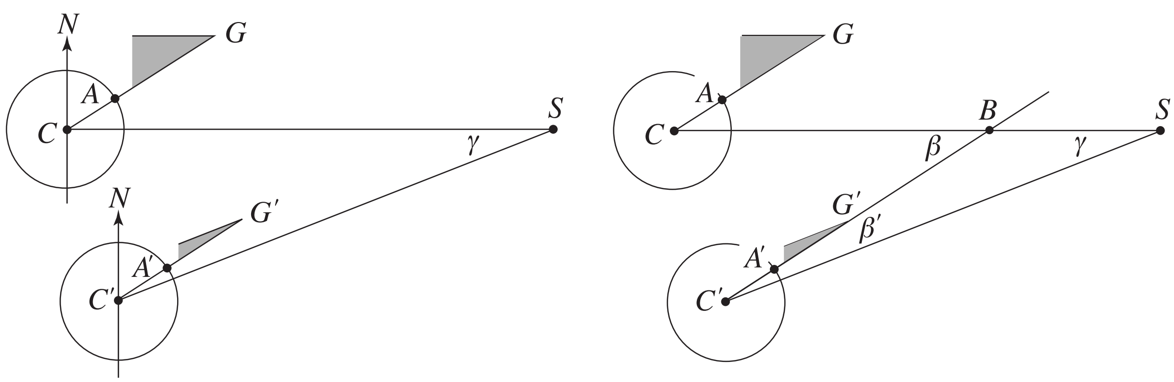
Solutions to Problems and Projects for Chapter 3

3.1. Figures (a) and (b) below are more explicit versions of Figure 3.35. **i.** The segments CAG and $C'A'G'$ are parallel because they represent the same gnomon at the same point $A = A'$ on the surface of the Earth.

ii. Since the angles α and α' at G and G' are by the shadows cast by the gnomon blocking the light rays of a very distant Sun (at S), it follows that $\alpha = \angle GCS$ and $\alpha' = \angle G'C'S = \beta'$. Because $C'B$ extends $C'A'G'$ it is parallel to CAG. Therefore, $\alpha = \beta$.

iii. From the triangle in Figure (b), $\gamma + \beta' + (\pi - \beta) = \pi$. Therefore, $\gamma = \beta - \beta' = \alpha - \alpha'$.



(a) (b)

This was the strategy used by the Greek astronomers to establish that the obliquity of the ecliptic is approximately 24° in their Earth-centered context.

3.2. The minute hand moves through two complete revolutions (during the motion of the hour hand from the one o'clock to the three o'clock positions) and then another $\frac{8}{12} = \frac{2}{3}$ of one revolution. Since the minute hand is 14 feet long, its tip moves

$$2\pi \cdot 14 + 2\pi \cdot 14 + \tfrac{2}{3}(2\pi \cdot 14) = (56 + \tfrac{56}{3})\pi = 74\tfrac{2}{3}\pi \approx 234.57 \text{ feet.}$$

The hour hand moves for $2\frac{2}{3}$ hours. So it moves $\frac{8}{3} \cdot \frac{1}{12} = \frac{2}{9}$ of one revolution and hence $\frac{2}{9} \cdot 2\pi \cdot 9 = 4\pi \approx 12.57$ feet.

3.3. The tip of the arrow takes 6 hours to move from B to C and therefore 1.5 hours to trace out arc $AB +$ arc CD. Since the arc $AB =$ arc CD, it takes the tip $\frac{3}{4}$ hours to trace arc CD. In one hour the tip traces out a distance of $\frac{1}{12} \cdot 2\pi \cdot 1 = \frac{1}{6}\pi$ meters. Therefore arc CD is $\frac{3}{4} \cdot \frac{1}{6}\pi = \frac{1}{8}\pi \approx 0.39$ meters long.

3.4. Since the clock loses 1 minute per hour, it looses 72 minutes or $1\frac{1}{5}$ hours in the 72 hour period. So the hour hand moves through $70\frac{4}{5}$ hours on the dial. Since the radius of the hour hand is $r = 2$ feet, its tip travels $\frac{1}{12} \cdot 2\pi r = \frac{1}{3} \cdot \pi \approx 1.047$ feet in one hour. So it travels a total of $(70\frac{4}{5})\frac{\pi}{3} \approx 74.14$ feet. For every one hour rotation of the hour hand along the dial, the minute hand does one complete revolution. Since it is $r = 2.5$ feet long, its tip will move through a distance of $70\frac{4}{5} \cdot 2\pi r = 354\pi \approx 1112.12$ feet.

3.5. We saw that during autumn $\mathbf{v}$ rotates, on average, faster than $\mathbf{r}$. During summer, E moves from B to C (summer solstice to autumn equinox) and both $\mathbf{r}$ and $\mathbf{v}$ rotate from B to C. The arrow $\mathbf{r}$ rotates through an angle greater the $90°$ while $\mathbf{v}$ rotates through exactly $90°$. Therefore

during summer $\mathbf{v}$ rotates, on average, more slowly than $\mathbf{r}$.

Since the rotational speed of $\mathbf{r}$ is constant, it follows that $\mathbf{v}$ has, on average, greater angular velocity during autumn than during summer. This observation is consistent with Kepler's second law. This says that a planet moves faster in its orbit when it is closer to the Sun than when it is farther away.

