Solutions to Problems and Projects for Chapter 4

4.1. $28 = 2^2 \cdot 7$; $143 = 11 \cdot 13$; $192 = 2^6 \cdot 3; 720 = 2^4 3^2 5$.

4.2. The hint provides everything except the conclusion. Since $mt^2 = s^2$, the primes in the factorization of mt^2 and hence of m occur with even powers only. So m is a square. So the assumption that $\sqrt{m}$ is rational implies that m is a square.

4.3. The number $\sqrt{n}$ is rational only if n is a square. In the first case, the largest rational number that arises is $\sqrt{100} = \sqrt{10^2} = 10$. All the others have the form $\sqrt{n}$ for n equal to $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2$, and 9^2. The rest are irrational. It follows that there are ten rational numbers of the form $\sqrt{n}$ with $1 \leq n \leq 100$. All are positive integers. The second situation is done in the same way. Since $1000^2 = 1{,}000{,}000$, exactly one thousand numbers of the form $\sqrt{n}$ with $1 \leq n \leq 1{,}000{,}000$ are rational numbers. They are those with n equal to $1^2, 2^2, 3^2, \ldots, 999^2$, and 1000^2. Again, all are positive integers.

4.4. a) Let $r = 1.\underline{7}7777\ldots$. So $10r = 17.\underline{7}7777\ldots$. Therefore $10r - r = 16$ and $r = \frac{16}{9}$.

b) Put $r = 2.\underline{67}6767\ldots$. So $100r = 267.\underline{67}67\ldots$. Hence $100r - r = 265$ and $r = \frac{265}{99}$. (Why would the use of $10r$ not be productive?)

c) Let $r = 4.\underline{728}728\ldots$. Since $1000r = 4728.\underline{728}728\ldots$, it follows that $1000r - r = 4724$. Therefore $r = \frac{4724}{999}$.

d) This computation is more challenging. As before, let $r = 35.34672\underline{638}638\ldots$. Notice that by moving the decimal point five places to the right, it ends up to the left of the first sequence of 638. So $100{,}000r = 3534672.\underline{638}638$ and hence $100{,}000{,}000r = 3534672638.\underline{638}638\ldots$. It follows that

$$100{,}000{,}000r - 100{,}000r = 3{,}534{,}672{,}638 - 3{,}534{,}672 = 3{,}531{,}137{,}966$$

and hence $r = \frac{3{,}531{,}137{,}966}{99{,}900{,}000}$.

4.5. $\frac{5}{4} = 1 + \frac{1}{4} = 1 + 0.25 = 1.25 = 1.25\underline{0}00$

$$\begin{array}{r|l}
 & 2.3636\ldots \\ \hline
198 & 468 \\
 & \underline{396} \\
 & \;720 \\
 & \;\underline{594} \\
 & \;1260 \\
 & \;\underline{1188} \\
 & \quad 720 \\
 & \quad \underline{594}
\end{array}$$

So $\frac{468}{192} = 2.\underline{36}36\ldots$

4.6. Let u and v be any two distinct real numbers. Let u be the smaller one and v the larger one. Whether they are large or small, there is a first digit (from the left) where they are different, say the billion digit, the 10 million digit, the 10 digit, the 10th digit, the millionth digit, or whatever. Let u_1 and v_1 be the two numbers in this "place." So $u_1 < v_1$. Let r be the number that is the same as both u and v up to the first place where they are different, let it equal r_1 with $u_1 < r_1 \leq v_1$ in this first place, and let all the digits be equal to zero thereafter. Given the way it is constructed, this number r satisfies $u < r \leq v$ and it is rational because it has the repeating 0000... at the end. All this is best illustrated with an example. Let $u = 91648327.1403528...$ and $v = 91648327.1403538....$ The first "place" at which u and v differ is the millionth place where u has a 2 and v a 3. The recipe described above tells us that $r = 91648327.1403530000...$ is a rational number between u and v.

4.7. Turn to Section 7.10 and see that $e = 2.7182818284590452....$ Consider the number $r = 2.7\underline{1828}18281828....$ Since $10r = 27.\underline{1828}1828...$ and $10{,}000(10r) = 271828.\underline{1828}1828...$, we see that $100{,}000r - 10r = 271828 - 27$. Therefore $99{,}990r = 271{,}801$ and hence $r = \frac{271{,}801}{99{,}990}$. A one step division tells us that $r = 2\,\frac{71{,}821}{99{,}990}$.

4.8. A 3×5 rectangle can be divided into 15 identical 1×1 squares. Each of these can be divided into 4 identical $\frac{1}{2} \times \frac{1}{2}$ squares. This provides a division of the rectangle into 60 identical squares. This process can be continued, to give subdivisions of the rectangle into more and more identical squares that are tinier and tinier. A rectangle with sides $\sqrt{2}$ and $\sqrt{8} = 2\sqrt{2}$ can be divided into 2 squares of size $\sqrt{2} \times \sqrt{2}$. As was just illustrates for 1×1 squares, these in turn can be subdivided identically into smaller and smaller squares.

4.9. Suppose that the rectangle R has been subdivided into such a finite array of identical squares. Let each of these squares have side length s. Suppose that there are n squares along the side a and m squares along the side b. Then $a = ns$ and $b = ms$. Hence $\frac{a}{b} = \frac{ns}{ms} = \frac{n}{m}$ is a rational number. Suppose, conversely, that $\frac{a}{b} = \frac{n}{m}$ is a rational number. Let $s = \frac{a}{n} = \frac{b}{m}$. Since $a = ns$ and $b = ms$, the rectangle R can be subdivided into nm identical squares of side s.

4.10. Neither $\frac{\sqrt{2}}{1}$ nor $\frac{\sqrt{6}}{\sqrt{3}} = \frac{\sqrt{2}\sqrt{3}}{\sqrt{3}} = \sqrt{2}$ is rational, so that neither of these rectangles can be subdivided into identical squares. Since $\frac{\sqrt{45}}{\sqrt{20}} = \frac{\sqrt{3^2 5}}{\sqrt{2^2 5}} = \frac{3\sqrt{5}}{2\sqrt{5}} = \frac{3}{2}$ is a rational number, the third rectangle can be subdivided into identical squares.

4.11. The steps are these:

$$\begin{aligned} 4x^2 - 8x - 12 &= 4[x^2 - 2x - 3] = 4[x^2 - 2x + (\tfrac{2}{2})^2 - (\tfrac{2}{2})^2 - 3] \\ &= 4[(x^2 - 2x + 1) - 1 - 3] = 4[(x-1)^2 - 4]. \end{aligned}$$

The smallest value that $4x^2 - 8x - 12$ can have is $4[0-4] = -16$ and it occurs for $x = 1$. The solutions of $4x^2 - 8x - 12 = 0$ are the same as those of $4[(x-1)^2 - 4] = 0$ and in turn those of $(x-1)^2 = 4$. Since $x - 1 = \pm 2$, the solutions are $x = -1, 3$.

4.12. Follow the recipe to get

$$\begin{aligned} -5x^2 + 3x + 4 &= -5(x^2 - \tfrac{3}{5}x - \tfrac{4}{5}) = -5\big(x^2 - \tfrac{3}{5}x + (\tfrac{3}{10})^2 - (\tfrac{3}{10})^2 - \tfrac{4}{5}\big) \\ &= -5\big((x - \tfrac{3}{10})^2 - (\tfrac{3}{10})^2 - \tfrac{4}{5}\big) = -5\big((x - \tfrac{3}{10})^2 - \tfrac{9}{100} - \tfrac{80}{100}\big) = -5\big((x - \tfrac{3}{10})^2 - \tfrac{89}{100}\big). \end{aligned}$$

So $-5x^2 + 3x + 4 = 0$ when $x - \frac{3}{10} = \pm\sqrt{\frac{89}{100}} = \pm\frac{\sqrt{89}}{10}$ and hence when $x = \frac{3}{10} \pm \frac{\sqrt{89}}{10}$.

Doing exactly the same thing with $-5x^2 + 3x - 4$, we get

$$-5x^2 + 3x - 4 = -5\big((x - \tfrac{3}{10})^2 - (\tfrac{3}{10})^2 + \tfrac{4}{5}\big) = -5\big((x - \tfrac{3}{10})^2 + \tfrac{71}{100}\big).$$

Since $(x - \frac{3}{10})^2 \geq 0$ for any x, $-5x^2 + 3x - 4 = -5\big((x - \frac{3}{10})^2 + \frac{71}{100}\big)$ is always negative and hence never 0.

4.13. Note that $3x^2 + 21x + 12 = 3(x^2 + 7x + 4)$. By completing the square,

$$x^2 + 7x + 4 = x^2 + 7x + (\tfrac{7}{2})^2 - (\tfrac{7}{2})^2 + 4 = (x + \tfrac{7}{2})^2 + 4 - \tfrac{49}{4} = (x + \tfrac{7}{2})^2 - \tfrac{33}{4}.$$

So $x^2 + 7x + 4$ is equal to 0 precisely when $x = -\frac{7}{2} \pm \frac{\sqrt{33}}{2}$.

4.14. By completing the square, we get

$$\begin{aligned} ax^2 + bx + c &= a[x^2 + \tfrac{b}{a}x + \tfrac{c}{a}] = a[x^2 + \tfrac{b}{a}x + (\tfrac{b}{2a})^2 - (\tfrac{b}{2a})^2 + \tfrac{c}{a}] \\ &= a[\big(x^2 + \tfrac{b}{a}x + (\tfrac{b}{2a})^2\big) - \tfrac{b^2}{(2a)^2} + \tfrac{4ac}{(2a)^2}] = a[(x + \tfrac{b}{2a})^2 - \tfrac{b^2-4ac}{(2a)^2}]. \end{aligned}$$

So $ax^2 + bx + c = 0$ translates to $a[(x + \frac{b}{2a})^2 - \frac{b^2-4ac}{(2a)^2}] = 0$ and hence $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{(2a)^2}$. Notice that $b^2 - 4ac \geq 0$. So we get, $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{(2a)^2}} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}$, and therefore $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $a = 0$, the equation $ax^2 + bx + c = 0$ reduces to $bx = -c$. If $b \neq 0$, then $x = -\frac{c}{b}$. And if $b = 0$?

4.15. Notice that the white areas of the three regions are equal. Therefore

$$x^2 + dx = x^2 + 2(\tfrac{d}{2} \cdot x) = (x + \tfrac{d}{2})^2 - (\tfrac{d}{2})^2.$$

This is exactly what you get if you complete the square for $x^2 + dx$.

4.16. If x is the intial number of apples, then $x = \frac{x}{5} + \frac{x}{12} + \frac{x}{8} + \frac{x}{20} + \frac{x}{4} + \frac{x}{7} + 30 + 120 + 300 + 50$. Therefore $x - \frac{x}{5} - \frac{x}{12} - \frac{x}{8} - \frac{x}{20} - \frac{x}{4} - \frac{x}{7} = 500$. Since $5 \cdot 12 \cdot 2 \cdot 7 = 840$ is a common denominator, we get

$$\frac{840x - 168x - 70x - 105x - 42x - 210x - 120x}{840} = 500.$$

So $125x = 840 \cdot 500$, and hence $x = 3360$.

4.17. Let x, y, z and w be the respective weights of the gold, brass, tin, and iron in the crown. Observe that

$$x+y=\tfrac{2}{3}\cdot 60=40;\ x+z=\tfrac{3}{4}\cdot 60=45;\ x+w=\tfrac{3}{5}\cdot 60=36;\ \text{and } x+y+z+w=60.$$

This system of 4 equations in 4 unknowns can be solved as follows: Since $y=40-x$, $z=45-x$, and $w=36-x$, we get that $x+(40-x)+(45-x)+(36-x)=60$. So $-2x+121=60$, and $x=30\frac{1}{2}$. Therefore, $y=9\frac{1}{2}, z=14\frac{1}{2}$, and $w=5\frac{1}{2}$.

