian measure of the angle β. Since the radian measure of $\alpha + \beta$ is $\frac{\pi}{2}$, it follows that the radian measure of α is equal to $\frac{\pi}{2} - \frac{d}{r}$. So $d = r(\frac{\pi}{2} - \alpha)$.

1.13. If α is equal to $53°$, then the radian measure of α is $53 \cdot \frac{\pi}{180}$. By the previous problem, $d = r(\frac{\pi}{2} - \alpha) = 6370(\frac{\pi}{2} - \frac{53\pi}{180}) = 6370(\frac{37\pi}{180}) \approx 4114$ kilometers.

1.14.
- **i.** By applying the Pythagorean theorem to Figure 1.47, we get $OT^2 + (\frac{1}{2}s_n)^2 = 1$.
- **ii.** Since $QT+OT = 1$, we see by an application of part (i) that $(1-QT)^2+(\frac{1}{2}s_n)^2 = 1$. So $1 - QT = \sqrt{1 - \frac{1}{4}s_n^2}$. Therefore, $QT = 1 - \frac{\sqrt{4-s_n^2}}{2}$.
- **iii.** By another application of the Pythagorean Theorem, $QT^2 + \frac{1}{4}s_n^2 = s_{2n}^2$. So by part (ii), $s_{2n}^2 = \big(1-\frac{\sqrt{4-s_n^2}}{2}\big)^2 + \frac{1}{4}s_n^2 = 1 - \sqrt{4 - s_n^2} + \frac{1}{4}(4-s_n^2) + \frac{1}{4}s_n^2 = 2 - \sqrt{4 - s_n^2}$.
- **iv.** The product ns_{2n} is an approximation for π because the $2n$ segments of length s_{2n} approximate the circumference 2π of the circle of radius 1. This last approximation becomes more and more accurate with increasing n because the $2n$ segments involved taken together approximate the circumference more and more tightly.

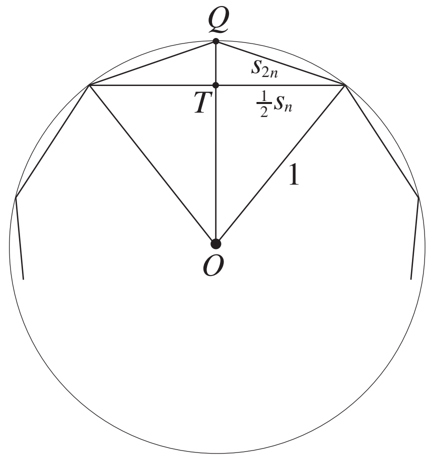


Fig. 1.47

1.15. **i.** In the given circle of radius 1, draw six radii so that the six angles that they form at the circle's center are each 60°. Connect consecutive endpoints of these radii to get the inscribed hexagon. The six triangles that are formed are equilateral. It follows that $s_6 = 1$.

ii. Taking $s_6 = 1$ in the equality of 1.14iii, we get $s_{12}^2 = 2 - \sqrt{4-1} = 2 - \sqrt{3}$. So $s_{12} = \sqrt{2-\sqrt{3}}$. By repeating this, $s_{24} = \sqrt{2-\sqrt{2+\sqrt{3}}}$, $s_{48} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$, and $s_{96} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$. With a calculator, $\pi \approx 48 s_{96} \approx 3.14103$. After one more step, $\pi \approx 96 s_{192} \approx 3.14145$.

iii. The square inscribed in a circle of radius 1 has diagonal 2, so that by the Pythagorean Theorem, $s_4 = \sqrt{2}$.

iv. With $s_4 = \sqrt{2}$ in the equality of 1.14iii, $s_8 = \sqrt{2-\sqrt{2}}$. By repetition, $s_{16} = \sqrt{2-\sqrt{2+\sqrt{2}}}$, $s_{32} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$, and $s_{64} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$. It follows that $\pi \approx 64 s_{128} \approx 3.141277$.

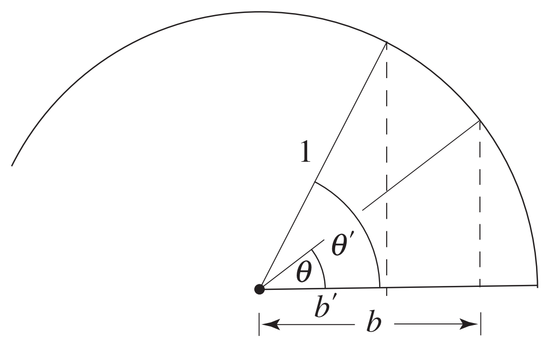
v. The correct decimal expansion of π starts with $\pi \approx 3.141592$.

1.16. At three items for a dollar, one item is worth exactly $33\frac{1}{3}$ cents. Since there is no coin worth $\frac{1}{3}$ of a cent, this amount falls beyond the capacity of our system.

1.17. Consider Figures 1.27a. Since the angle at the bottom left is $\frac{\pi}{4}$, we get $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\tan\frac{\pi}{4} = 1$. Turn to 1.27b. The right triangle has angle $\frac{\pi}{3}$ on the bottom left and $\frac{\pi}{6}$ at the top. It follows that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, and $\tan\frac{\pi}{3} = \sqrt{3}$.