3.6. The only thing to do in this problem is to understand.

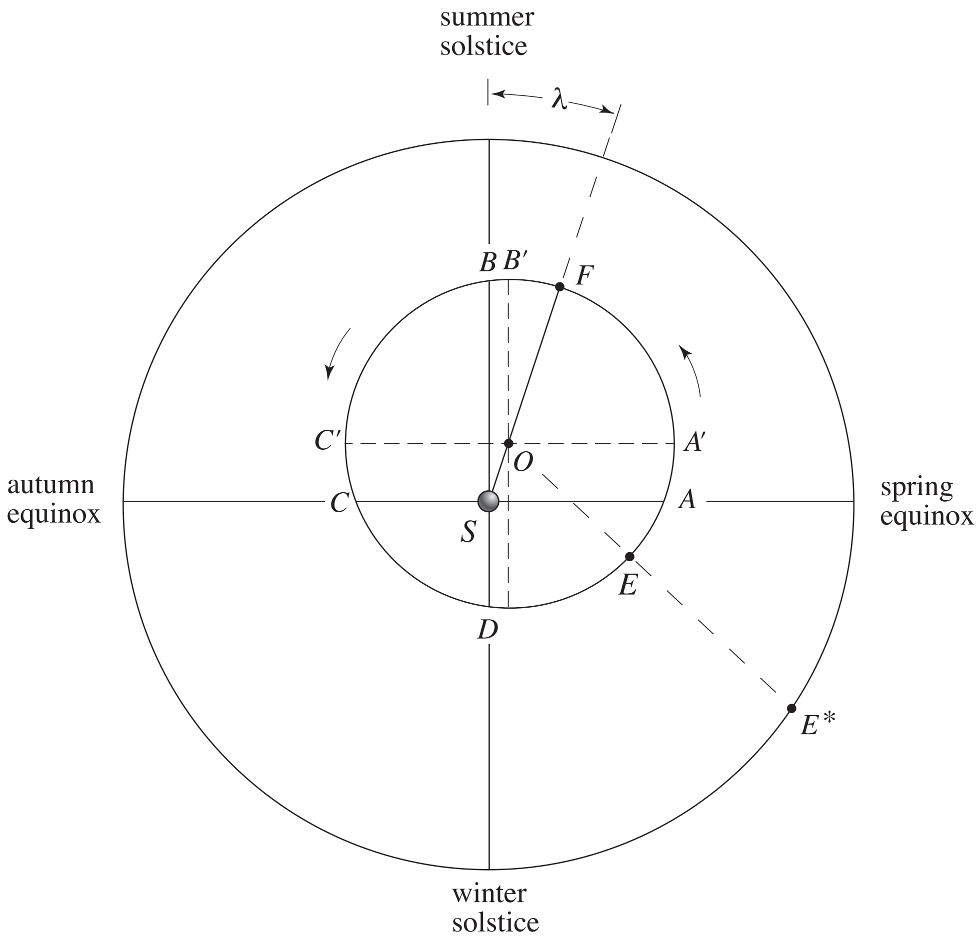
3.7. Since arc $B'B = 0.0035r$, $\beta = \frac{\text{arc}B'B}{r} = 0.0035$ radians, and hence $\beta = 0.0035 \cdot \frac{180°}{\pi} \approx 0.20°$. Since this angle is small, we know from the pattern that Table 1.2 establishes that $\sin\beta$ is essentially equal to 0.0035. (A calculator confirms that $\sin 0.0035 \approx 0.00349999$.) Applied to the circular sector $OB'B$, this means that $\frac{b}{r}$ is essentially equal to $\frac{\text{arc}B'B}{r}$, and hence that the same is true for b and $B'B$.

3.8. Use the data of Hipparchus's time: spring: $94\frac{1}{2}$ days, summer: $92\frac{1}{2}$ days, autumn: $88\frac{1}{8}$ days, winter: $90\frac{1}{8}$ days, and let $r = OE$ and $c = SO$ as before. Adapting the discussion of Section 3.2 to these data, provides the modified versions of Figures 3.7 and 3.8 sketched below. The lengths of spring and summer together tell us that Earth E takes $94\frac{1}{2} + 92\frac{1}{2} = 187$ days to travel from point A to point C. This means that $\angle AOC = 187 \cdot 0.0172 = 3.2164$ radians. So arc $AC = 3.2164r$. Therefore

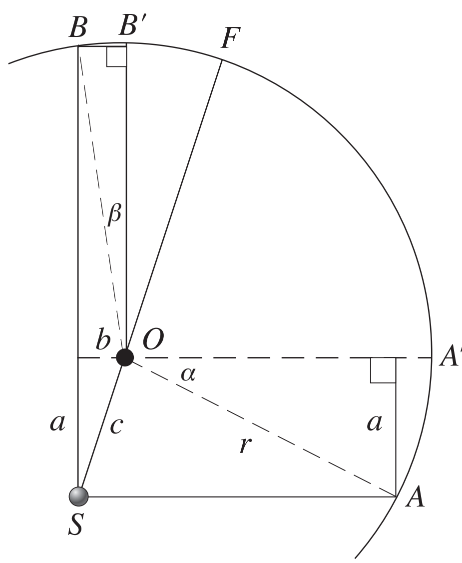
$$\begin{aligned}\text{arc } AA' &= \tfrac{1}{2}(\text{arc } AC - \text{arc } A'C') = \tfrac{1}{2}(3.2164 - \pi)r \\ &= \tfrac{1}{2}(3.2164 - 3.1416)r = \tfrac{1}{2}(0.0748)r = 0.0374r.\end{aligned}$$

Notice that arc AB = arc AA' + arc $A'B'$ + arc $B'B$. Since arc $AB = 94.5 \cdot 0.0172r = 1.6254r$, it follows that

$$\begin{aligned}\text{arc } B'B &= \text{arc } AB - \text{arc } AA' - \text{arc } A'B' = (1.6254 - 0.0374 - \tfrac{\pi}{2})r \\ &= (1.6254 - 0.0374 - 1.5708)r = 0.0172r.\end{aligned}$$



Since $\alpha = 0.0374 \cdot \frac{180}{\pi} = 2.14°$ is a small angle, $\sin \alpha = \frac{a}{r}$ is approximately equal to α in radian measure (see the information in Table 1.2), and we can take $a = 0.0374r$. In the same way, $\beta = 0.0172 \cdot \frac{180}{\pi} = 0.99°$ is small, so that $\sin \beta = \frac{b}{r}$ is approximately