4.18. We will assume that the daily production of the son and son-in-law is 200 and 250 bricks respectively (even though they do not appear to have worked a full day). So the three together can make $300+200+250=750$ bricks in one day. Therefore, they will require $\frac{300}{750}=\frac{2}{5}$ of a day to make 300. Observe that the answer needs to be given in hours. While the wording of the problem suggests that "day" means "working day," it is not clear how many hours such a working day has. If it has eight hours, then the three brick makers will need $\frac{2}{5}\cdot 8=\frac{16}{5}=3\frac{1}{5}$ hours. And if it has twelve?

4.19. Let x be "thy" age in years. The determination of x is the only possible question. A translation tells us that $x=\frac{1}{6}x+\frac{1}{8}x+\frac{1}{3}x+27$. So $x=\frac{4x+3x+8x}{24}+27$ and hence $24x-4x-3x-8x=(24)(27)$. So $x=3(24)=72$ years. The interpretation that "likewise a wife and a later-born son" might involve an additional $\frac{1}{3}x$ would result in the equation $24x-4x-3x-8x-8x=(24)(27)$ or $x=648$ years. Not likely.

4.20. To bring the problem to a "common denominator," notice that in 12 days, the first spout will fill 12 tanks, the second 6 tanks, the third 4 tanks, and the fourth will fill 3 tanks. So in 12 days the four spouts together will fill $12+6+4+3=25$ tanks. So together, they will fill one tank in $\frac{12}{25}$ of a day.

4.21. Two questions might be: How many apples did each Grace have in her basket and how many apples did each of them give to each Muse? The phrase "the nine and the three" tells us that there are three Graces. The hint tells us to let each of the three Graces have x apples and to let each Grace give y apples to the Muses. So the Muses get a total of $3y$ apples. Since each Muse gets the same number of apples, each Muse must get $\frac{y}{3}$ apples. Since they all have the same number of apples at the end, $x-y=\frac{y}{3}$, so that $y=\frac{3}{4}x$. The equality $y=\frac{3}{4}x$ meets all the requirements if x is taken to be a multiple of 4. If $x=4n$ then $y=3n$ and each of the 12 goddesses would have n apples after the exchange.

An aside: In Greek mythology, the Graces were three (or more) minor deities, commonly regarded to be daughters of Zeus. They were patrons of various pleasures in life, such as play, amusement, rest, happiness, and relaxation. The Muses, also daughters of Zeus, were Greek goddesses who presided over literature, science and the arts. They were often invoked at the beginning of various lyrical poems—such as in the Homeric epics—to inspire the poet or to speak through the poet's words. The nine Muses were Calliope (epic poetry), Clio (history), Euterpe (lyric poetry), Thalia (comedy and pastoral poetry), Melpomene (tragedy), Terpsichore (dance), Erato (love poetry), Polyhymnia (sacred poetry, and Urania (astronomy).

4.22. Let x be Diophantus's age when he died. Check that

$$x = \tfrac{x}{6} + \tfrac{x}{12} + \tfrac{x}{7} + 5 + \left(\tfrac{x}{2} + 4\right) = \tfrac{x}{6} + \tfrac{x}{12} + \tfrac{x}{7} + \tfrac{x}{2} + 9.$$

After multiplying through by the common denominator $7{\cdot}12 = 84$, we get

$$84x = 14x + 7x + 12x + 42x + 9{\cdot}84 = 75x + 9{\cdot}84.$$

So $x = 84$.

4.23. Since $\frac{x^2-y^2}{x+y} = \frac{(x+y)(x-y)}{x+y} = x - y = b$, we get $x = y + b$. Since $\frac{x}{y} = a$, x is also equal to $x = ay$. Therefore $ay = y + b$ and $(a-1)y = b$. Since $b > 0$, $a \neq 1$. Therefore $y = \frac{b}{a-1}$ and $x = \frac{ab}{a-1}$.

4.24. Since $a \neq 0$, we know that $x \neq 0$ and $y = \frac{a}{x}$. Since $x + \frac{a}{x} = b$, it follows that $\frac{x^2+a}{x} = b$, and hence that $x^2 - bx + a = 0$. By the quadratic formula $x = \frac{b \pm \sqrt{b^2-4a}}{2}$. Using $y = \frac{a}{x}$ we get $y = \frac{2a}{b \pm \sqrt{b^2-4a}}$.

4.25. Let x, y, z, and w be the four numbers. We know that

$$x + y + z = 22,\ x + y + w = 24,\ x + z + w = 27, \text{and } y + z + w = 20.$$

Substracting the last equation from each of the other three gives us:

$$(a)\ x - w = 2,\ (b)\ x - z = 4,\ \text{and } (c)\ x - y = 7.$$

Subtracting (a) from (b) and then from (c), gives $w - z = 2$ and $w - y = 5$. So $z = w - 2$ and $y = w - 5$. Therefore using $y + z + w = 20$, we get $(w-5) + (w-2) + w = 20$. So $3w = 27$. Hence $w = 9$. It follows that $z = 7, y = 4$, and $x = 11$.

4.26. Since the ratios $DC : CA : AD$ are equal to $3 : 4 : 5$, we let $DC = 3x$, and get that $CA = 4x$ and $AD = 5x$. Since DC and CA are integers, $CA - DC = x$ is an integer. Let $DB = y$ and $AB = z$. Putting all this into the triangle we get Figure 4.31. Observe that $\tan \frac{\theta}{2} = \frac{3x}{4x} = \frac{3}{4}$, and $\tan\ \theta = \frac{3x+y}{4x}$. Using the formula, $\tan \theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}$, we get

$$\tfrac{3x+y}{4x} = \tfrac{\frac{3}{2}}{1-\frac{9}{16}} = \tfrac{3}{2}{\cdot}\tfrac{16}{7} = \tfrac{24}{7}.$$

So $21x + 7y = 96x$, and $75x = 7y$. Since 7 is a prime that divides $75x$, it must divide 75 or x. But it does not divide 75, so 7 divides x. Therefore, 7 is the smallest possible value for x. Does $x = 7$ work? With $x = 7, DC = 21, CA = 28$, and $AD = 35$. Since $75x = 7y$, $y = 75$. By Pythagoras's theorem:

$$\begin{aligned} z^2 &= 16x^2 + (3x+y)^2 = 16{\cdot}49 + (21+75)^2 = 16{\cdot}49 + (6{\cdot}16)^2 \\ &= 16(49 + 6^2 16) = 16(49+576) = 16{\cdot}625 = 4^2{\cdot}25^2 = 100^2. \end{aligned}$$

Therefore $z = 100$. Finally, we need to check whether the condition that AD bisects the angle θ at A is met. Let $\alpha = \angle CAD, \alpha' = \angle DAB$, and let β be the angle at B. Notice that $\sin\alpha = \frac{3}{5}$. If we can show that $\sin\alpha' = \frac{3}{5}$, then $\alpha = \alpha'$ because both α and α' are acute angles. By the law of sines, $\frac{\sin\alpha'}{y} = \frac{\sin\beta}{5x}$. So $\sin\alpha' = \frac{75}{35}\sin\beta = \frac{15}{7}\sin\beta$. Since $\sin\beta = \frac{4x}{z} = \frac{28}{100} = \frac{7}{25}$, we get $\sin\alpha' = \frac{15}{7}\cdot\frac{7}{25} = \frac{3}{5}$.

4.27. The hint is easily carried out. Since $c^2 = x^2 + (2x-c)^2 = x^2 + 4x^2 - 4xc + c^2$, we get $5x^2 - 4xc = 0$. With $x \neq 0$, we get $5x = 4c$. So $x = \frac{4}{5}c$. In this way, Diophantus comes up with $c^2 = (\frac{4}{5})^2c^2 + (\frac{3}{5})^2c^2$. If c is rational, then c^2 is rational, so that c^2 is the sum of the two rational numbers $(\frac{4}{5})^2c^2$ and $(\frac{3}{5})^2c^2$. (Observe that $(\frac{4}{5})^2 + (\frac{3}{5})^2 = \frac{16+9}{25} = 1$.)

Assume that the word "quantity" in Problems 4.28 and 4.29 refers to the same number in each case. (Otherwise there would be two different variables involved, and it would not be possible to solve the equations that arise.)

4.28. With x the quantity in question, we get $(\frac{x}{3}+1)(\frac{x}{4}+1) = 20$. So $\frac{x^2}{12} + \frac{x}{3} + \frac{x}{4} + 1 = 20$, and hence

$$\tfrac{x^2}{12} + \tfrac{4x+3x}{12} + 1 = 20 \text{ and } x^2 + 7x - 228 = 0.$$

By the quadratic formula, $x = \frac{-7\pm\sqrt{49-4(-228)}}{2} = \frac{-7\pm\sqrt{961)}}{2} = \frac{-7\pm 31}{2} = -19, 12$.

4.29. With x the quantity, we get $(\frac{x}{3})(\frac{x}{4}) = x + 24$. So $\frac{x^2}{12} - x - 24 = 0$. Hence $x^2 - 12x - 288 = 0$, and by the quadratic formula

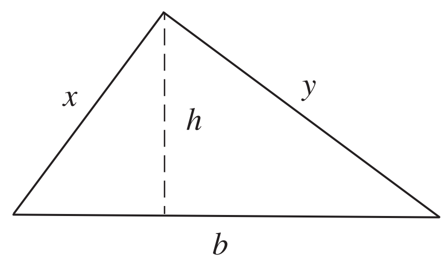
$$x = \tfrac{12\pm\sqrt{144-4(-288)}}{2} = \tfrac{12\pm\sqrt{144+1152}}{2} = \tfrac{12\pm\sqrt{1296}}{2} = \tfrac{12\pm 36}{2} = -12, 24.$$

4.30. There is an ambiguity here. The instructions have two interpretations. The equality $(x+7)(\sqrt{3}x) = 10x$ is one of them and $(x+7)(\sqrt{3x}) = 10x$ is another. The second one is more interesting and we will pursue it. Dividing both sides by $\sqrt{x}$, we get $\sqrt{3}(x+7) = 10\sqrt{x}$. After squaring both sides $3(x^2 + 14x + 49) = 100x$, and hence $3x^2 - 58x + 147 = 0$. By the Pythagorean theorem

$$x = \tfrac{58\pm\sqrt{58^2-4\cdot3\cdot147}}{6} = \tfrac{58\pm\sqrt{58^2-1764}}{6} = \tfrac{58\pm\sqrt{1600}}{6} = \tfrac{58\pm 40}{6} = 3, \tfrac{49}{3}.$$

4.31. By the Pythagorean theorem, the height h of an equilateral triangle of side a satisfies $h^2 + (\frac{a}{2})^2 = a^2$. Therefore $h^2 = \frac{3}{4}a^2$ and hence $h = \frac{\sqrt{3}}{2}a$. So the area of this triangle is $\frac{1}{2}ah = \frac{\sqrt{3}}{4}a^2$. Gerbert's value $\frac{a}{2}(a - \frac{a}{7}) = \frac{a}{2}(\frac{6}{7}a) = \frac{3}{7}a^2$ for the area leads to the approximation $\frac{\sqrt{3}}{4} \approx \frac{3}{7}$, or $0.4330 \approx 0.4286$.

4.32. We'll consider the right triangle shown below. The two perpendicular sides are x and y, its base is the hypothenuse $b = \sqrt{x^2 + y^2}$ and h is its height. The area A of the triangle is both $A = \frac{1}{2}h\sqrt{x^2 + y^2}$ and $A = \frac{1}{2}xy$. We need to solve these equations for



x and y in terms of A and h. Since $y = \frac{2A}{x}$ and $4A^2 = h^2(x^2 + y^2)$, we get $4A^2 = h^2(x^2 + \frac{4A^2}{x^2})$. After multiplying through by x^2, we get $4x^2A^2 = h^2(x^4 + 4A^2)$ and therefore, $h^2(x^2)^2 - 4A^2x^2 + 4h^2A^2 = 0$. By the quadratic formula,

$$x^2 = \frac{4A^2 \pm \sqrt{16A^4 - 4h^2(4h^2A^2)}}{2h^2} = \frac{4A^2 \pm 4A\sqrt{A^2 - h^4}}{2h^2} = \frac{2A^2 \pm 2A\sqrt{A^2 - h^4}}{h^2} = \frac{2A}{h^2}(A \pm \sqrt{A^2 - h^4}).$$

So $x = \frac{\sqrt{2A}}{h}\sqrt{A \pm \sqrt{A^2 - h^4}}$ and $y = \frac{2A}{x} = \frac{\sqrt{2A}\,h}{\sqrt{A \pm \sqrt{A^2 - h^4}}}$ and we are done. Since $A > \sqrt{A^2 - h^4}$, both $+$ and $-$ can occur.