1.18. Using a calculator *in radian mode*, we get $\sin 0.1 \approx 0.099833$, $\tan 0.1 \approx 0.100335$, $\sin 0.01 \approx 0.00999983$, $\tan 0.01 \approx 0.01000033$, $\sin 0.001 \approx 0.000999999833$, $\tan 0.001 \approx 0.001000000333$.

1.19. The figure below shows a part of a circle of radius 1 as well as its center. From the figure, $\cos\theta' = b' < b = \cos\theta$.



1.20. We'll elaborate on the hint. Let θ be an angle given in degrees. The same angle in radians is $\theta \cdot \frac{\pi}{180}$. By the limit equality that concludes Section 1.6,

$$\lim_{\theta\to 0} \frac{\sin\left(\theta \cdot \frac{\pi}{180}\right)}{\left(\theta \cdot \frac{\pi}{180}\right)} = 1.$$

Since the angles θ in degrees and $\theta \cdot \frac{\pi}{180}$ in radians are the same, $\sin\theta = \sin\left(\theta \cdot \frac{\pi}{180}\right)$. So with θ in degrees,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\left(\theta \cdot \frac{\pi}{180}\right)} = 1$$

and therefore, $\frac{1}{\frac{\pi}{180}} \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. Multiply through by $\frac{\pi}{180}$ to get the result.

1.21. With θ in radians, $\lim\limits_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}\right) = 1 \cdot \frac{1}{1} = 1$.

1.22. Dividing the identity $\sin^2\theta + \cos^2\theta = 1$ through by $\cos^2\theta$, gives $\tan^2\theta + 1 = \frac{1}{\cos^2\theta} = \sec^2\theta$. In the same way, dividing the identity $\sin^2\theta + \cos^2\theta = 1$ through by $\sin^2\theta$, gives $1 + \cot^2\theta = \csc^2\theta$.

1.23. The radius of the semicircle in Figure 1.48 is equal to 1, P is a point on the diameter AB, and CP is the perpendicular to AB that determines the angle θ. By results in

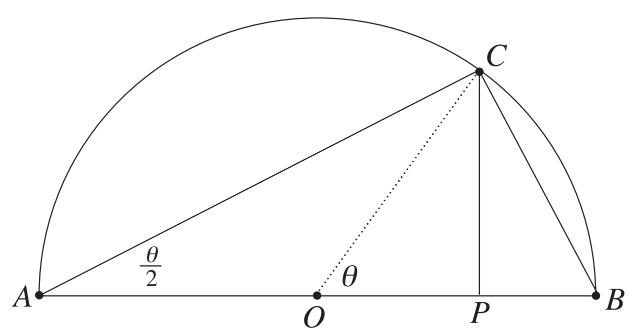


Fig. 1.48

Sections 1.3 and 1.6, ΔABC is a right triangle and $\angle CAO = \frac{\theta}{2}$.

Notice that $\sin\theta = CP$ and $\cos\theta = OP$. By the Pythagorean Theorem, $AC^2 = CP^2 + (1+OP)^2 = \sin^2\theta + (1+\cos\theta)^2 = \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta = 2(1+\cos\theta)$.

i. Since $\sin\frac{\theta}{2} = \frac{CP}{AC}$, we get $\sin^2\frac{\theta}{2} = \frac{CP^2}{AC^2} = \frac{\sin^2\theta}{2(1+\cos\theta)}$. Therefore,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}\frac{\sin^2\theta}{1+\cos\theta} \times \frac{1-\cos\theta}{1-\cos\theta} = \frac{1}{2}\frac{\sin^2\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{1}{2}\frac{\sin^2\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1}{2}(1-\cos\theta).$$

Figure 1.48 tells us that $\cos\frac{\theta}{2} = \frac{1+OP}{AC}$. So $\cos^2\frac{\theta}{2} = \frac{(1+\cos\theta)^2}{2(1+\cos\theta)} = \frac{1}{2}(1+\cos\theta)$. The equality $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$ follows from the definition of the tangent.

ii. $\tan\frac{\theta}{2} = \frac{CP}{1+OP} = \frac{\sin\theta}{1+\cos\theta}$. It follows that $\tan^2\frac{\theta}{2} = \frac{\sin^2\theta}{1+2\cos\theta+\cos^2\theta}$ and therefore that $1-\tan^2\frac{\theta}{2} = \frac{1+2\cos\theta+\cos^2\theta-\sin^2\theta}{1+2\cos\theta+\cos^2\theta} = \frac{2\cos\theta+2\cos^2\theta}{(1+\cos\theta)^2} = \frac{2\cos\theta(1+\cos\theta)}{(1+\cos\theta)^2} = \frac{2\cos\theta}{1+\cos\theta}$.

By combining what we know, $\frac{\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{2\cos\theta} = \frac{\sin\theta}{2\cos\theta} = \frac{1}{2}\tan\theta$.