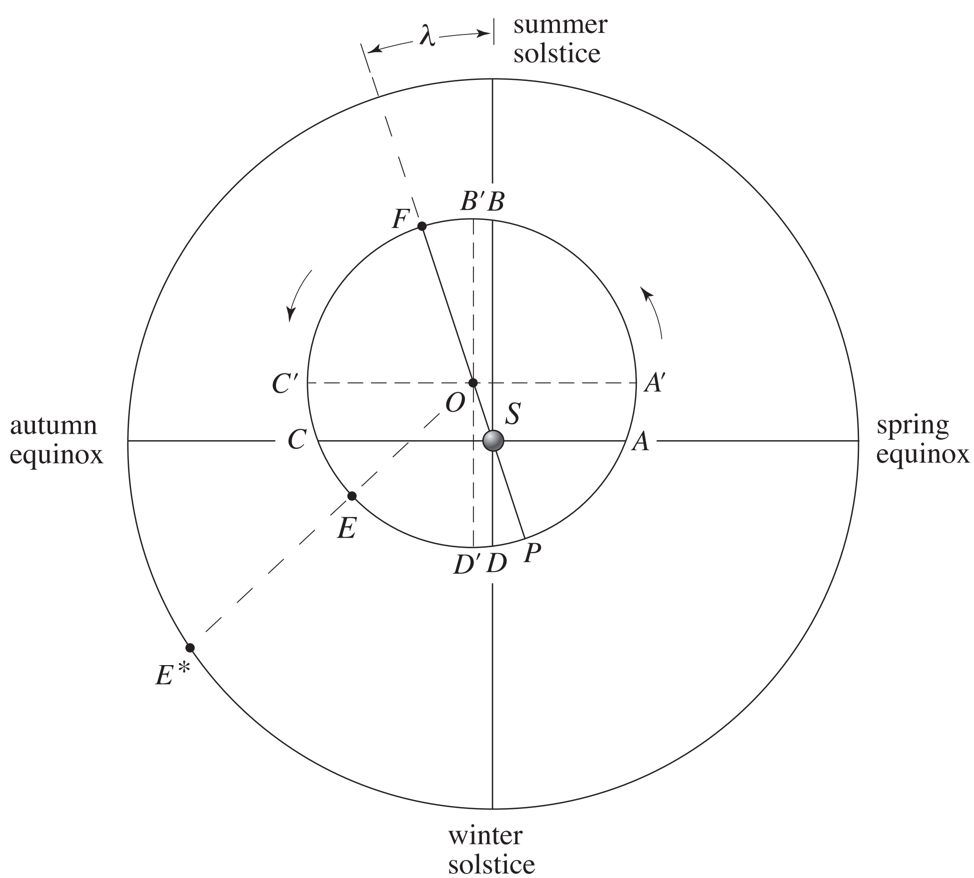


equal to β in radian measure, and we can take $b = 0.0172r$. The Pythagorean theorem determines the hypotenuse $c = SO$ of the small triangle with sides a and b to be $c = \sqrt{(0.0374r)^2 + (0.0172r)^2} = r\sqrt{0.0374^2 + 0.0172^2} = 0.0412r$. Since $\tan\lambda = \frac{0.0172}{0.0374} = 0.4599$, $\lambda = 24.70°$.

3.9. Approximate values for the current lengths of the seasons are: spring, 92 days, 18 hours, 20 minutes, or 92.764 days; summer, 93 days, 15 hours, 31 minutes, or 93.647 days; autumn, 89 days, 20 hours, 4 minutes, or 89.836 days; and winter, 88 days, 23 hours, 56 minutes, or 88.997 days. This and more recent information can be derived from data provided by the United States Naval Observatory (USNO) on the website

http://aa.usno.navy.mil/data/docs/EarthSeasons.php

These values for the lengths of the seasons tell us that the starting diagram for this analysis is the same as that of Figure 3.7. This figure is reproduced below. The



points D' and the perihelion position P have been added. As before, we'll let $r = OE$ and $c = OS$.

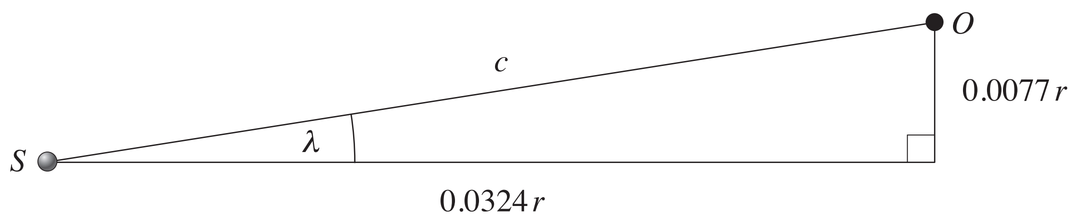
The lengths of spring and summer tell us that the Earth E takes 186.411 days to travel from point A to point C. Since the segment EO rotates at a rate of 0.0172 radians per day, $\angle AOC = 186.411 \cdot 0.0172 = 3.2063$ radians. So arc $AC = 3.2063r$. Therefore

$$\begin{aligned}\text{arc } AA' &= \tfrac{1}{2}(\text{arc } AC - \text{arc } A'C') = \tfrac{1}{2}(3.2063 - \pi)r \\ &= \tfrac{1}{2}(3.2063 - 3.1416)r = \tfrac{1}{2}(0.0748)r = 0.0324r.\end{aligned}$$