4.33. The roots of $-2x^2 + 7x - 5$ are $\frac{-7 \pm \sqrt{7^2 - 4(-2)(-5)}}{-4} = \frac{7 \pm 3}{4}$. So they are 1 and $\frac{5}{2}$. Since the terms $x - 1$ and $x - \frac{5}{2}$ both divide $-2x^2 + 7x - 5$ their product does as well. Check that $-2(x^2 - \frac{7}{2}x + \frac{5}{2}) = -2(x - 1)(x - \frac{5}{2})$.

4.34. The roots of $3x^2 - 9x + 8$ are $\frac{9 \pm \sqrt{9^2 - 4(3)(8)}}{6} = \frac{9 \pm \sqrt{-15}}{6}$. Since $\sqrt{-15}$ does not make sense within the real numbers, there are no real roots, and there is no factorization (with real coefficients).

4.35. The computation is

$$\begin{aligned}\left(x + \tfrac{b - \sqrt{b^2 - 4ac}}{2a}\right) \cdot \left(x + \tfrac{b + \sqrt{b^2 - 4ac}}{2a}\right) &= x^2 + \left(\tfrac{b - \sqrt{b^2 - 4ac}}{2a} + \tfrac{b + \sqrt{b^2 - 4ac}}{2a}\right)x + \left(\tfrac{b - \sqrt{b^2 - 4ac}}{2a}\right)\left(\tfrac{b + \sqrt{b^2 - 4ac}}{2a}\right) \\ &= x^2 + \tfrac{b}{a}x + \tfrac{1}{4a^2}(b^2 - (b^2 - 4ac)) = x^2 + \tfrac{b}{a}x + \tfrac{c}{a}.\end{aligned}$$

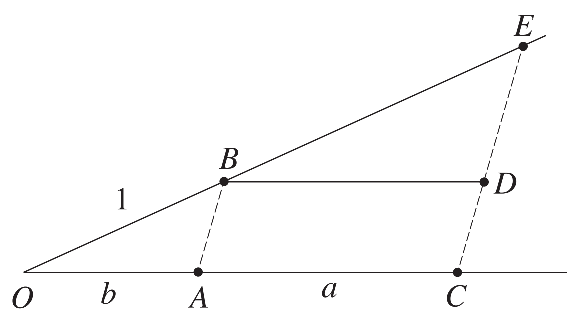
Since $a(x^2 + \frac{b}{a}x + \frac{c}{a}) = ax^2 + bx + c$, we are done.

4.36. If the polynomial $x^3 + 6x^2 - 9x - 14$ factors and has a root that is an integer, then this integer must divide 14. Start by checking $x = \pm 1$. At $x = 1$, the value of the polynomial is -16 ± 1, but at $x = -1$, the value is zero. So -1 is a root. Checking $x = \pm 2$, we find 2 is a root (but that -2 is not). Let's check $x = \pm 7$ next. A calculator confirms that $x = -7$ is a root. So the terms $x+1, x-2$, and $x+7$ all divide $x^3 + 6x^2 - 9x - 14$ and hence their product does also. Since $(x+1)(x-2)(x+7) = (x^2 - x - 2)(x+7) = x^3 - x^2 - 2x + 7x^2 - 7x - 14 = x^3 + 6x^2 - 9x - 14$, we are done.

4.37. Notice that $125 - 4 \cdot 25 - 4 \cdot 5 - 5 = 0$, so that $x = 5$ is a root of $x^3 - 4x^2 - 4x - 5$. A polynomial division confirms that $x^3 - 4x^2 - 4x - 5 = (x-5)(x^2 + x + 1)$. Consider the factor $x^2 + x + 1$. Since the term $b^2 - 4ac$ of the quadratic formula is $1 - 4(1)(1)$ and therefore negative, $x^2 + x + 1$ has no real roots and therefore no factors of the from $x - d$ with d real. So over the real numbers, the factorization $x^3 - 4x^2 - 4x - 5 = (x-5)(x^2 + x + 1)$ is complete.

4.38. Complete the segments BA and AC to a parallelogram $BACD$ with D a point on the segment CE. Since AC and BD are parallel, the triangles ΔOAB and ΔBDE are similar. So $\frac{b}{1} = \frac{BE}{BD} = \frac{BE}{a}$ and hence $ab = BE$.

4.39. The relabeled version of Figure 4.33 is shown below. The ratio of segments $\frac{a}{b}$ is defined to be the segment BE. Now regard $1, a, b$, and BE to be the lengths of the segments of the figure. Complete the segments BA and AC to the parallelogram $BACD$. By



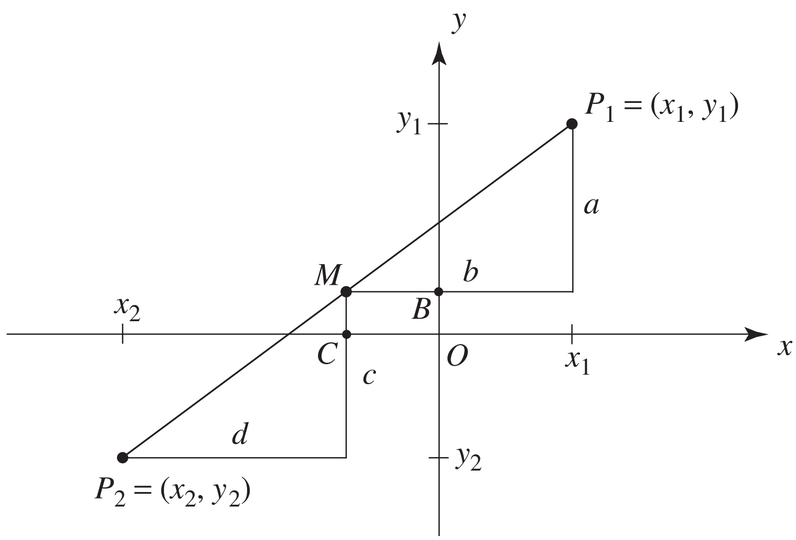
similar triangles, the lengths $1, b, a$, and BE satisfy $\frac{1}{b} = \frac{BE}{a}$. Since the length of BE is equal to $\frac{a}{b}$, the definition of the ratio of two segments agrees with the definition of the ratio of real numbers.

4.40. $\sqrt{(1-4)^2 + (1-5)^2} = \sqrt{3^2 + 4^2} = 5$ and $\sqrt{(1-(-1))^2 + (-6-(-3))^2} = \sqrt{13}$.

4.41. This length is the distance from $(-3, -7)$ to $(6, 8)$. So it is $\sqrt{(-3-6)^2 + (-7-8)^2} = \sqrt{9^2 + 15^2} = \sqrt{81 + 225} = \sqrt{306} \approx 17.49$.

4.42. Let's suppose that $a \leq b$ (because the argument in the other case is the same). It follows that $a \leq \frac{a+b}{2} \leq b$. The distance between a and $\frac{a+b}{2}$ is $\frac{a+b}{2} - a = \frac{b}{2} - \frac{a}{2}$ and that between b and $\frac{a+b}{2}$ is $b - \frac{a+b}{2} = \frac{b}{2} - \frac{a}{2}$. Since $c = \frac{a+b}{2}$ is the same distance from a and b, it is the midpoint of the segment on the number line that a and b determine.

For the situation in the plane, consider the figure below. Let $P_1 = (x_1, y_1)$ and $P_2 =$



(x_2, y_2) be two points in the coordinate plane and let M be the midpoint of the segment that connects them. The two triangles that P_1M and P_2M along with the accompanying vertical segments (with lengths a and c) and horizontal segments (with lengths b and d) determine are similar. By applying the "ratio of corresponding sides property" of similar triangles,we get the equalities $\frac{a}{P_1M} = \frac{c}{P_2M}$ and $\frac{b}{P_1M} = \frac{d}{P_2M}$. Since $P_1M = P_2M$, it follows that $a = c$ and $b = d$. But this means that the point B is the midpoint of the segment determined by the coordinates y_1 and y_2 and that C is the midpoint of the segment determined by the coordinates x_1 and x_2. From the earlier case of the number line, it follows that the y-coordinate of B is $\frac{y_1+y_2}{2}$ and that the x-coordinate of C is $\frac{x_1+x_2}{2}$. It follows that M is the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

4.43. By applying the conclusion of Problem 4.42, we get

i. The midpoint is $\left(\frac{1+7}{2}, \frac{3+15}{2}\right) = (4, 9)$.

ii. The midpoint is $\left(\frac{-1+8}{2}, \frac{6-12}{2}\right) = \left(\frac{7}{2}, -3\right)$.

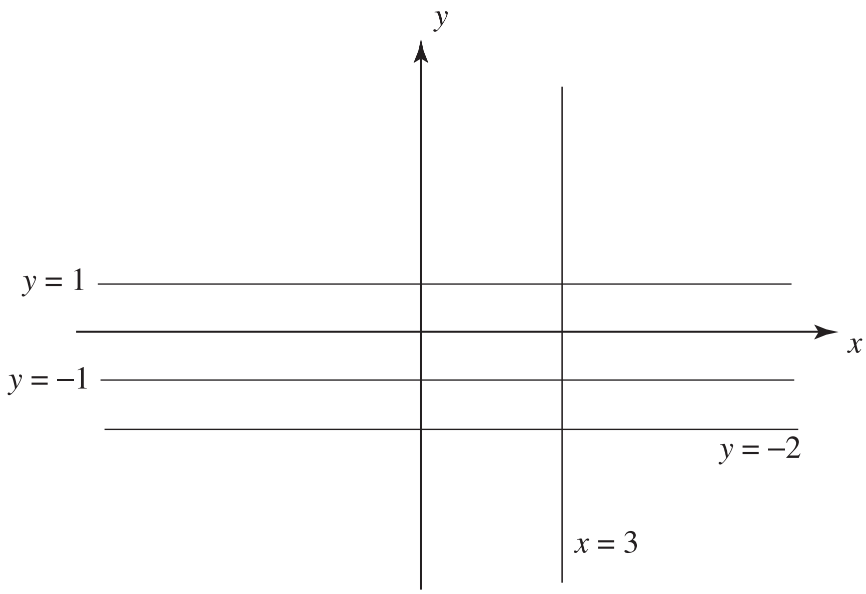
4.44. Using the distance formula, we get that the lengths of the segments AB, AC, and BC are

$$AB = \sqrt{(6-11)^2 + (-7-(-3))^2} = \sqrt{25+16} = \sqrt{41},$$
$$AC = \sqrt{(6-2)^2 + (-7-(-2))^2} = \sqrt{16+25} = \sqrt{41}, \text{ and}$$
$$BC = \sqrt{(11-2)^2 + (-3-(-2))^2} = \sqrt{81+1} = \sqrt{82}.$$

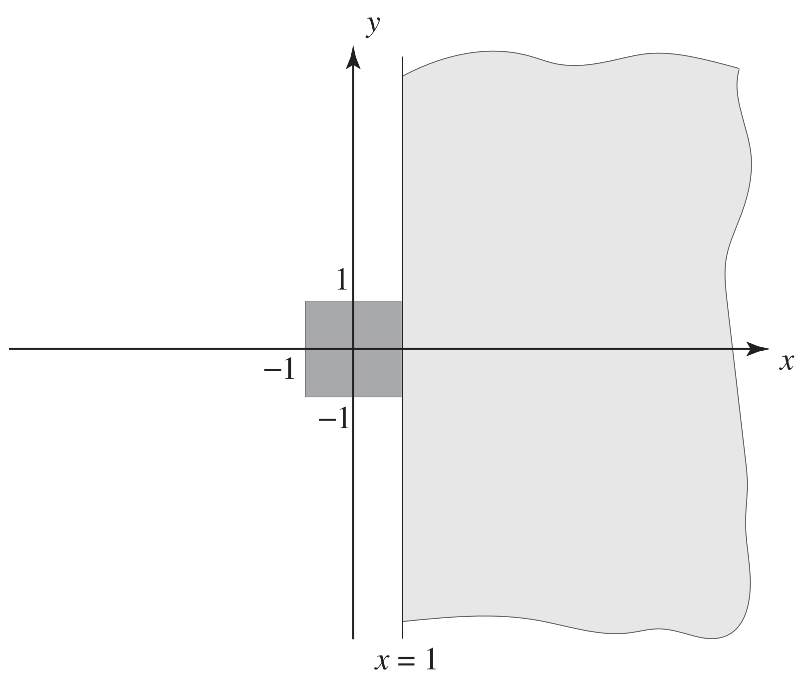
So $AB^2 + AC^2 = BC^2$, and therefore the triangle ΔABC is a right triangle with hypotenuse BC.