1.24. The focus is on Figure 1.49b. The angles $\angle CAD$ and $\angle BAD$ are both acute. So by the acute case already verified, $\angle CAD = \frac{1}{2}\angle COD$ and $\angle BAD = \frac{1}{2}\angle BOD$. Notice next that $\theta = \angle COD + \angle BOD$. Therefore, $\angle CAB = \angle CAD + \angle BAD = \frac{1}{2}\angle COD + \frac{1}{2}\angle BOD = \frac{1}{2}\theta$.

1.25. Let α and β be two angles with sum less than $\frac{\pi}{2}$ and place them into the two right triangles ΔABC and ΔBDC shown in Figure 1.50. Complete the diagram to the

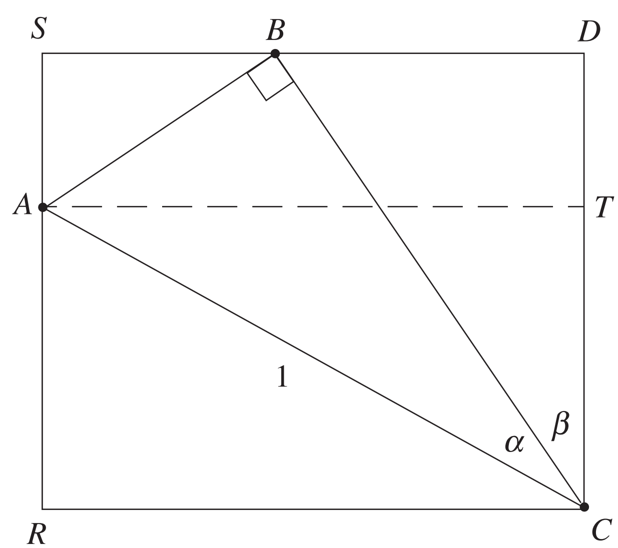


Fig. 1.50

rectangle $RSDC$ and draw in the parallel AT to SD.

A look at the angles around the point B tells us that $\angle SBA + \frac{\pi}{2} + \angle DBC = \pi$ and a look at the right triangle ΔBDC informs us that $\angle DBC + \frac{\pi}{2} + \beta = \pi$. So, $\angle SBA = \frac{\pi}{2} - \angle DBC$ and $\beta = \frac{\pi}{2} - \angle DBC$. Therefore, $\angle SBA = \beta$.

i. Note that $\cos\beta = \cos\angle SBA = \frac{SB}{AB}$ and $\sin\beta = \frac{BD}{BC}$. It follows therefore that $AT = SB + BD = AB\cos\beta + BC\sin\beta$. Since $AB = \sin\alpha, BC = \cos\alpha$, and $\sin(\alpha+\beta) = \frac{AT}{1} = AT$ we can conclude that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

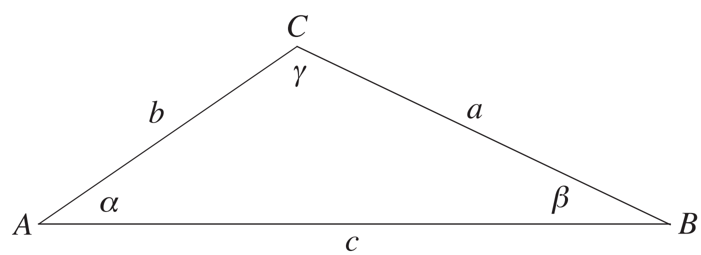
ii. Since $\cos\beta = \frac{DC}{BC}$ and $\sin\beta = \sin\angle SBA = \frac{SA}{AB}$, we get $DC = BC\cos\beta$ and $SA = AB\sin\beta$. Therefore, $TC = DC - SA = BC\cos\beta - AB\sin\beta$. Another look at Figure 1.50 tells us that $\cos\alpha = \frac{BC}{1} = BC$ and $\sin\alpha = \frac{AB}{1} = AB$. Since $\cos(\alpha+\beta) = \frac{TC}{1} = TC$, we get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

iii. $\tan(\alpha+\beta) = \dfrac{\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}}{\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta}} = \dfrac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \dfrac{\tan\alpha+\tan\beta}{1-(\tan\alpha)(\tan\beta)}$.

1.26. Formulas i, ii, and iii follow by taking $\alpha = \beta$ in the formulas of 1.25 and applying $\sin^2\alpha + \cos^2\alpha = 1$.

The next three problems make use of the triangle in the figure below



1.27. Since α and β are $\frac{\pi}{5}$ and $\frac{\pi}{7}$ respectively, γ is equal to $\pi - \frac{\pi}{5} - \frac{\pi}{7} = \frac{(35-7-5)\pi}{35} = \frac{23\pi}{35}$. So $\gamma > \frac{\pi}{2}$ is obtuse. Since $c = 8$,

$$\frac{\sin\frac{23\pi}{35}}{8} = \frac{\sin\frac{\pi}{5}}{a} = \frac{\sin\frac{\pi}{7}}{b}$$

by applying the law of sines to the figure below. A calculator (in radian mode) informs us that $a \approx 5.34$ and $b \approx 3.94$.