Notice that arc $AB' =$ arc $AA' +$ arc $A'B' = 0.0324r + \frac{\pi}{2}r = 0.0324r + 1.5708r = 1.6032r$. Since arc $AB = 92.764 \cdot 0.0172r = 1.5955r$, we get that

$$\text{arc } BB' = \text{arc } AB' - \text{arc } AB = (1.6032 - 1.5955)r = 0.0077r.$$

As in Section 3.2, this provides the information for the right triangle at the center of the figure. It follows that $\tan\lambda = \frac{0.0077}{0.0324} = 0.2377$, and hence that $\lambda = 0.2333$. Since



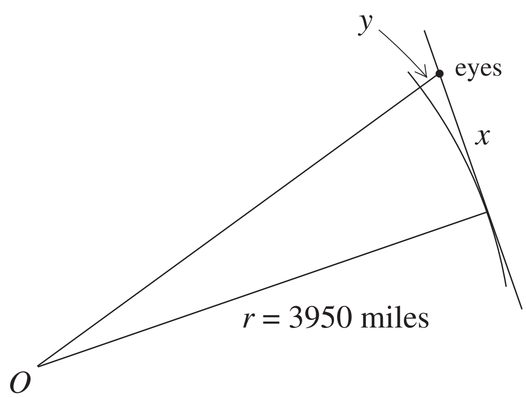
$\angle FOB' = \lambda$, arc $FB' = 0.2333r$. So arc $FB = 0.2333r + 0.0077r = 0.2410r$. It follows that Earth takes $\frac{0.2410r}{0.0172r} \approx 14$ days to travel from the summer solstice position B to the aphelion position F. Since $\angle D'OP$ is also equal to λ, arc $D'P = 0.2333r$. The fact that arc $D'D =$ arc $BB' = 0.0077r$, means that arc $DP = 0.2333r - 0.0077r = 0.2256r$. Since $\frac{0.2256r}{0.0172r} \approx 13$, Earth is at perihelion 13 days after it arrives at the winter solstice position D.

3.10. In Figure 3.40, $r = OA$ is a radius and L is the tangent at A. Suppose that the angle between them is not $90°$. So the segment from L to O that is perpendicular to L is different from OA. It is designated by OB in the figure. Notice that OA is the hypotenuse of the right triangle ΔOBA. Since OA is the hypotenuse, $OA > OB$. It follows from the figure that $r = OA > OB > OC = r$. This contradicts the assumption that OA is not perpendicular to L.

3.11. In the figure below, r is Earth's radius and O is its center. Let x be the length of the line of sight of the person to the horizon and let y be the distance of the eyes above the ground. By the Pythagorean theorem, $(r+y)^2 = r^2 + x^2$. So $r^2 + 2ry + y^2 = r^2 + x^2$ and hence $x^2 = 2ry + y^2$. We'll work in miles. So $y = 6 \cdot \frac{1}{5280} \approx 0.00114$ miles and therefore,

$$x = \sqrt{2ry + y^2} \approx \sqrt{2(3950)(0.00114) + 0.00114^2}.$$

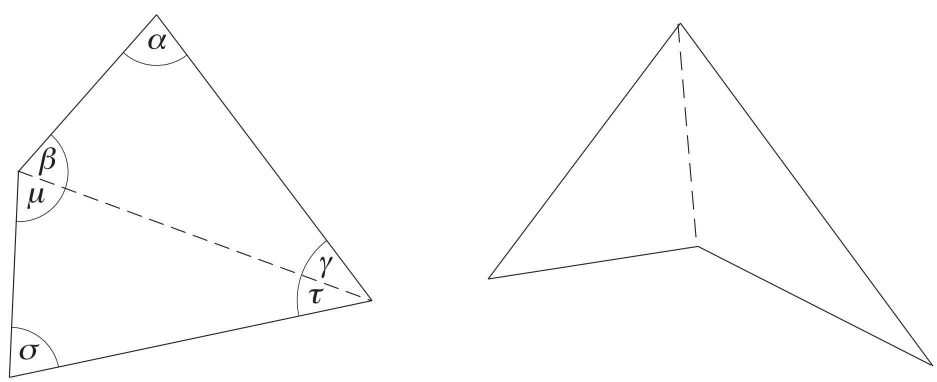
Since $0.00114^2 \approx 0.000001$ is tiny compared to $2(3950)(0.00114) \approx 9.006$, we find that



$x = \sqrt{2ry + y^2} \approx \sqrt{2(3950)(0.00114)} = \sqrt{9.006} \approx 3$ miles.

3.12. Let r be the Earth's radius in miles. If the angle that the geometer measures between his line of sight to the horizon and a plumb line is θ, then by Figure 3.41, $\sin\theta = \frac{r}{r+1.5}$. So $(r+1.5)\sin\theta = r$ and hence $r(1-\sin\theta) = 1.5\sin\theta$, so that $r = \frac{1.5\sin\theta}{1-\sin\theta}$. With θ equal to $88\frac{1}{6}^\circ, 88\frac{1}{2}^\circ$, and $88\frac{1}{3}^\circ$, we get the respective values of about 2929, 4376, and 3544 miles. We know from the previous problem that today's accurate value for this radius is 3950 miles. It is obvious that this method of determining the radius r is very susceptible to errors in the measurement of the angle θ.

3.13. The figure below depicts two typical quadrilaterals. The quadrilateral on the left is



split into two triangles and it follows that its interior angles add to

$$\alpha + (\beta + \mu) + \sigma + (\tau + \gamma) = (\alpha + \beta + \gamma) + (\mu + \sigma + \tau) = 180^\circ + 180^\circ = 360^\circ.$$

A similar thing works for the quadrilateral on the right.