4.45. Notice that the x-coordinates of the points A, B, and C occur in increasing order. If the sum of the distance from A to B plus the distance from B to C is equal to the distance from A to C, then the points must lie on the same line. Let's check. Since $AB = \sqrt{(-1-3)^2 + (3-11)^2} = \sqrt{80} = 2\sqrt{20}$, $BC = \sqrt{(3-5)^2 + (11-15)^2} = \sqrt{20}$, and $AC = \sqrt{(-1-5)^2 + (3-15)^2} = \sqrt{180} = 3\sqrt{20}$, this is indeed the case.

4.46. The graphs of $x = 3$ and $y = -2$ are the lines labeled in the figure below. The graph of $y = |1|$ consists of the two lines $y = 1$ and $y = -1$. For xy to equal 0, either $x = 0$ or $y = 0$. So the graph of $xy = 0$ consists of the x-axis together with the y-axis.



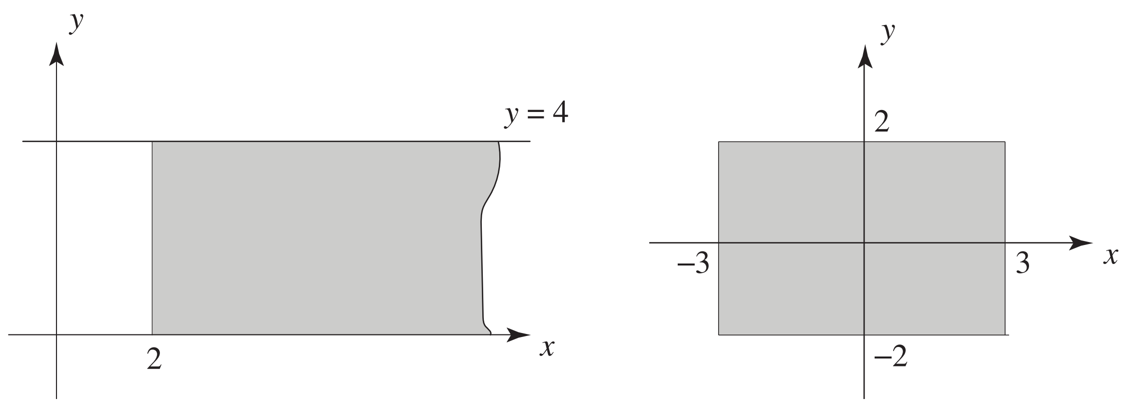
4.47. **i.** The set $\{(x, y) \mid x \geq 1\}$ of "all points (x, y) with the property that $x \geq 1$" is everything in the plane on the line $x = 1$ and to its right. See the region in light gray in the figure below.



ii. The set $\{(x, y) \mid |x| < 1 \text{ and } |y| < 1\}$ of "all points (x, y) with the property that $|x| < 1$ and $|y| < 1$" consists of all points (x, y) with $-1 < x < 1$ and $-1 < y < 1$. This is everything in the box depicted in dark gray in the figure (but not the boundary of the box).

4.48. **i.** The set $\{(x, y) \mid xy < 0\}$ consists of all (x, y) such that x and y are either both positive or both negative. In the context of Figure 4.4, this consists of the first and third quadrants (without the two coordinate axes).

ii. The set $\{(x, y) \mid 0 \leq y \leq 4 \text{ and } x \leq 2\}$ is depicted in gray on the left in the figure below. There is no restriction on this strip on the right (it goes on forever).



iii. The set $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$ consists of everything inside the gray box shown in the figure above on the right (except the boundary of the box).

4.49. This is easily sketched. The center of the circle is the point $(3, -5)$ and its radius is $\sqrt{7} \approx 2.65$.

4.50. This is $(x-3)^2 + (x-(-1))^2 = 5^2$ or $(x-3)^2 + (x+1)^2 = 25$.

4.51. This circle has center $(3, -7)$ and radius $\sqrt{9} = 3$. So its equation is $(x-3)^2+(x+7)^2 = 9$.

4.52. Rewrite the given equation as $x^2 - 4x + y^2 + 10y = -13$. Completing both squares and the balancing things by adding the appropriate constants on the right, we get

$$x^2 - 4x + (\tfrac{4}{2})^2 + y^2 + 10y + (\tfrac{10}{2})^2 = -13 + (\tfrac{4}{2})^2 + (\tfrac{10}{2})^2,$$
$$x^2 - 4x + 2^2 + y^2 + 10y + 5^2 = -13 + 4 + 25, \text{ and finally}$$
$$(x-2)^2 + (y+5)^2 = 16.$$

This is a circle with center $(2, -5)$ and radius 4.

4.53. By completing the square, first for the x-terms and then for the y-terms, we get

$$\begin{aligned} x^2+ax+y^2+by+c &= x^2+ax+\left(\tfrac{a}{2}\right)^2-\left(\tfrac{a}{2}\right)^2+y^2+bx+\left(\tfrac{b}{2}\right)^2-\left(\tfrac{b}{2}\right)^2+c \\ &= (x+\tfrac{a}{2})^2+(y+\tfrac{b}{2})^2+c-\left(\tfrac{a}{2}\right)^2-\left(\tfrac{b}{2}\right)^2 \\ &= (x+\tfrac{a}{2})^2+(y+\tfrac{b}{2})^2+c-\left(\tfrac{a}{2}\right)^2-\left(\tfrac{b}{2}\right)^2. \end{aligned}$$

So $x^2+y^2+ax+by+c=0$ translates to $(x+\frac{a}{2})^2+(y+\frac{b}{2})^2=\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2-c$. For this last equality to hold, we must have $\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2-c\geq 0$. For the equation $x^2+y^2+ax+by+c=0$ to represent a circle, a, b, and c need to satisfy the condition $\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2-c>0$. If this is so, then $\left(-\frac{a}{2},-\frac{b}{2}\right)$ is the center of the circle and $\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2-c}$ is its radius.

4.54. Completing the square for the term x^2+4x+7 does the trick. Since $x^2+4x+7=x^2+4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+7=(x+2)^2+3$, the equation of the parabola in rewritten form is $y=(x+2)^2+3$. So the smallest possible y-coordinate on the graph of the parabola is 3. The corresponding x-coordinate is -2. So $(-2,3)$ is the lowest point on the parabola. The parabola crosses the y axis at the point $(0,7)$. Since the point $(-4,7)$ is also on the parabola, it is now easy to sketch its graph.

4.55. The form of the equation $y=3x^2-2x+5$ tells us that the graph is a parabola with horizontal directrix. The discussion in Section 4.3 informs us that the general equation of such a parabola is

$$y=\left(\tfrac{1}{2(b-c)}\right)x^2-\left(\tfrac{a}{b-c}\right)x+\left(\tfrac{a^2+b^2-c^2}{2(b-c)}\right),$$

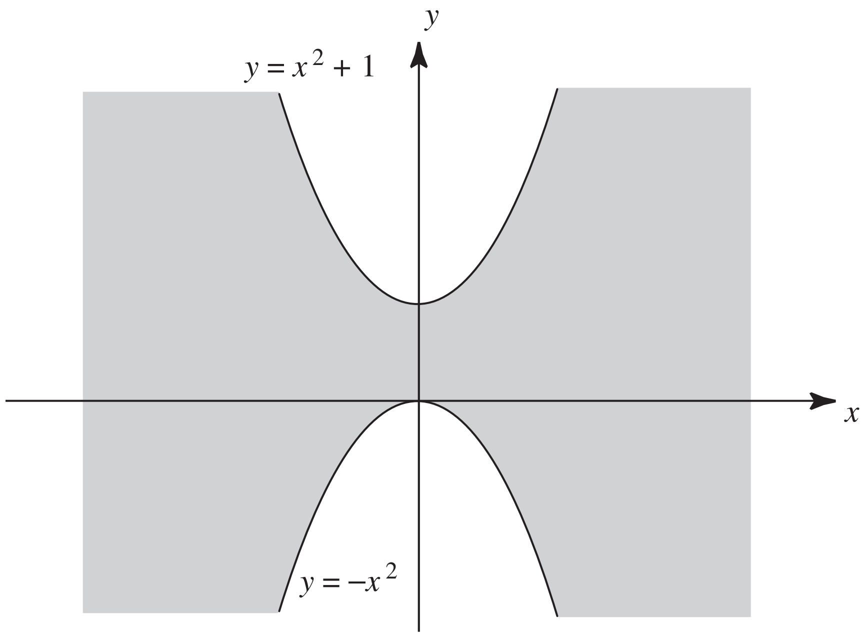
where (a,b) is the focus and $y=c$ is the directrix. In order to locate the focus and directrix of the parabola $y=3x^2-2x+5$, we need to set $\frac{1}{2(b-c)}=3$, $\frac{a}{b-c}=2$, and $\frac{a^2+b^2-c^2}{2(b-c)}=5$ and solve for a, b and c. The first equality tells us that $b-c=\frac{1}{6}$. So $\frac{a}{b-c}=6a=2$, and hence $a=\frac{1}{3}$. Since

$$5=\tfrac{a^2+b^2-c^2}{2(b-c)}=\tfrac{a^2}{2(b-c)}+\tfrac{(b-c)(b+c)}{2(b-c)}=\tfrac{1}{3^2\cdot 2(\frac{1}{6})}+\tfrac{b+c}{2}=\tfrac{1}{3}+\tfrac{b+c}{2},$$

it follows that $2+3(b+c)=30$, and hence that $b+c=\frac{28}{3}$. Since $b-c=\frac{1}{6}$, $2b=\frac{28}{3}+\frac{1}{6}=\frac{57}{6}=\frac{19}{2}$ and $b=\frac{19}{4}$. Finally, $c=b-\frac{1}{6}=\frac{19}{4}-\frac{1}{6}=\frac{57-2}{12}=\frac{55}{12}$. Therefore the focus of the parabola $y=3x^2-2x+5$ is the point $(a,b)=(\frac{1}{3},\frac{19}{4})$ and its directrix is the line $y=\frac{55}{12}$.

4.56. A look at the Figure 4.34 tells us that a point $P=(x,y)$ is on the parabola precisely when $\sqrt{(x-a)^2+(y-b)^2}=-x+c=-(x-c)$. So $(x-a)^2+(y-b)^2=(x-c)^2$ and therefore $x^2-2ax+a^2+y^2-2by+b^2=x^2-2cx+c^2$. Solving for x we get, $2cx-2ax=-y^2+2by-a^2-b^2+c^2$ or $2ax-2cx=y^2-2by+a^2+b^2-c^2$ and therefore $x=\frac{1}{2(a-c)}y^2-\frac{b}{a-c}y+\frac{a^2+b^2-c^2}{2(a-c)}$.