1.28. Let $\alpha = \frac{\pi}{5}, b = 7$, and $c = 11$. By the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos\alpha = 7^2 + 11^2 - 2\cdot 77\cos\frac{\pi}{5} \approx 16.01$.

1.29. If the three sides a, b, and c satisfy $a^2 + b^2 = c^2$, then by the law of cosines $a^2 + b^2 = a^2 + b^2 - 2ab\cos\gamma$, so that $\cos\gamma = 0$. The definition of the cosine in Section 1.6 tells us that $\gamma = \frac{\pi}{2}$.

The solutions of some of the "inverse" trigonometry problems that follow require a calculator with inverse trig feature. (Use three-decimal-accuracy.)

1.30. It follows quickly from Figures 1.27a and 1.27b, that $\alpha = 30^\circ = \frac{\pi}{6}$, $\beta = 45^\circ = \frac{\pi}{4}$, $\gamma = 60^\circ = \frac{\pi}{3}$, and $\varphi = 60^\circ = \frac{\pi}{3}$.

1.31. These answers are better provided in terms of approximations than equalities. The inverse trig function of a calculator shows that:

i. $\sin(12.65^\circ) \approx 0.219$

ii. $\sin(56.51^\circ) \approx 0.834$

iii. For the angle 0.002 in radians, $\sin(0.002) \approx 0.002$

iv. For the angle 0.7262 in radians, $\sin(0.7262) \approx 0.664$

v. $\tan(37.74^\circ) \approx 0.774$

vi. $\tan(55.92^\circ) \approx 1.478$

vii. For the angle 1.4756 in radians, $\tan(1.4756) \approx 10.473$

viii. For the angle 1.53466 in radians, $\tan(1.53466) \approx 27.664$

1.32. In triangle (a) let a and b be the sides opposite the angles 40° and 35° respectively. By the law of sines, $\frac{\sin 40^\circ}{a} = \frac{\sin 35^\circ}{b} = \frac{\sin 105^\circ}{135}$. So $a = 135\,\frac{\sin 40^\circ}{\sin 105^\circ} \approx 89.84$ and $b = 135\,\frac{\sin 35^\circ}{\sin 105^\circ} \approx 80.16$.

In triangle (b) let c be the remaining side. By the law of cosines, $c^2 = 30^2 + 45^2 - 2(30)(45)\cos 130^\circ \approx 4660.53$, so that $c \approx 68.27$.

In triangle (c) let a and b be the sides opposite the angles 55° and 65°. By the law of sines, $\frac{\sin 55^\circ}{a} = \frac{\sin 65^\circ}{b} = \frac{\sin 60^\circ}{85}$. So $a = 85\,\frac{\sin 55^\circ}{\sin 60^\circ} \approx 80.40$ and $b = 85\,\frac{\sin 65^\circ}{\sin 60^\circ} \approx 88.95$.

1.33. Let α, β, and γ be the angles opposite the sides of lengths 11, 7, and 5, respectively. Then α, β, and γ satisfy $11^2 = 7^2 + 5^2 - 2(7)(5)\cos\alpha, 7^2 = 11^2 + 5^2 - 2(11)(5)\cos\beta$ and $5^2 = 11^2 + 7^2 - 2(11)(7)\cos\gamma$, respectively. So

$$\cos\alpha = -\tfrac{121-49-25}{70},\ \cos\beta = -\tfrac{49-121-25}{110},\ \text{and}\ \cos\gamma = -\tfrac{25-121-49}{154}.$$

It follows that $\alpha \approx 132.18^\circ, \beta \approx 28.14^\circ$, and $\gamma \approx 19.69^\circ$.

1.34. We will take the value $\pi = 3.1420$ and work with 4 decimal accuracy. We'll also rely on the discussion that ends Section 1.6 about the close agreement between $\sin\theta$ and θ for small angles θ (in radians). In Figure 1.35, 3° is replaced by $\frac{1}{6}^\circ$. This angle equals $\frac{1}{6}\frac{\pi}{180} = \frac{\pi}{1080} = 0.0029$ in radians. Since this is a small angle, we'll take $\sin 0.0029 = 0.0029$. Therefore, take $\frac{D_M}{D_S} = \frac{r_M}{r_S} = 0.0029$. In Figure 1.34, 1° is replaced by $\frac{1}{4}^\circ$. In radian measure $\frac{1}{4}^\circ$ is equal to $\frac{1}{4}\frac{\pi}{180} = \frac{\pi}{720} = 0.0044$ radians. This is a small angle, so we'll take $\sin 0.0044 = 0.0044$. Therefore, $\frac{r_M}{D_M} = 0.0044$. If $4r_M$ is replaced by $5r_M$, and hence $2r_M$ by $2.5r_M$ in Figure 1.36, then a repetition of the analysis that led to the formula $\frac{r_E}{r_M} + \frac{r_E}{r_S} = 3$ will give $\frac{r_E}{r_M} + \frac{r_E}{r_S} = 3.5$ instead. From $\frac{r_M}{r_S} = 0.0029$, we get $r_S = 345 r_M$. Inserting $r_S = 345 r_M$, shows that $3.5 = \frac{r_E}{r_M} + \frac{r_E}{345 r_M} = \frac{346 r_E}{345 r_M}$, and therefore that $r_M = \frac{346 r_E}{(3.5)(345)} = 0.2865 r_E$. Taking Eratosthenes's value $r_E = 6200$ kilometers, this gives $r_M = 1780$ kilometers. Since $r_S = 345 r_M$, we find that $r_S =$ 614,000 kilometers. Since $\frac{r_M}{D_M} = 0.0044, \frac{D_M}{r_M} = 227$. So $D_M = 404{,}000$ kilometers. Finally, $D_S = \frac{D_M}{0.0029} = \frac{404{,}000}{0.0029}$, so $D_S = 139 \times 10^6$ kilometers. A look at Table 1.3 shows that the distances r_M, r_S, D_M, and D_S just derived are within the "ball park."