3.14. Focus on Figure 3.42. From the point A you have sited the points B and C so that the segments AB and AC that you have drawn are tangent to the tank. Since the segments OB and OC are both radii of the circular cross section of the tank, the angles $\angle ABO$ and $\angle ACO$ are both 90°. Since you have measured $\angle BAC$, you now know—thanks to

Problem 3.13—what $\angle BOC$ is equal to. In addition, you can measure the length of arc BC by walking it off. With $\angle BOC$ in radians and r the radius of the tank, you have $\angle BOC = \frac{\text{arc}\,BC}{r}$. So the value of r is now in hand.

3.15. From the discussion in Section 3.3, we know in the case of Mercury M and its distance MS from the Sun, that $\sin\alpha = \frac{MS}{ES}$. Therefore Copernicus's estimate is $MS = (\sin 22°)ES \approx 0.3746\,ES$.

3.16. Copernicus studies the orbit of Jupiter at opposition and then again at quadrature. See Figure 3.43. His finding that $\alpha - \beta \approx 79°$ tells him that $\cos 79° \approx \frac{ES}{JS}$. So $JS \approx \frac{ES}{\cos 79°} \approx 5.24\,ES$.

3.17. This problem uses Figure 3.44 and explains how the law of sines leads to the formula

$$\frac{MS}{ES} = \frac{\sin\alpha}{\sin\beta}\,.$$

3.18. Copernicus knew the period of the orbit of Mars to be about 1.9 years or $1.9 \cdot 12 = 22.8$ months. Since Mars traces out its full orbit of $360°$ in 22.8 months it, it traces out $360° \cdot \frac{2}{22.8} = 31.6°$ in two months. This is the angle $\angle M'SM$ in Figure 3.44. In the same two months Earth traces out an angle of $360° \cdot \frac{2}{12} = 60°$. Therefore $\angle MSE = 60° - 31.6° = 28.4°$. Since he measured the angle α to be $114.9°$, he had the estimate $\beta \approx 180° - 28.4° - 114.9° = 36.7°$. Therefore

$$\frac{MS}{ES} = \frac{\sin\alpha}{\sin\beta} = \frac{\sin 114.9°}{\sin 36.7°} = 1.52.$$

Table 3.1 tells us that this is exactly the value that Copernicus achieves.

3.19. **i.** Let E be a typical position of Earth on its circular orbit. Since the rotational speed of OE is constant and Earth takes $365\frac{1}{4}$ days to complete its orbit, its rotational speed is $\frac{360}{365.25}$ degrees per day. Earth is at E_1 on March 5, 1590, returns to E_1 one year later and then moves for another $321\frac{3}{4}$ days to reach E_2 on January 21, 1592. In reference to Figure 3.14, the assumption that Earth moves clockwise around its circular orbit is not consistent with what has just been described.

ii. Suppose that O, E, and Mars M lie in a line. Since the Earth moves counterclockwise in its orbit it is in position E shortly after it arrives at E_4. By knowing the time involved, Tycho can estimate the angle $\angle E_4OE = \angle E_4OM$. Knowing the times involved, he can then estimate $\angle E_3OE, \angle E_2OE$, and $\angle E_1OE$, and therefore, $\angle E_3OM, \angle E_2OM$, and $\angle E_1OM$.

iii. The discussion of Section 3.2 informs us that in Copernicus's model, the distance OS is small compared to the radius of Earth's circular orbit. So the angles $\angle OE_1M$ to $\angle OE_4M$ are approximated by $\angle SE_1M$ to $\angle SE_4M$, respectively.

iv. By knowing that the angles of any triangle add to $180°$.

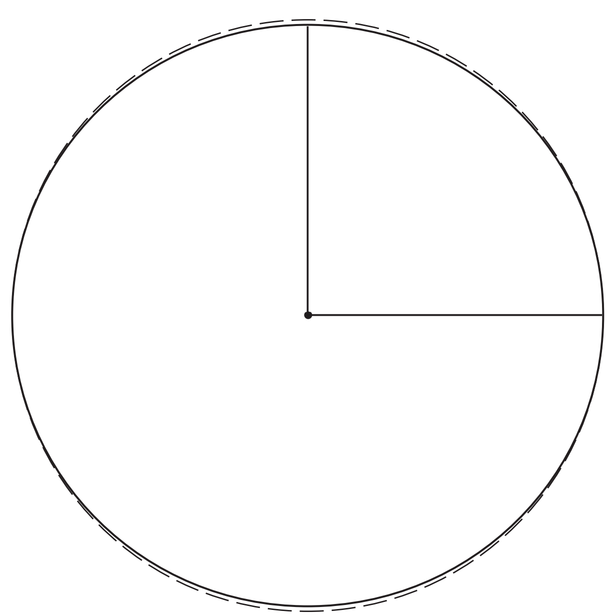
3.20. Kepler knew that the center C of Mars's circular orbit has to lie on the perpendicular bisectors of the segments M_1M_2 and M_2M_3. (This follows from the conclusion of Problem 1.9, a fact already known to the Greeks.) So the correct position of C is the intersection of the two bisectors.

3.21. It follows from Figure 3.19 that max $= a + c$ and min $= a - c$, and therefore that avg $= \frac{1}{2}[(a+c)+(a-c)] = a$. So $\frac{\text{max} - \text{avg}}{\text{avg}} = \frac{\text{avg} - \text{min}}{\text{avg}} = \frac{c}{a} = \varepsilon$, the eccentricity of the orbit.

3.22. Since $(a - c) + (a + c) = 2a$ is the sum of the distances AF_1 and AF_2, it follows that $k = 2a$. Since the semiminor axis lies on the perpendicular bisector F_1F_2, the symmetry of the situation implies that $BF_1 = BF_2$. Therefore, $2BF_1 = 2a$, and hence $BF_1 = BF_2 = a$. The Pythagorean theorem tells us that $a^2 = b^2 + c^2$, and hence that $b = a\sqrt{1-\varepsilon^2}$. This last equality tells us that if ε is close to 0, then b is close to a, so the ellipse is close to a circle. The closer ε is to 0, the closer the ellipse to a circle. If ε is close to 1, then b is close to 0, so the ellipse is flat. The closer ε is to 1, the flatter the ellipse.