4.57. The figure below tells us that this infinite region consists of all points outside the two parabolas.



4.58. Let $P = (x, y)$ be any point in the plane. If $y = x^2 + 3x + 4$, then (x, y) is on the parabola of Figure 4.12. It follows that if $y > x^2 + 3x + 4$, then (x, y) lies above the parabola. If in addition $y < 4$, then (x, y) lies below the line $y = 4$. So if the point (x, y) is to lie inside the parabolic section of the figure, its coordinates must satisfy $x^2 + 3x + 4 < y < 4$. If the boundaries of the parabolic section are included, then the condition on the coordinates is $x^2 + 3x + 4 \le y \le 4$.

4.59. By completing the square we get

$$y = 3x^2 + 6x + 7 = 3(x^2 + 2x + \tfrac{7}{3}) = 3(x^2 + 2x + 1 - 1 + \tfrac{7}{3}) = 3((x + 1)^2 + \tfrac{4}{3}).$$

It follows that $(-1, 4)$ is the lowest point on the parabola. Note that $3x^2 + 6x + 7 = 8$ is equivalent to $3((x + 1)^2 + \frac{4}{3}) = 8$. Solving this equation for x, gives us $(x + 1)^2 = \frac{4}{3}$ and hence that $x = -1 \pm \frac{2}{\sqrt{3}}$. It follows that the parabola crosses the cut $y = 8$ at the points $(x, 8)$ with $x = -1 - \frac{2}{\sqrt{3}} \approx -2.15$ and $x = -1 + \frac{2}{\sqrt{3}} \approx 0.15$. Notice that the x-coordinate -1 of the lowest point of the parabola lies halfway between the x-coordinates of its two points of intersection with the line $y = 8$. It is now easy to sketch the parabolic section. Let (x, y) be any point in the plane. The point lies on the parabola if $y = 3x^2 + 6x + 7$ and above the parabola if $3x^2 + 6x + 7 < y$. It follows that the point lies within the parabolic section if $3x^2 + 6x + 7 < y < 8$.

4.60. Completing the square for $y = x^2 + x - 11$, we get

$$y = x^2 + x - 11 = x^2 + x + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 - 11 = \left(x + \tfrac{1}{2}\right)^2 - 11\tfrac{1}{4}$$

and doing so for $y = 2x^2 - 4x - 7$, we get

$$y = 2(x^2 - 2x - \tfrac{7}{2}) = 2(x^2 - 2x + 1 - 1 - \tfrac{7}{2}) = 2\big((x-1)^2 - \tfrac{9}{2}\big).$$

The lowest point of the parabola $y = x^2 + x - 11$ is $\big(-\frac{1}{2}, -11\frac{1}{4}\big)$. To find its x-intercepts, set

$$y = x^2 + x - 11 = \big(x + \tfrac{1}{2}\big)^2 - 11\tfrac{1}{4} = 0$$

to get $x = -\frac{1}{2} \pm \frac{\sqrt{45}}{2}$. So the x-intercepts are $x = -\frac{1}{2} - \frac{\sqrt{45}}{2} \approx -3.85$ and $x = -\frac{1}{2} + \frac{\sqrt{45}}{2} \approx 2.85$. The y-intercept is the point $(0, -11)$.

The lowest point of $y = 2x^2 - 4x - 7$ is $(1, -9)$. Setting $2x^2 - 4x - 7 = 0$ and solving for x, we get $2\big((x-1)^2 - \frac{9}{2}\big) = 0$ and $x = 1 \pm \frac{3}{\sqrt{2}}$ for the x-intercepts of this parabola. So the x-intercepts are $x = 1 - \frac{3}{\sqrt{2}} \approx 1.12$ and $x = 1 + \frac{3}{\sqrt{2}} \approx 3.12$. The y-intercept is the point $(0, -7)$.

To find the x-coordinates of the points of intersection of the two parabolas set $2x^2 - 4x - 7 = x^2 + x - 11$ and solve for x. So $x^2 - 5x + 4 = 0$. An easy factorization tells us that $(x-1)(x-4) = 0$, so that $x = 1$ or $x = 4$. It follows that the points of intersection are $(1, -9)$ and $(4, 9)$.

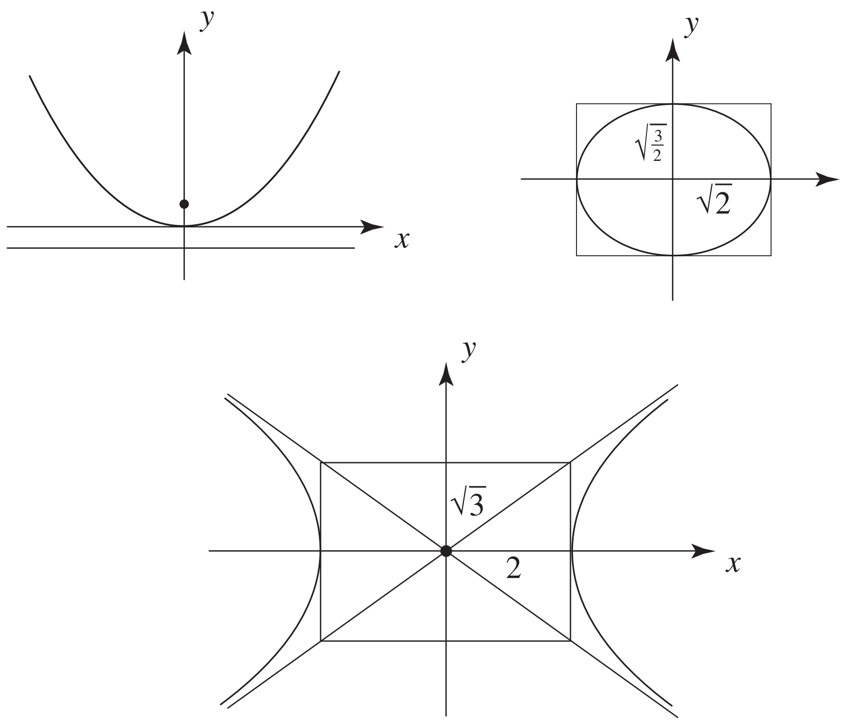
The two parabolas intersect for $x = 1$ and $x = 4$. But what happens for $1 < x < 4$? Checking at $x = 2$, we get $x^2 + x - 11 = -5$ and $2x^2 - 4x - 7 = -7$. It is not hard to check (use the quadratic formula) that $2x^2 - 4x - 7 < x^2 + x - 11$ for any x with $1 < x < 4$. Outside the interval $1 \le x \le 4$, things are reversed and $x^2 + x - 11 < 2x^2 - 4x - 7$. Given all that has been established, it is now fairly routine (plot additional points if needed) to sketch the graphs of both $y = x^2 + x - 11$ and $y = 2x^2 - 4x - 7$ and to see the relationship of the two graphs to each other. The set of points that the two parabolas enclose is the set of points above the parabola $y = 2x^2 - 4x - 7$ and below the parabola $y = x^2 + x - 11$. In set theoretic notation this is $\big\{(x, y) \,\big|\, 2x^2 - 4x - 7 \le y \le x^2 + x - 11\big\}$.

4.61. There is not much left to do. Only a little algebra remains. From $y = (\tan\theta)x - d$ and $d = bt^2$, it follows that $y = -bt^2 + (\tan\theta)x$. From $\cos\theta = \frac{x}{v_0 t}$, we get $t^2 = \frac{1}{(v_0\cos\theta)^2}x^2$ and therefore $y = \frac{-b}{(v_0\cos\theta)^2}\,x^2 + (\tan\theta)x$.

4.62. Comparing the equation $y = \frac{1}{2}x^2$ with equation $(*)$ of Section 4.3, we see that $\frac{1}{2(b-c)} = \frac{1}{2}$, $a = 0$, and $a^2 + b^2 - c^2 = 0$. It follows that $b - c = 1$ and $b^2 - c^2 = (b-c)(b+c) = b + c = 0$. So $2b = 1$ and hence $b = \frac{1}{2}$, and $c = -\frac{1}{2}$. Therefore the focal point of the parabola is $F = (a, b) = (0, \frac{1}{2})$ and the directrix is the horizontal line $y = -\frac{1}{2}$.

After dividing $3x^2 + 4y^2 = 6$ through by 6 we get $\frac{x^2}{2} + \frac{y^2}{\frac{3}{2}} = 1$. This is equal to $\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{\frac{3}{2}})^2} = 1$. So the ellipse has semimajor axis $a = \sqrt{2}$ and semiminor axis $b = \sqrt{\frac{3}{2}}$.

Dividing $3x^2 - 4y^2 = 12$ by 12, we get $\frac{x^2}{4} - \frac{y^2}{3} = 1$ and therefore $\frac{x^2}{2^2} - \frac{y^2}{(\sqrt{3})^2} = 1$. With $a = 2$ and $b = \sqrt{3}$, we see from Section 4.5 that the asymptotes of the hyperbola are the two lines $y = \frac{\sqrt{3}}{2}$ and $y = -\frac{\sqrt{3}}{2}$. Since $c^2 = a^2 + b^2 = 4 + 3 = 7$, the x-intercepts of the hyperbola are $\pm\sqrt{7}$.



The three graphs along with the relevant information are sketched above.

4.63. Since the ellipse has equation $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$, the analysis in Section 4.4 tells us that $a = 5$ is the semimajor axis and $b = 2$ is the semiminor axis. Form $a^2 = b^2 + c^2$ follows that $c = \sqrt{5^2 - 2^2} = \sqrt{21}$, so that the eccentricity is equal to $\varepsilon = \frac{c}{a} = \frac{\sqrt{21}}{5} \approx 0.92$.

4.64. Since the semimajor axis and semiminor axis are $a = 5$ and $b = 3$, respectively, the distance between the center $(0,0)$ and either focal point is $c = \sqrt{a^2 - b^2} = \sqrt{5^2 - 3^2} = 4$. So the two focal points are $(\pm 4, 0)$. The circle with center $(4,0)$ and radius 2 has equation $(x-4)^2 + y^2 = 4$. By multiplying $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ through by $5^2 \cdot 3^2$, we get $9x^2 + 25y^2 = 9 \cdot 25 = 225$. Using the equations of the circle and the ellipse together we see that the x-coordinate of a point of intersection satisfies

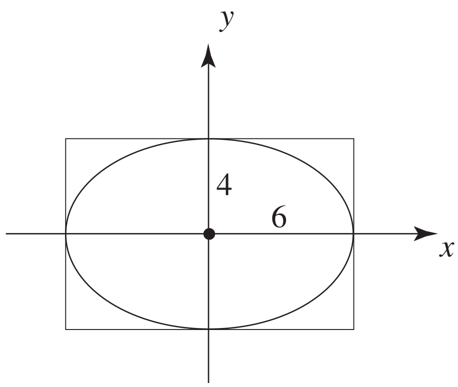
$$9x^2 + 25(4 - (x-4)^2) = 9x^2 + 25(4 - x^2 + 8x - 16) = -16x^2 + 8{\cdot}25x - 12{\cdot}25 = 9{\cdot}25$$

and hence that $-16x^2 + 8{\cdot}25x - 21{\cdot}25 = 0$. By the quadratic formula

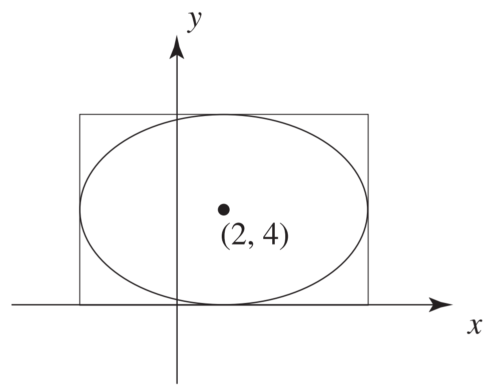
$$x = \frac{-8{\cdot}25 \pm \sqrt{8^2{\cdot}25^2 - 4{\cdot}16{\cdot}21{\cdot}25}}{-32} = \frac{-8{\cdot}25 \pm 8{\cdot}5\sqrt{25-21}}{-32} = \frac{8{\cdot}5(5\pm 2)}{32} = \frac{5(5\pm 2)}{4} = \frac{15}{4} \text{ or } \frac{35}{4}.$$

Since $-5 \le x \le 5$ for any point (x, y) on the ellipse, $x = \frac{35}{4}$ cannot arise. So $x = \frac{15}{4} = 3\frac{3}{4}$ is the only possibility. How to get the corresponding y-coordinates is clear.