1.35. In Figure 1.52, let $\gamma = \angle MAE$. Since $\alpha + \gamma = 180°$ and $\beta + \mu + \gamma = 180°$, it follows that $\alpha = \beta + \mu$. By the law of sines, $\frac{\sin\gamma}{EM} = \frac{\sin\mu}{EA}$. Since $\sin\gamma = \sin(180° - \alpha) = \sin\alpha$ by an equality established in Section 1.6, it follows that $D_M = EM = EA \cdot \frac{\sin\alpha}{\sin\mu} = r_E \cdot \frac{\sin\alpha}{\sin\mu}$. Taking Eratosthenes value $r_E = 6200$ kilometers for Earth's radius and noticing that $\mu = \alpha - \beta = 50°55' - 49°48' = 1°7'$, Ptolemy obtains the estimate $D_M = 6200 \frac{\sin 50°55'}{\sin 1°7'} \approx 6200 \frac{0.776}{0.0195} \approx 247{,}000$ kilometers. A look at Table 1.3 tells us that while this value is much better than what Aristarchus achieved, it falls far short of today's accurate value.

1.36. Given the differences in notation it is better to repeat the argument of the solution of Problem 1.24 rather than to apply the result. So let the angle θ in Figure 1.53 be greater than 180° and choose the point D on the circle of the figure such that POD is a diameter. Notice that the angles $\angle AOD = \theta_1$ and $\angle BOD = \theta_2$ are both less than 180°. So by the case already verified, $\angle APD = \frac{1}{2}\theta_1$ and $\angle BPD = \frac{1}{2}\theta_2$. Therefore $\angle APB = \frac{1}{2}(\theta_1 + \theta_2) = \frac{\theta}{2}$.

1.37. In Figure 1.59, F is chosen so that $\angle DAF = \angle BAE = \angle BAC$. By the Corollary, $\angle ADB = \angle ACB$. These two equalities imply that the triangles ΔADF and ΔACB of Figure 1.57 are similar in the situation of Figure 1.59 as well. Since $\angle DAC + \angle EAF = \angle DAF = \angle BAE = \angle BAF + \angle FAE$, it follows that $\angle DAC = \angle BAF$. By the Corollary, $\angle DCA = \angle ABD = \angle ABF$. It follows that the triangles ΔABF and ΔACD of Figure 1.58 are also similar in this new context. Therefore, the equalities **(a)** and **(b)** both hold again, and the conclusion $(AC)\cdot(BD) = (AB)\cdot(CD) + (AD)\cdot(BC)$ follows as in the case of Figure 1.56.

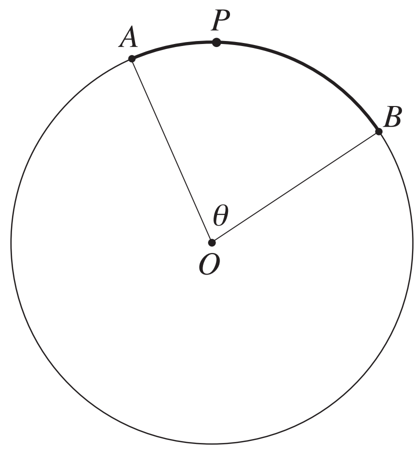
1.38. Consider a right triangle with sides a and b and hypothenuse c. Take two copies of this triangle to form a rectangle with sides a and b and diagonal c. The center O of the rectangle is the point of intersection of the two diagonals. The circle with center O and radius $\frac{1}{2}c$ and center the center of the rectangle has all four corners of the rectangle on it. By Ptolemy's theorem, $c \cdot c = a \cdot a + b \cdot b$.

1.39. By Ptolemy's theorem, $AB \cdot CD = AD \cdot CB + AC \cdot BD$. By a result of Section 1.3, any triangle that has a diameter as side is a right triangle. Since all diagonals are equal to 1, Figure 1.60 tells us that $AB = \sin(\alpha + \beta), CD = 1, AD = \sin\alpha, CB = \cos\beta, AC = \cos\alpha$, and $BD = \sin\beta$. The result follows. (Note that Ptolemy's corollary was not needed.)