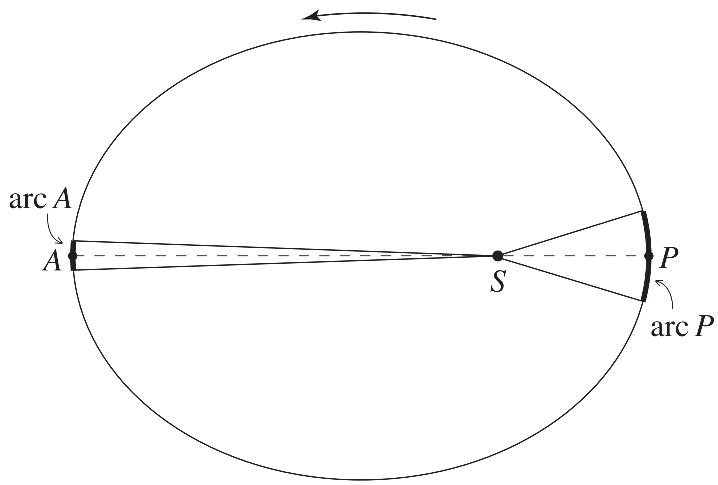
3.23. By one of the conclusions of Problem 3.22, the focal points of an ellipse can be located as follows. Take a segment of length equal to the semimajor axis, place one endpoint at the top of the ellipse, and rotate it until the other endpoint lies on the focal axis. This other endpoint—note that there are two possibilities, one to the left, the other to the right of the center of the ellipse—determines the focal points. This is how the focal points 1, 2, and 3 of the three ellipses of Figure 3.46 are determined.

3.24. From Table 3.3, $a = 0.387$ au and $\varepsilon = 0.206$. Since $b = a\sqrt{1-\varepsilon^2}$, $b \approx 0.379$ au. At a scale 1 au $= 25$ cm, $a = 25(0.387) = 9.675$ cm and $b = 25(0.379) = 9.475$ cm. The difference $a - b$ is 2 millimeters and the distance between the center of the ellipse and



either of its focal points is $c = \varepsilon a = 0.206(9.675) \approx 2$ cm. The figure above (its scale depends on the screen size) shows both the ellipse with semimajor axis a and semiminor axis b and—as a dashed curve—the circle of radius a with the same center. The difference between the two—if they were drawn in the same way and placed side by side—would probably not be detectable visually.

3.25. For any two identical time intervals I_P and I_A the areas of the two corresponding elliptical sectors in the figure below are the same by Kepler's second law. Since P and A are the points of the planet's closest and farthest distances from S, it follows (for the given time interval) that the arc at P is the longest and the arc at A the shortest. Hence the speed of the planet at P is greatest and that at A is least. The point of the



problem is to draw a numerical consequence from this observation. Let a and ε be the semimajor axis and eccentricity of the orbit of the planet and let arc P and arc A designate the lengths of the two arcs depicted in the figure above.

i. Since arc P and arc A are short, they are approximated by circular arcs both centered at S. Since $SP = a(1-\varepsilon)$ and $SA = a(1+\varepsilon)$, it follows from the study of the circular sector in Section 2.2 that the areas of the two elliptical sectors in the figure are approximately equal to $\frac{1}{2}a(1-\varepsilon)(\text{arc } P)$ and $\frac{1}{2}a(1+\varepsilon)(\text{arc } A)$, respectively. Therefore,

$$\tfrac{1}{2}a(1-\varepsilon)(\text{arc } P) \approx \tfrac{1}{2}a(1+\varepsilon)(\text{arc } A).$$

ii. The fact that average speed is equal to distance traveled divided by the time it takes, tells us that the average speeds of the planets in traversing the two arcs are $\frac{\text{arc } P}{I_P}$ and $\frac{\text{arc } A}{I_A}$, respectively. Since the arcs are both small, these averages are approximately equal to v_{max} and v_{min}, respectively.

iii. The approximations

$$\tfrac{v_{\text{max}}}{v_{\text{min}}} \approx \tfrac{\text{arc } P}{\text{arc } A} \approx \tfrac{1+\varepsilon}{1-\varepsilon}$$

follow from (i) and (ii) above.

iv. When the time interval $I_P = I_A$ is pushed to zero, the approximations above get tighter and tighter, so that $\frac{v_{\max}}{v_{\min}} = \frac{1+\varepsilon}{1-\varepsilon}$.

v. If follows directly from the data in Table 3.3 that $\frac{v_{\max}}{v_{\min}}$ is approximately equal to 1.03 for Earth, 1.20 for Mars, and 1.52 for Mercury.

3.26. Since $\varepsilon \approx 0$, $1+\varepsilon \approx 1-\varepsilon$, so that by the result of Problem 3.25iv, $v_{\min} \approx v_{\max}$. Since the orbital speed v of any planet satisfies $v_{\min} \leq v \leq v_{\max}$, all these speeds are nearly constant. If the orbit of a comet satisfies, $0.999 < \varepsilon < 1$, then $1+\varepsilon > 1.999$. Also, $-\varepsilon < -0.999$ and $1-\varepsilon < 1-0.999 = 0.001$. It follows that $\frac{v_{\max}}{v_{\min}} = \frac{1+\varepsilon}{1-\varepsilon} > \frac{1.999}{0.001} = 1999$. So $v_{\max} > 1999 v_{\min}$.

3.27. If $\varepsilon_A > \varepsilon_B$, then $1+\varepsilon_A > 1+\varepsilon_B$ and $1-\varepsilon_B > 1-\varepsilon_A$ (since $-\varepsilon_B > -\varepsilon_A$). It follows that $\frac{1+\varepsilon_A}{1-\varepsilon_A} > \frac{1+\varepsilon_B}{1-\varepsilon_A} > \frac{1+\varepsilon_B}{1-\varepsilon_B}$.