4.65. The ellipse of the first equation has semimajor axis $a = 6$ and semiminor axis $b = 4$. The center of the ellipse is the origin $(0, 0)$ and it's focal axis is the x-axis. The ellipse is shown in Figure (a) below. The second equation is that of the ellipse shown in Figure (a) shifted without rotating it so that its center ends up at the point $(2, 4)$.

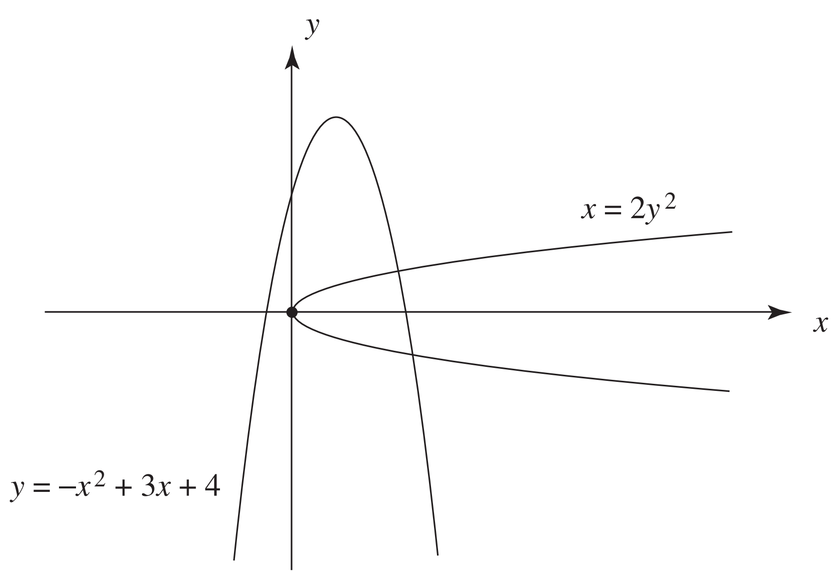


(a)

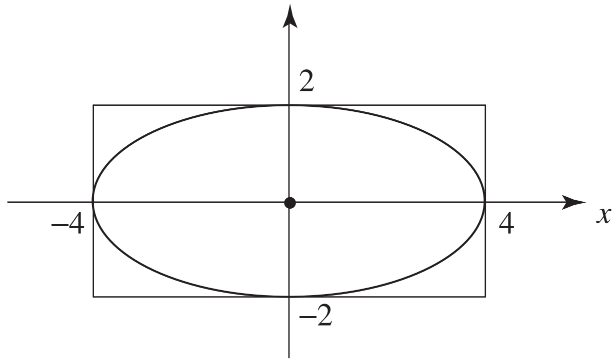


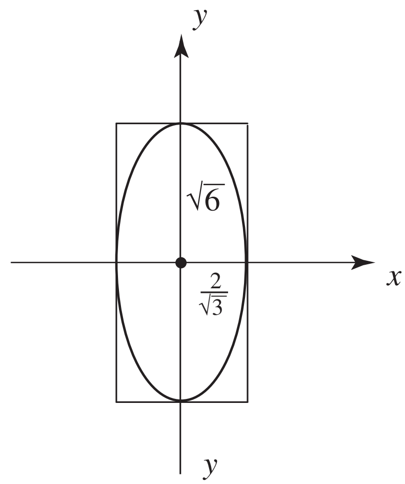
(b)

4.66. It follows from Section 4.3 that **i.** $y = -x^2 + 3x + 4$ is a parabola with horizontal directrix. The equation **ii.** $x = 2y^2$ represents a parabola with vertical directrix. The two parabolas are sketched below.

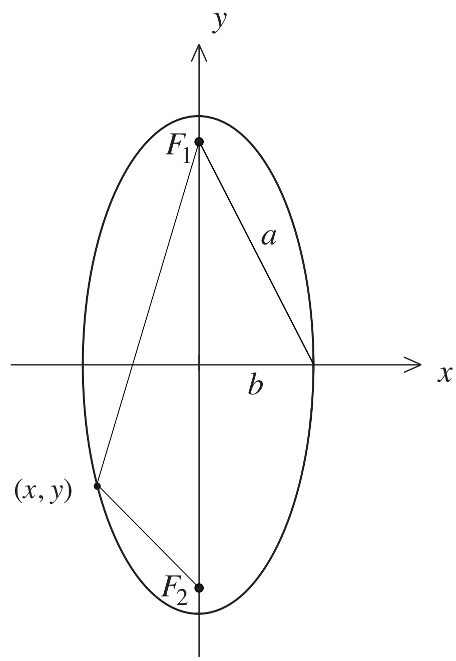


Since **ii.** $x^2 + 4y^2 = 16$ is equivalent to $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and hence to $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$, this is an ellipse of the sort studied in Section 4.4 with horizontal focal axis. The equation **iv.** $9x^2 + 2y^2 = 12$ is equivalent to $\frac{9}{12}x^2 + \frac{2}{12}y^2 = 1$ and in turn to $\frac{3}{4}x^2 + \frac{1}{6}y^2 = 1$ and $\frac{x^2}{\frac{4}{3}} + \frac{y^2}{6} = 1$ and $\frac{x^2}{(\frac{2}{\sqrt{3}})^2} + \frac{y^2}{(\sqrt{6})^2} = 1$. The two ellipses are sketched below.





4.67. For $c = \sqrt{a^2 - b^2}$ we'll take the points $F_1 = (0, c)$ and $F_2 = (0, -c)$ in the coordinate plane and show that the ellipse determined by these two points and the number $k = 2a$ has equation $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. This will imply that F_1 and F_2 are the focal points of this ellipse and that $k = 2a$ is the defining constant. The nature of the equation tells us in turn that the graph of the ellipse is as depicted in Figure 4.36. The derivation of the equation is in essence the same as the one provided in Section 4.4.



Let $P = (x, y)$ be a point in the plane. Then P is on the ellipse precisely when the sum $PF_1 + PF_2 = k = 2a$. In view of the figure above, we get the first equation below. The other equations follow in successive steps via algebraic maneuvers:

$$
\begin{aligned}
\sqrt{x^2+(y-c)^2}+\sqrt{x^2+(y+c)^2} &= 2a,\\
\sqrt{x^2+(y-c)^2} &= 2a-\sqrt{x^2+(y+c)^2},\\
x^2+(y-c)^2 &= 4a^2-4a\sqrt{x^2+(y+c)^2}+x^2+(y+c)^2,\\
(y-c)^2 &= 4a^2-4a\sqrt{x^2+(y+c)^2}+(y+c)^2,\\
y^2-2cy+c^2 &= 4a^2-4a\sqrt{x^2+(y+c)^2}+y^2+2cy+c^2,\\
a\sqrt{x^2+(y+c)^2} &= a^2+cy,\\
a^2\big(x^2+(y+c)^2\big) &= a^4+2a^2cy+c^2y^2,\\
a^2(x^2+y^2+2cy+c^2) &= a^4+2a^2cy+c^2y^2,\\
a^2x^2+a^2y^2+2a^2cy+a^2c^2 &= a^4+2a^2cy+c^2y^2,\\
a^2x^2+a^2y^2-c^2y^2 &= a^4-a^2c^2,\\
a^2x^2+(a^2-c^2)y^2 &= a^2(a^2-c^2).
\end{aligned}
$$

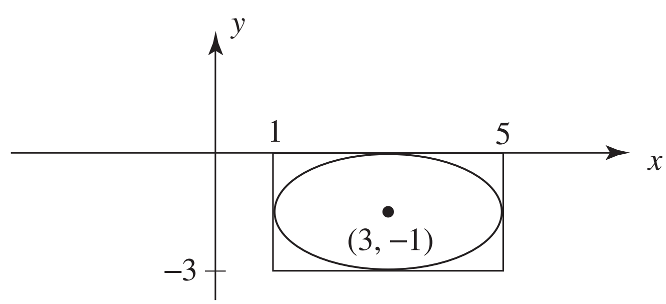
Since $b^2=a^2-c^2$, this last equation becomes $a^2x^2+b^2y^2=a^2b^2$. Because $a>c\geq 0$ and $b>0$, we can divide $a^2x^2+b^2y^2=a^2b^2$ by a^2b^2 to get

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

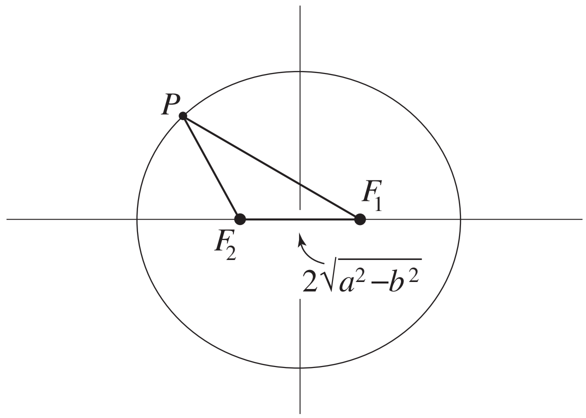
4.68. Since $B^2-4AC=0-4\cdot 1\cdot 4=-16$ is negative, the graph of $x^2+4y^2-6x+8y+9=0$ is an ellipse (if the equation is not degenerate). Completing squares gives us

$$x^2+4y^2-6x+8y+9=x^2-6x+3^2-3^2+4(y^2+2y+1^2-1^2)+9=(x-3)^2+4(y+1)^2-4,$$

so that $(x-3)^2+4(y+1)^2=4$. Therefore, $\frac{(x-3)^2}{2^2}+(y+1)^2=1$. It follows that this is indeed the equation of an ellipse. Its graph is obtained by ishifting the graph of the ellipse $\frac{x^2}{2^2}+y^2=1$, 3 units to the right and 1 unit down. So the center of the given ellipse is $(3,-1)$ and its semimajor and semiminor axes are 2 and 1, respectively.

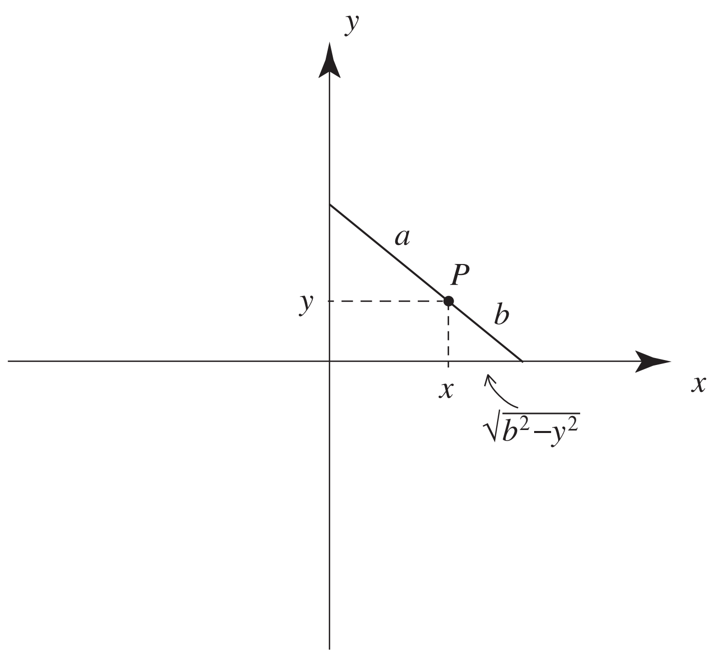


4.69. Let the points F_1 and F_2 and two lengths a and b with $a\geq b$ be given. Put a sheet of graph paper on your board and place the two points so that $F_1=(c,0)$ and $F_2=(-c,0)$ for some where $c=\sqrt{a^2-b^2}$. So the distance $2c=2\sqrt{a^2-b^2}$ is determined by a and b. With a string of length $2a+2\sqrt{a^2-b^2}$ in a loop and stretched as described, any point P that the pencil marks out has the property that $PF_1+PF_2=(2a+2\sqrt{a^2-b^2})-(2\sqrt{a^2-b^2})=2a$. But this means that we have



drawn an ellipse with defining constant $k = 2a$ and focal points F_1 and F_2. Since $2a$ is the defining constant, a is the semimajor axis and since $c = \sqrt{a^2 - b^2}$, $b = \sqrt{a^2 + c^2}$ is the semiminor axis.