1.40. Consider the generic quadrilateral of Figure 1.55 and let O be the point of intersection of the two diagonals. To see that ΔADO is similar to ΔBOC, observe first that $\angle AOD = \angle BOC$. By Ptolemy's corollary, $\angle ADO = \angle ADB = \angle ACB = \angle OCB$. So two of the angles of ΔADO and ΔBOC are equal. Therefore the remaining angles are equal as well and the triangles are similar. The similarity of the other pair of triangles is verified in the same way.

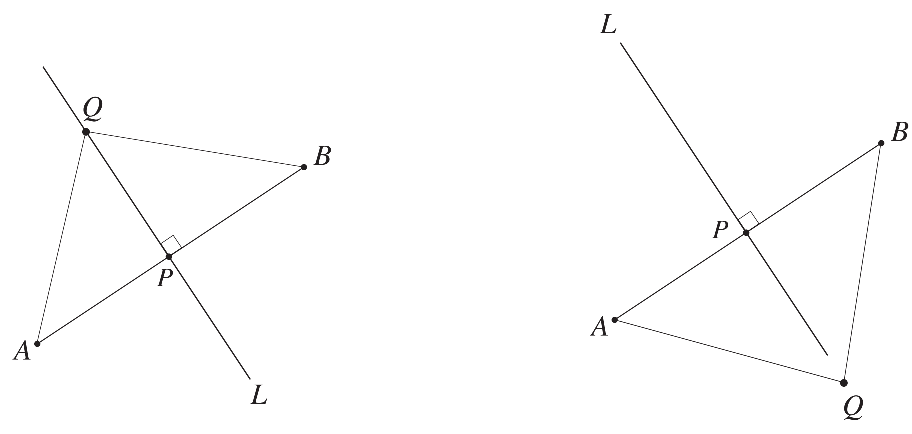
1.41. Let O be the center of the circle. The angle $\angle AOB = \frac{4}{3}$ radians. By Ptolemy's proposition, $\angle APB = \frac{2}{3}$ radians. Hence $\angle APB = \frac{2}{3} \cdot \frac{180}{\pi} = \frac{120}{\pi}$ degrees.

1.42. The figure below depicts a typical situation. The arc complementary to the highlighted



arc AB determines the angle $2\pi - \theta$. By Ptolemy's proposition, $\angle APB = \frac{1}{2}(2\pi - \theta) = \pi - \frac{\theta}{2}$.

1.43. A typical situation is depicted below. Let Q be any point on the perpendicular bisector L. Since $AP = BP$ and QP is perpendicular to AB, it follows that the triangles ΔAPQ

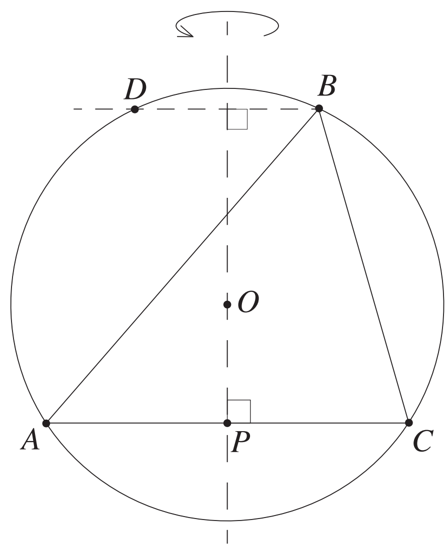


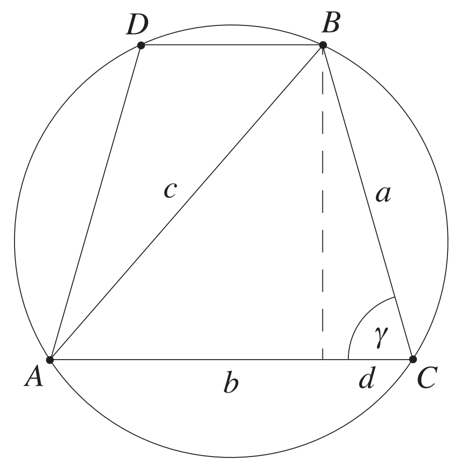
and ΔBPQ are congruent. So $QA = QB$. Conversely, suppose that Q is any point such that $QA = QB$ and refer to the figure on the right. We need to show that Q is on L. Consider the triangle ΔABQ. The fact that it is isosceles, tells us that

$\angle BAQ = \angle ABQ$. Since the segments AP and AQ and the angles between them are respectively equal to the segments BP and BQ and the angle between them, the triangles ΔPAQ and ΔPBQ are congruent. Since $\angle APQ$ and $\angle BPQ$ are equal and add to $180°$, it follows that $\angle APQ = \angle BPQ = 90°$. Hence Q is on L. (Note that the second part of this problem was already taken up in Problem 1.9.)