3.28. From the given, $v_{\max} \approx v_{\min} + (0.1)v_{\min} = (1.1)v_{\min}$. So $\frac{1+\varepsilon}{1-\varepsilon} = \frac{v_{\max}}{v_{\min}} \approx 1.1$ and hence $1+\varepsilon \approx 1.1(1-\varepsilon) = 1.1 - 1.1\varepsilon$. So $2.1\varepsilon \approx 0.1$ and hence $\varepsilon \approx \frac{0.1}{2.1} \approx 0.048$. A look at Table 3.3 tells us that the planet is Jupiter.

3.29. Consider Earth's orbit and let a be its semimajor axis and T its period. We know that $a = 1$ au and that $T = 1$ year. It follows from Kepler's third law that in the units au and years, the ratio $\frac{a^3}{T^2}$ is equal to 1 for every planet. So if the period T_J of a planet is known in years, then its semimajor axis is $a_J = (T_J^2)^{\frac{1}{3}} = T_J^{\frac{2}{3}}$ au.

3.30. The problem here is simply to understand the derivation—as it is described—of the formula

$$1+2+3+\cdots+(k-1)+k = \frac{k(k+1)}{2},$$

where k can be any positive integer.

3.31. Start by adding the sum $1+3+5+\cdots+(2k-3)+(2k-1)$ twice:

$$1+3+5+\cdots+(2k-3)+(2k-1)\ +\ 1+3+5+\cdots+(2k-3)+(2k-1).$$

As in the solution of Problem 3.30, add two terms at a time from the inside out to get $2k+2k+\cdots+2k$. Since $1, 3, 5, \ldots, 2k-3, (2k-1)$ are the first k odd positive integers, there are k of these $2k$ in this sum. Therefore

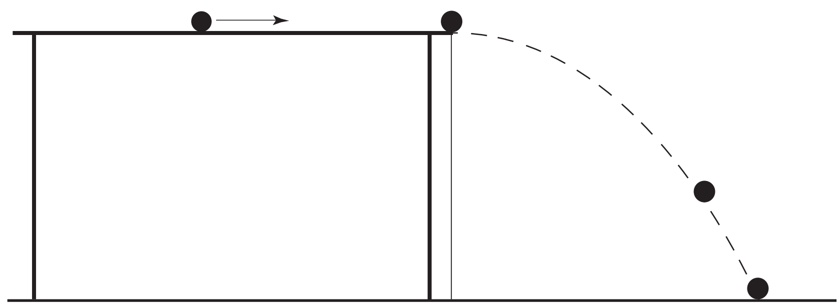
$$1+3+5+\cdots+(2k-3)+(2k-1) = \tfrac{1}{2}(k\cdot 2k) = k^2.$$

With the principle of induction, the verification of this formula proceeds as follows. Let S_k be the statement, $1+3+5+\cdots+(2k-3)+(2k-1) = k^2$. For $k=1$, the sum stops at $(2k-1) = 1$ and $k^2 = 1$. So statement S_1 is true. Assuming that S_k or $1+3+5+\cdots+(2k-3)+(2k-1) = k^2$ is true, it follows that

$$1+3+5+\cdots+(2k-3)+(2k-1)+(2k+1)=k^2+2k+1=(k+1)^2$$

and therefore that S_{k+1} is true. We have shown that for any $k \geq 1$, the truth of S_k implies the truth of S_{k+1}. Therefore the principle of induction with $m=1$ tells us that the formula $1+3+5+\cdots+(2k-3)+(2k-1)=k^2$ is valid for all $k \geq 1$.

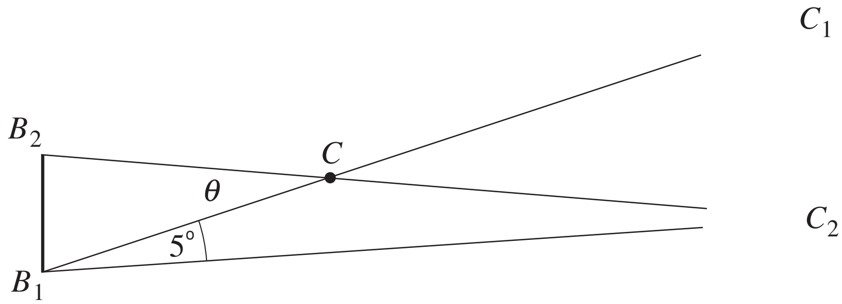
3.32. Galileo conceives the motion of a projectile as a simultaneously and separately occurring composite of a vertical and horizontal motion. In reference to the ball's parabolic fall from the table, this vertical component is the same as the motion of a ball that is dropped from rest at the edge of the table. Given Galileo's principle of horizontal



inertia, the horizontal component of the ball's parabolic fall is a continuation of its horizontal roll on the table at 3 meters per second. It follows from this conceptual separation, that it will take the parabolic ball $\frac{2}{5}$ of a second to hit the floor. Since the ball's horizontal speed during its fall is a constant 3 meters per second, this means that the ball will hit the ground a distance of $\frac{2}{5}\cdot 3=\frac{6}{5}$ feet from the foot of the table.

3.33.
- **i.** Galileo knew that $d \propto t^2$ from the experiment described in connection with Figure 3.21. Since we have already observed that $y \propto d$, it follows that $y \propto t^2$.
- **ii.** Since $\frac{1}{2}v\cdot t=d$ and $d\propto t^2$, it follows that $v\cdot t\propto d\propto t^2$. Therefore $v\propto t$ and in view of (i), $v\propto\sqrt{y}$.
- **iii.** Let t_0 be the time it takes for the ball to fall from rest straight down to the floor from the edge of the table. This fall is the vertical component of the ball's flight no matter what velocity v the ball has after its descent from the inclined plane. By Galileo's law of horizontal inertia, the horizontal component of the ball's fall to the ground has constant speed v. It follows that $x=v\cdot t_0$. So $x\propto v$.
- **iv.** Galileo's conclusion $x\propto\sqrt{y}$ follows by combining the results of (ii) and (iii).