4.70. The segment of fixed length and the fixed point P on it are shown. The distances from P to the two endpoints of the segment are a and b respectively. We'll assume that $a \geq b$. Suppose that P is in the first quadrant and let x and y be its coordinates. Consider the two right triangles with hypotenuse a and b that P determines and refer to the figure below. Since these two triangles are similar, and (by the Pythagorean



theorem) the base of this second triangle has length $\sqrt{b^2 - y^2}$, it follows that $\frac{x}{a} = \frac{\sqrt{b^2-y^2}}{b}$. Therefore

$$\tfrac{x^2}{a^2} = \tfrac{b^2-y^2}{b^2} = 1 - \tfrac{y^2}{b^2}.$$

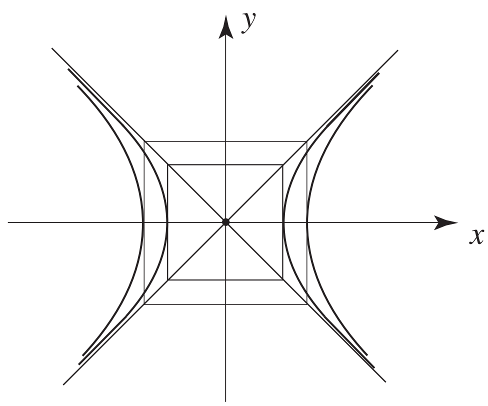
It follows that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This argument is easily modified for the situation where P is in the second (replace x by $-x$), third (replace x and y by $-x$ and $-y$), and fourth quadrants. Carry out the details for the fourth.

4.71. Completing the square is the key. Doing so, we get

$$\begin{aligned}16x^2 - 96x - 25y^2 - 100y - 356 &= 16(x^2 - 6x + 3^2 - 3^2) - 25(y^2 + 4y + 2^2 - 2^2) - 356\\ &= 16(x-3)^2 - 25(y+2)^2 - 16 \cdot 3^2 + 25 \cdot 2^2.\end{aligned}$$

Therefore $16(x-3)^2 - 25(y+2)^2 - 144 + 100 = 356$ and hence $4^2(x-3)^2 - 5^2(y+2)^2 = 400$. After dividing by $4^2 \cdot 5^2 = 16 \cdot 25 = 400$, we finally get $\frac{(x-3)^2}{5^2} - \frac{(y+2)^2}{4^2} = 1$. Taking $c = \sqrt{5^2 + 4^2} = \sqrt{41}$ in the discussion of Section 4.5, we get that the focal points are $(-\sqrt{41}, 0)$ and $(\sqrt{41}, 0)$ and that the eccentricity is $\frac{c}{a} = \frac{\sqrt{41}}{5}$.

4.72. The figure below shows the two defining rectangles for the two hyperbolas. They are both squares. The smaller square is the 2×2 square of the inner hyperbola $x^2 - y^2 = 1$ and the larger square is the $2\sqrt{2} \times 2\sqrt{2}$ square of the outer hyperbola $\frac{x^2}{(\sqrt{2})^2} - \frac{y^2}{(\sqrt{2})^2} = 1$. Since the diagonals of the two squares determine the same pair of lines, the asymptotes of the hyperbolas $x^2 - y^2 = 1$ and $\frac{x^2}{2} - \frac{y^2}{2} = 1$ are the same. They are the lines



$y = \pm\frac{1}{1}x = \pm\frac{\sqrt{2}}{\sqrt{2}}x$. The focal points are determined by the diagonals of the squares. For $x^2 - y^2 = 1$ are the points $(\pm\sqrt{2}, 0)$. Since $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$, the focal points of $\frac{x^2}{2} - \frac{y^2}{2} = 1$ are $(\pm 2, 0)$. The focal axis is the x-axis in either case.

4.73. Let's consider a point $P = (x, y)$ in the xy-plane of Figure 4.17. As shown in Section 4.5, such a point is on the right branch of the hyperbola precisely if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a.$$

By simplifying, squaring both sides, canceling, moving things around, and simplifying once more, this equation is transformed in successive steps to

$$\begin{gathered}
\sqrt{(x+c)^2+y^2} = 2a + \sqrt{(x-c)^2+y^2}, \\
(x+c)^2+y^2 = 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2, \\
(x+c)^2 = 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2, \\
x^2+2cx+c^2 = 4a^2 + 4a\sqrt{(x-c)^2+y^2} + x^2-2cx+c^2, \\
a\sqrt{(x-c)^2+y^2} = -a^2+cx, \\
a^2\big((x-c)^2+y^2\big) = a^4-2a^2cx+c^2x^2, \\
a^2(x^2-2cx+c^2+y^2) = a^4-2a^2cx+c^2x^2, \\
a^2x^2-2a^2cx+a^2c^2+a^2y^2 = a^4-2a^2cx+c^2x^2, \\
a^2x^2-c^2x^2+a^2y^2 = a^4-a^2c^2, \\
(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2).
\end{gathered}$$

Since $c^2 = a^2+b^2$ and hence $a^2-c^2 = -b^2$, this last equation becomes $b^2x^2-a^2y^2 = a^2b^2$. Because $a > 0$ and $b > 0$, we can divide $b^2x^2-a^2y^2 = a^2b^2$ by a^2b^2 to get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

For the left branch, start with the equation $\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = 2a$ and proceed as above.

4.74. Since any line has the form $ax+by+c=0$, the hint suggests that the equation $x^2+4xy+4y^2+6x+12y+9=0$ might be of the form $(ax+by+c)^2=0$. Since

$$(ax+by+c)^2 = a^2x^2+2ax(by+c)+(by+c)^2 = a^2x^2+2abxy+2acx+b^2y^2+2bcy+c^2,$$

we'll set $a^2x^2+2abxy+b^2y^2+2acx+2bcy+c^2 = x^2+4xy+4y^2+6x+12y+9$. To get $a^2=1, ab=2, b^2=4, ac=3, bc=6$, and $c^2=9$, we can take $a=1, b=2$, and $c=3$, and see that all six equations are satisfied. It follows that with the line $x+2y+3=0$, $x^2+4xy+4y^2+6x+12y+9=(x+2y+3)^2=0$. So the equation is degenerate and its graph is the single line $x+2y+3=0$.

4.75. If what is asserted is correct, then x and y satisfy $3x^2+19x-2xy-y^2+9y+14=0$ precisely if either $x-y+7=0$ or $3x+y-2=0$. This suggests that

$$(x-y+7)(3x+y-2) = 3x^2+19x-2xy-y^2+9y+14.$$

Since $(x-y+7)(3x+y-2) = 3x^2+xy-2x-3xy-y^2+2y+21x+7y-14 = 3x^2+19x-2xy-y^2+9y-14$, this equality holds and solves the problem.

4.76. Let $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be the equation of a conic section and consider the term $B^2 - 4AC$. If the conic section is a parabola, then it is always the case that $B^2 - 4AC = 1$.

For the circle $x^2 + y^2 - 1 = 0$, $A = 1, B = 0$, and $C = 1$, so that $B^2 - 4AC = -4$. Now let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the standard equation of an ellipse. After multiplying through by a^2b^2, we get $b^2x^2 + a^2y^2 - a^2b^2 = 0$. For this equation, $A = b^2, B = 0$, and $C = a^2$, so that $B^2 - 4AC = -4a^2b^2$. Take $a > 0$ very large and set $b = \frac{1}{a}$. Since a is large, b is small, so that this is a flat ellipse. The larger the choice of a, the flatter this ellipse. In all cases, $B^2 - 4AC = -4$.

4.77. Completing squares is again the key. Suppose first that $A \neq 0$ and $C \neq 0$. After completing squares,

$$\begin{aligned}
Ax^2 + Cy^2 + Dx + Ey + F &= A(x^2 + \tfrac{D}{A}x) + C(y^2 + \tfrac{E}{C}y) + F \\
&= A\big(x^2 + \tfrac{D}{A}x + (\tfrac{D}{2A})^2 - (\tfrac{D}{2A})^2\big) + C\big(y^2 + \tfrac{E}{C}y + (\tfrac{E}{2C})^2 - (\tfrac{E}{2C})^2\big) + F \\
&= A\big((x + \tfrac{D}{2A})^2 - (\tfrac{D}{2A})^2\big) + C\big((y + \tfrac{E}{2C})^2 - (\tfrac{E}{2C})^2\big) + F \\
&= A(x + \tfrac{D}{2A})^2 + C(y + \tfrac{E}{2C})^2 + F - \tfrac{D^2}{4A} - \tfrac{E^2}{4C}.
\end{aligned}$$

We must have $A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C})^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F$ with $\frac{D^2}{4A} + \frac{E^2}{4C} - F \geq 0$ since the original equation is that of a conic section. If $\frac{D^2}{4A} + \frac{E^2}{4C} - F = 0$, then $A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C})^2 = 0$. If A and C have the same sign, then we are in the degenerate case of the point $(x, y) = \big(-\frac{D}{2A}, -\frac{E}{2C}\big)$. If A and C have opposite signs, say $A > 0$ and $C < 0$, then $\sqrt{A}(x + \frac{D}{2A}) = \pm\sqrt{-C}(y + \frac{E}{2C})$ and we are in the degenerate situation of a line.

It follows that $A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C})^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F$ with $\frac{D^2}{4A} + \frac{E^2}{4C} - F > 0$. The next step is to divide this equation through by $\frac{D^2}{4A} + \frac{E^2}{4C} - F$. If both $A > 0$ and $C > 0$, we get (a positive constant p can be put in the form $p = (\sqrt{p})^2$) the rewritten version

$$\begin{aligned}
\frac{A(x+\frac{D}{2A})^2}{\frac{D^2}{4A}+\frac{E^2}{4C}-F} + \frac{C(y+\frac{E}{2C})^2}{\frac{D^2}{4A}+\frac{E^2}{4C}-F} &= \frac{(x+\frac{D}{2A})^2}{\frac{D^2}{4A^2}+\frac{E^2}{4AC}-\frac{F}{A}} + \frac{(y+\frac{E}{2C})^2}{\frac{D^2}{4AC}+\frac{E^2}{4C^2}-\frac{F}{C}} \\
&= \frac{(x+\frac{D}{2A})^2}{\Big(\sqrt{\frac{D^2}{4A^2}+\frac{E^2}{4AC}-\frac{F}{A}}\Big)^2} + \frac{(y+\frac{E}{2C})^2}{\Big(\sqrt{\frac{D^2}{4AC}+\frac{E^2}{4C^2}-\frac{F}{C}}\Big)^2} = 1.
\end{aligned}$$

of the original equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$. A look back at Section 4.4 tells us that in this rewritten form we are dealing with the shifted version of an ellipse with focal axis either the x- or the y-axis.

If A and B are both negative, then apply the above progression of computations to the equation $-Ax^2 - Cy^2 - Dx - Ey - F = 0$. If A and B are nonzero with opposite sign, the same argument tells us that we are dealing with the shifted version of a hyperbola that has either the x- or y-axis as focal axis (or degenerate situations).

If either $A = 0$ or $C = 0$, then it is easy to see that the equation is either degenerate or that it represents the shift of a parabola that has focal axis either the x-axis or the y-axis.