1.44. This problem was already considered in Problem 1.11.

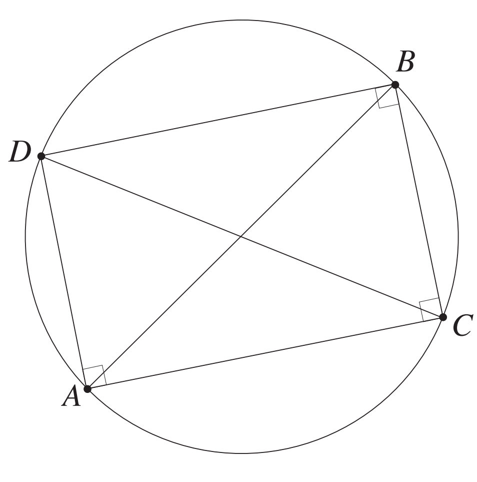
1.45. Start with the figure below on the left. Consider the base AC of the triangle and let P be its midpoint. By Problem 1.43, the perpendicular bisector of AC goes through





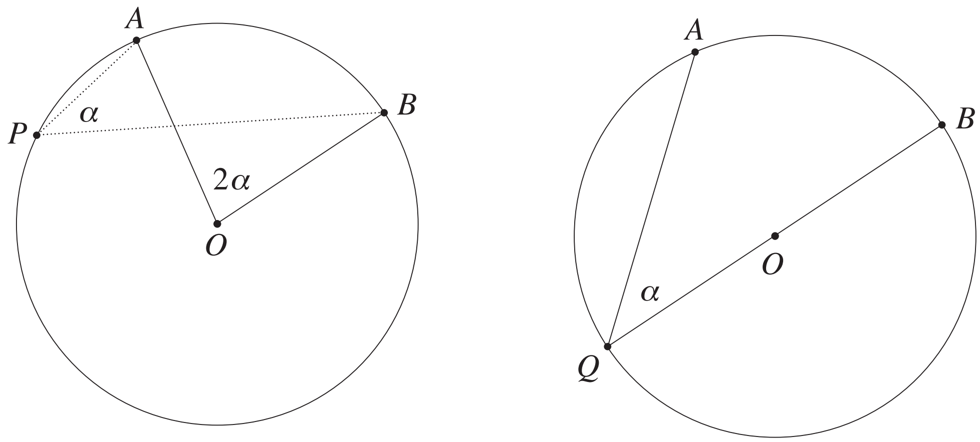
the center O of the circle. It follows that the trapezoid on the right is obtained by revolving the triangle ΔACB around the diameter through P and O. So $AD = a, DC = c$, and $b = AC = 2d + DB$. By Ptolemy's theorem, $c^2 = a^2 + b \cdot DB = a^2 + b(b - 2d) = a^2 + b^2 - 2bd$. Since $\cos\gamma = \frac{d}{a}$, we get $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

1.46. Following the hint, label the triangle ΔABC and let the right angle be at C. The right triangle is inscribed in the circle of the figure below. Let O be the center of the circle



and extend CO to a diameter CD of the circle. Since the $\angle ACB$ is a right angle and since the triangles ΔCAD and ΔCBD have right angles at A and B respectively (this follows from the fact that CD is a diameter of the circle), we know that $ACBD$ is a rectangle with diagonals CD and AB. Because CD is a diameter and the lengths of AB and CD are equal, AB is a diameter too.

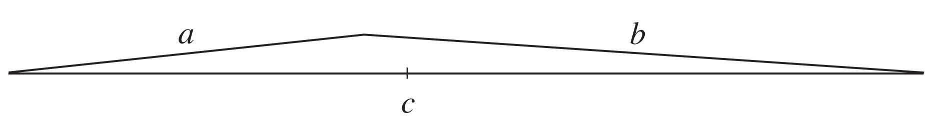
1.47. The circle with center O and radius r, the arc with endpoints A and B, the point P on the circle (but not on the arc), and the angle $\angle APB = \alpha$ (in radians) are shown in



the figure on the left. That $\angle AOB = 2\alpha$ follows from Ptolemy's proposition. Therefore, $2\alpha = \frac{\text{arc } AB}{r}$ and hence arc $AB = 2r\alpha$. Choose the point Q so that QOB is a diameter of the circle. By Ptolemy's corollary, $\angle AQB = \alpha$. Since QOB is a diameter, $\angle QAB = 90°$. Therefore, $AB = 2r(\sin\alpha)$.

1.48. Refer to Figure 1.62. That $\sin\alpha = \frac{h}{c}$ follows directly. With r the radius of the circle, $\sin\alpha = \frac{a}{2r}$ by Problem 1.47. So $\frac{2r}{a} = \frac{c}{h}$. It follows that $r = \frac{ac}{2h}$.

1.49. Let a, b, and c be any three positive numbers. Is there a triangle that has a, b, and c as the lengths of its sides? Show that this is so if $a + b > c$. Start by taking segments of lengths a and b and aligning them in a straight line, as shown in Figure 1.63. If c is the length of this segment, then $c = a + b$. If a and or b are increased so that $a + b > c$, then segments of lengths a, b, and c can be arranged to form a triangle as the figure



shows. It follows also that if a, b and c are positive numbers such that such that $a + b > c$, then also $a + c > b$ and $b + c > a$.