3.34. The figure below captures what Urania has observed and recorded. The Greek column is positioned at C and C_1 and C_2 are the two distinctive features of Mount Olympus that she sees it against. The points B_1 and B_2 are the positions from which she observes C_1 and C_2 respectively and $5°$ is the angle that Urania measured. Since the baseline $B_1B_2 = 12$ is short and the distances to C_1 and C_2 long, the



lines of sight B_1C_2 and B_2C_2 are close to being parallel. Therefore θ is close to $5°$. Since $5°$ is equal to $5 \cdot \frac{\pi}{180}$ radians, it follows from the discussion about parallax in Section 3.7, that $5 \cdot \frac{\pi}{180} \approx \frac{12}{d(C)}$ where $d(C)$ is the distance from B_1 or B_2 to the column C. It follows that $d(C) \approx \frac{12 \cdot 180}{5\pi} \approx 137.5$ strides.

3.35. The specifics of what the students need to do are identical to the solution of Problem 3.34 described above.

3.36. With $EE' = 2ES = 2$ au and the angle of stellar parallax $p(A) = \frac{1}{2}\angle EAE'$ in seconds, we get the approximation $d(A,E) \approx \frac{2}{9.7p(A)} \times 10^6$ au for the distance of A from Earth E by applying the last formula of page 106. Since 1 ly = 63,241 au,

$$d(A,E) \approx \frac{2}{9.7p(A)} \times \frac{10^6}{6.3 \times 10^4} \approx \frac{3.3}{p(A)} \text{ ly}.$$

Putting in 0.29, 0.13, and 0.75 for $p(A)$, results in the distance estimates

$$11.4, \; 25.4, \; \text{and } 4.4 \text{ light-years}$$

for the stars 61 Cygni, Vega, and Alpha Centauri, respectively. Note that the largest of these angles of parallax is a mere $0.75 \cdot \frac{1}{3600} \approx 0.0002$ degrees.

3.37. Taking $p(A) = 60$ seconds in the approximation $d(A,E) \approx \frac{2}{9.7p(A)} \times 10^6$ au tells us that Tycho Brahe might have been able to detect the parallax of stars that are a distance of around 3400 to 3500 au away. He would have been able to conclude that such stars are very far from Earth, but given that the Earth-Sun distance had not been determined he would have had no measure of comparison. In any case, there were and are no such stars. The nearest stars are more than 4 light-years or $4 \times 63{,}241 \approx 250{,}000$ au away from Earth. In particular, the largest angles of stellar parallax—for example the 0.75 seconds of Alpha Centauri—were much too small for Tycho to detect.

3.38. **i.** Let r be the radius of the circle in Figure 3.50b. A look at Figure 3.50a tells us that $\cos\varphi = \frac{r}{r_E}$. Hence $r = r_E \cos\varphi$.

ii. The conclusion of Problem 1.9 tells us that the center O of the circle in Figure 3.50b lies on the perpendicular bisector of the segment BB'. This bisector bisects the angle θ as well (because the two triangles that are created are congruent). It follows that $\sin\frac{\theta}{2} = \frac{\frac{1}{2}BB'}{r}$. Therefore $BB' = 2r\sin\frac{\theta}{2}$. By formula (i) of Problem 1.23, $\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$. So $4\sin^2\frac{\theta}{2} = 2(1-\cos\theta)$, and therefore $2\sin\frac{\theta}{2} = \sqrt{2(1-\cos\theta)}$. That $BB' = (r_E\cos\varphi)\sqrt{2(1-\cos\theta)}$ follows by combining this result with the conclusion of (i).

iii. Let's turn to the baseline BB' of Flamsteed's calculation of the parallax of Mars and its depiction in Figure 3.50b. Given that the elapsed time between his measurements was 6 hours and 10 minutes, $\theta = \frac{6\frac{1}{6}}{24} \times 360° = 92.5°$. The latitude of Flamsteed's town of Derby is known to be $\varphi = 52.92°$. Using the value $r_E = 6370$ kilometers, we get that Flamsteed's baseline BB' is given by

$$BB' = (r_E\cos\varphi)\sqrt{2(1-\cos\theta)} = (6370\cos 52.92°)\sqrt{2(1-\cos 92.5°)} \approx 5549\,\text{km}$$

The value $BB' \approx 5300$ kilometers is implicit in Flamsteed's calculations.

3.39. The figure below depicts Mars in opposition and at perihelion. With perihelion and aphelion distances for Earth of 147,098,291 kilometers and 152,098,233 kilometers,

respectively, and a perihelion distance for Mars of 206,655,216 kilometers, it follows that 147,098,291 km $\leq d(E,S) \leq$ 152,098,233 km and $d + d(E,S) =$ 206,655,216 km. So $d =$ 206,655,216 km $- d(E,S)$, and hence

$$206{,}655{,}216 - 152{,}098{,}233 \leq d \leq 206{,}655{,}216 - 147{,}098{,}291$$

and the conclusion that the distance d between Earth and Mars lies in the range 54,556,983 km $\leq d \leq$ 59,556,925 km follows. Notice in particular that the estimates for the distance d between Earth and Mars (with Mars in opposition as in Figure 3.28) of both Flamsteed—$d \approx$ 52,000,000 kilometers—and Cassini—$d \approx$ 53,000,000 kilometers—came up short. It follows from the parallax distance formula that their measurements for the parallax $p(M)$ of Mars were a too large.