4.78. Let's start with $\theta = 50$. Since $\frac{50}{2\pi} \approx 7.96$, we know that $50 \approx 7.96(2\pi) \approx 16\pi$, so $P_\theta \approx (1, 0)$, so that $\cos 50 \approx 1$ and $\sin 50 \approx 0$. A calculator provides the more accurate $\cos 50 \approx 0.965$ and $\sin 50 \approx -0.262$. To play this game more accurately, we can write

$$\begin{aligned} 50 &\approx 7.96(2\pi) \approx 8(2\pi) - 0.04(2\pi) = 8(2\pi) - 0.16(\tfrac{\pi}{2}) \\ &\approx 8(2\pi) - \tfrac{1}{6}(\tfrac{\pi}{2}) = 8(2\pi) - \tfrac{1}{2}(\tfrac{\pi}{6}). \end{aligned}$$

So a close approximation of P_{50} can be gotten by going around the unit circle counterclockwise and stopping $\frac{1}{2} \cdot (\frac{\pi}{6})$ or $\frac{1}{2} \cdot 30°$ short of 8 complete revolutions. To estimate $\cos 50$ and $\sin 50$ with this approximation of P_{50}, one can make use of formulas

$$\cos\big(8(2\pi) - \tfrac{1}{2} \cdot \tfrac{\pi}{6}\big) = \cos(\tfrac{1}{2} \cdot \tfrac{\pi}{6}) \;\text{ and }\; \sin\big(8(2\pi) - \tfrac{1}{2} \cdot \tfrac{\pi}{6}\big) = -\sin(\tfrac{1}{2} \cdot \tfrac{\pi}{6})$$

in combination with the formulas $\sin^2 \frac{\phi}{2} = \frac{1}{2}(1 - \cos\phi)$ and $\cos^2 \frac{\phi}{2} = \frac{1}{2}(1 + \cos\phi)$ from Problem 1.23i. So

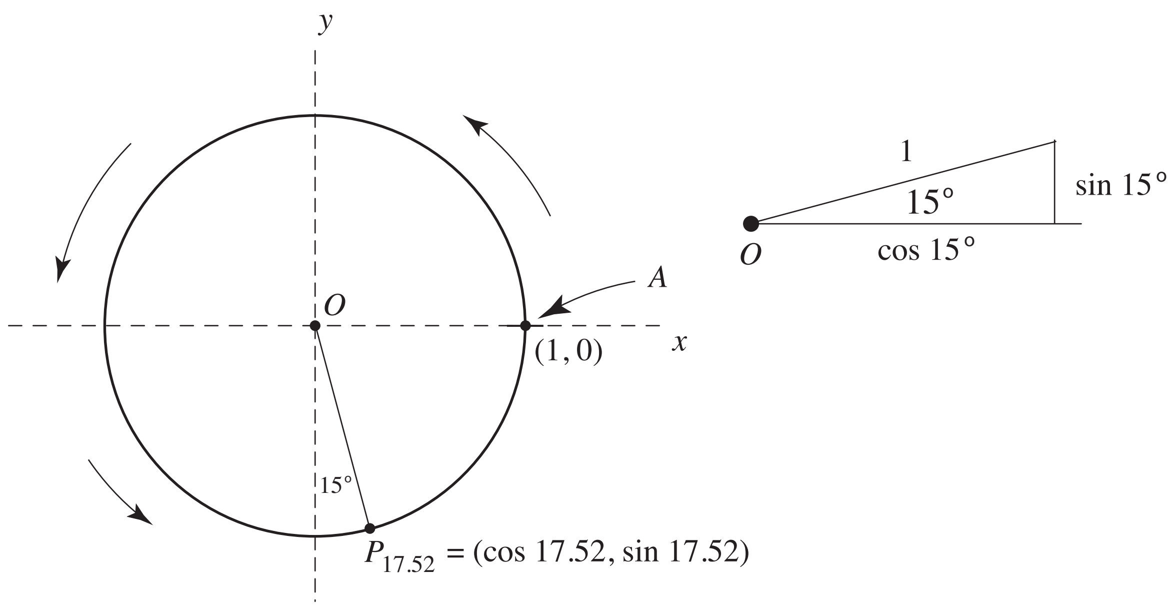
$$\begin{aligned} \cos^2 50 &\approx \cos^2(\tfrac{1}{2} \cdot \tfrac{\pi}{6}) = \tfrac{1}{2}(1 + \cos\tfrac{\pi}{6}) = \tfrac{1}{2}(1 + \tfrac{\sqrt{3}}{2}) = \tfrac{2+\sqrt{3}}{4} \approx \tfrac{3.73}{4} \;\text{ and} \\ \sin^2 50 &\approx \sin^2(\tfrac{1}{2} \cdot \tfrac{\pi}{6}) = \tfrac{1}{2}(1 - \cos\tfrac{\pi}{6}) = \tfrac{1}{2}(1 - \tfrac{\sqrt{3}}{2}) = \tfrac{2-\sqrt{3}}{4} \approx \tfrac{0.27}{4}. \end{aligned}$$

It follows that $\cos 50 \approx 0.966$ and $\sin 50 \approx -0.259$. Now to $\theta = -25$ radians. From above, $\frac{-25}{2\pi} \approx -3.98$, so that $-25 \approx -3.98(2\pi) \approx -8\pi$. It follows that P_{-25} is closely approximated by going around the unit circle almost 4 complete revolutions clockwise. So $P_{-25} \approx (1, 0)$. Therefore $\cos(-25) \approx 1$ and $\sin(-25) \approx 0$. A calculator shows that $\cos(-25) \approx 0.99$ and $\sin(-25) \approx 0.13$.

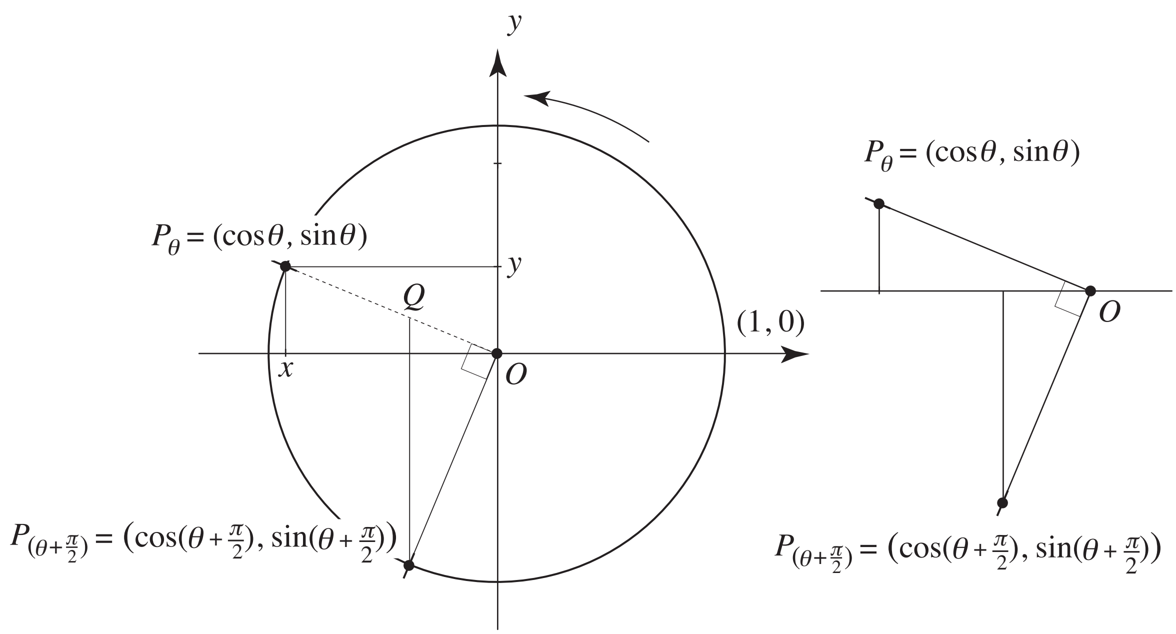
4.79. Since $\frac{17.52}{2\pi} \approx 2.79$, we see that

$$\begin{aligned} 17.52 &\approx 2.79(2\pi) = 2(2\pi) + 0.79(2\pi) = 2(2\pi) + 1.58\pi = 2(2\pi) + \pi + 0.58\pi \\ &= 2(2\pi) + \pi + 1.16(\tfrac{\pi}{2}) = 2(2\pi) + \pi + \tfrac{\pi}{2} + 0.16(\tfrac{\pi}{2}) \approx 2(2\pi) + \pi + \tfrac{\pi}{2} + \tfrac{1}{6}(\tfrac{\pi}{2}) \\ &= 2(2\pi) + \pi + \tfrac{\pi}{2} + \tfrac{\pi}{12}. \end{aligned}$$

So $P_{17.52}$ is near the point obtained by starting at $(1, 0)$ on the unit circle, then proceeding around it counterclockwise for two complete revolutions, then for another half revolution, then a quarter of a revolution, and finally for $\big(\frac{180}{12}\big)^\circ = 15°$ more degrees. It follows—see the figure below—that the x- and y-coordinates of $P_{17.52}$ are approximately, $x = \sin 15°$ and $y = -\cos 15°$. Using the half-angle formulas of Problem 1.23i, we know that $\sin 15° = \sqrt{\frac{1}{2}(1 - \cos 30°)} \approx 0.26$ and $\cos 15° = \sqrt{\frac{1}{2}(1 + \cos 30°)} \approx 0.97$. A calculator provides the approximations $\cos 17.52 \approx 0.239$ and $\sin 17.52 \approx -0.971$.



4.80. The figure below depicts a situation of an angle θ such that the point P_θ falls into the second quadrant. Since the right triangles of the figure with hypotenuse $P_{(\theta+\frac{\pi}{2})}O$, $P_{(\theta+\frac{\pi}{2})}Q$, and $P_\theta O$ have an additional angle in common, they are all similar. It follows



that the two triangles on the right side of the figure are similar. So $\cos(\theta+\frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta+\frac{\pi}{2}) = \cos\theta$.

4.81. The conclusion $\sin(\theta-\frac{\pi}{2}) = -\cos\theta$ and $\cos(\theta-\frac{\pi}{2}) = \sin\theta$ is easily verified by modifying Figure 4.25b.

4.82. Since $\sec\theta = \frac{1}{\cos\theta}$, the equalities for the secant follow from those for the cosine. To see that $\sec^2\theta = \tan^2\theta + 1$, take the identity $\sin^2\theta + \cos^2\theta = 1$ and divide it through by $\cos^2\theta$.

4.83. That the point (x, y) with $x = r\cos\theta$ and $y = r\sin\theta$ is on the circle $x^2+y^2 = r^2$ follows immediately from the identity $\sin^2\theta + \cos^2\theta = 1$. That every point on this circle has this form is demonstrated in the discussion of Section 4.6 that precedes Example 4.19. Let θ increase from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Over this stretch $x = r\cos\theta$ decreases from $r \cdot 1$ to $r \cdot 0$ and $x = r\sin\theta$ increases from $r \cdot 0$ to $r \cdot 1$. So the point (x, y) moves counterclockwise from $(r, 0)$ on the circle to $(0, r)$. Continuing in this way for seven more quarter circles does the rest.

4.84. Let a and b be a positive numbers, and consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let $x = a\cos\theta$ and $y = b\sin\theta$. Since $\left(\frac{x}{a}\right)^2 = \cos^2\theta$ and $\left(\frac{y}{b}\right)^2 = \sin^2\theta$, it follows that $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. Therefore the point $(x = a\cos\theta, y = b\sin\theta)$ is on the ellipse for any θ. With $\theta = 0$, $(x = a\cos\theta, y = b\sin\theta) = (a, 0)$. For θ increasing from 0 to $\frac{\pi}{2}$, $\cos\theta$ decreases from 1 to 0 and $\sin\theta$ increases from 0 to 1. In the process, $(a\cos\theta, b\sin\theta)$ moves from $(a.0)$ to $(0, b)$ and traces out the top right quarter of the ellipse. Continuing in this way shows that for $0 \leq \theta \leq 2\pi$ the point traces out the entire ellipse.