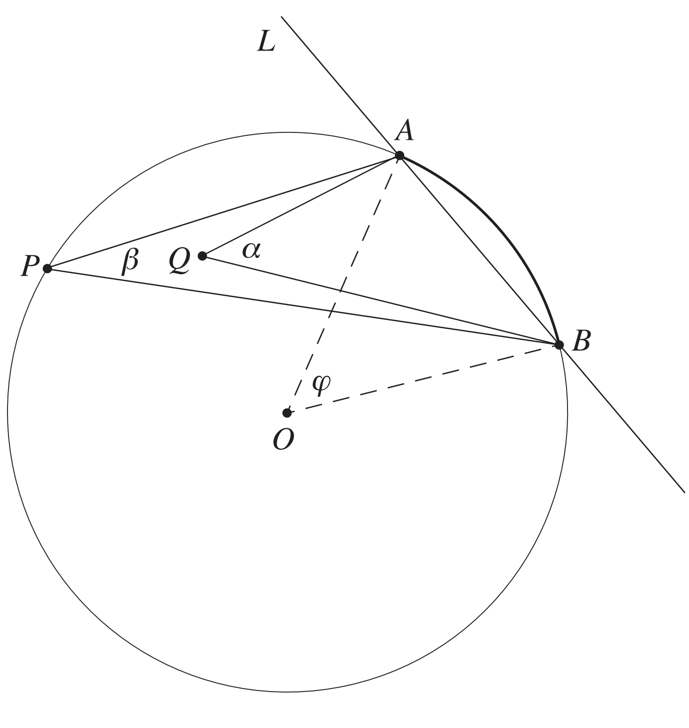
1.50. A triangle has sides of lengths 7 and 11 and (with the third side as base) height 4. Refer to Problem 1.48 and Figure 1.62 and think of $a = 7, c = 11$, and $h = 4$. Then

b corresponds to the third side. Refer to the two right triangles of Figure 1.62, let b_1 and b_2, respectively, be their third sides and observe that $b = b_1 + b_2$. By the Pythagorean theorem, $b_1^2 = 11^2 - 4^2$ and $b_2 = 7^2 - 4^2$. Since $b_1 = \sqrt{105} \approx 10.25$ and $b_2 = \sqrt{33} \approx 5.74$, $b = b_1 + b_2 \approx 10.25 + 5.74 \approx 15.99$. By the conclusion of Problem 1.48, we get that the radius r of the circle on which the vertices of the triangle lie is $r = \frac{ac}{2h} = \frac{7\cdot 11}{2\cdot 4} = \frac{77}{8} = 9.625$.

1.51. That there is a triangle with side lengths $7, 11$, and 17 follows from Problem 1.49. Let θ be the angle between the sides of lengths 7 and 11. By the law of cosines, $17^2 = 11^2 + 7^2 - 2(11)(7)\cos\theta$. So $\cos\theta = \frac{11^2+7^2-17^2}{2(11)(7)} \approx -0.7727$. Therefore $\theta \approx 140.60°$. Let α be the angle between the sides of lengths 7 and 17. By the law of sines, $\frac{\sin\alpha}{11} = \frac{\sin\theta}{17}$. So $\sin\alpha \approx \frac{11}{17}\sin 140.60° \approx 0.41$. So $\alpha \approx 24.20°$ If h is the height of the triangle with respect to the base 17, then $\sin\alpha = \frac{h}{7}$. So $h \approx 7\sin 24.25 \approx 2.87$. By Problem 1.48 and Figure 1.62, the radius r of the circle is $r = \frac{ac}{2h} \approx \frac{7\cdot 11}{2(2.87)} \approx 13.41$.

We now consider the question of the billboard and the car. How is the point on the highway S determined at which the passenger's lines of sight to the left and right edges A and B of the billboard attains a maximum value? Why is the intuitively answer that this point is the intersection of S with the perpendicular bisector of AB wrong? The problem that follows provides the key to the answer.

1.52. Turn to Figure 1.65, consider any point Q in the plane on the side of L opposite the arc AB, and let $\alpha = \angle AQB$. If Q is on the circle, then $\alpha = \angle AQB = \frac{\varphi}{2}$ by Ptolemy's proposition. For a point Q outside the circle, a point P has been chosen on the circle such that ΔPBA lies inside ΔQBA. By Ptolemy's proposition, $\beta = \angle APB = \frac{\varphi}{2}$.



Since $\alpha = \angle AQB < \beta$, it follows that $\alpha < \frac{\varphi}{2}$. If Q is inside the circle, chose a point P on the circle such that ΔQBA lies inside ΔPBA. See the figure above. This time, $\alpha = \angle AQB > \beta = \frac{\varphi}{2}$.

Now to the question about the highway and the billboard.

1.53. Suppose that a circle has been chosen that has the points A and B on it and has the line S as a tangent. Let P on the circle be the point of tangency. See Figure 1.66. Let C be the center of the circle. By Ptolemy's proposition, $\angle APB = \frac{1}{2}\angle ACB$. Since all other points Q on S lie outside the circle, it follows from Problem 1.52 for any such Q that $\angle AQB < \frac{1}{2}\angle ACB$. It follows that P is the point on the highway S at which the passenger's lines of sight to the left and right edges A and B of the billboard attains a maximum value.