Solutions to Problems and Projects for Chapter 6

6.1. Use Newton's derivative formula to show that

i. $g(x) = 4x^5$ has derivative $g'(x) = 20x^4$.

ii. $h(x) = \frac{7}{x^{-\frac{2}{3}}} = 7x^{\frac{2}{3}}$ has derivative $h'(x) = \frac{14}{3}x^{-\frac{1}{3}}$.

iii. $f(x) = 5x^{\frac{1}{100}} - 4x^{\frac{1}{3}}$ has derivative $f'(x) = \frac{1}{20}x^{-\frac{99}{100}} - \frac{4}{3}x^{-\frac{2}{3}}$.

iv. $g(x) = -2x^{\frac{1}{3}} + 3x^5 - 6$ has derivative $g'(x) = -\frac{2}{3}x^{-\frac{2}{3}} + 15x^4$.

v. $f(x) = 3(\sqrt{x})^7 = 3(x^{\frac{1}{2}})^7 = 3x^{\frac{7}{2}}$ has derivative $f'(x) = \frac{21}{2}x^{\frac{5}{2}}$.

vi. $y = x^{\frac{2}{7}} + 30x^4 - \frac{1}{4}x^{\frac{5}{3}}$ has derivative $\frac{dy}{dx} = \frac{2}{7}x^{-\frac{5}{7}} + 120x^3 - \frac{5}{12}x^{\frac{2}{3}}$.

6.2. The required generic antiderivative is

i. $F(x) = \frac{1}{2}x^4 + C$ for $f(x) = 2x^3$.

ii. $F(x) = 5 \cdot \frac{3}{4}x^{\frac{4}{3}} + C = \frac{15}{4}x^{\frac{4}{3}} + C$ for $f(x) = 5x^{\frac{1}{3}}$.

iii. $F(x) = 3 \cdot \frac{1}{6}x^6 + \frac{1}{4} \cdot \frac{7}{9}x^{\frac{9}{7}} + C = \frac{1}{2}x^6 + \frac{7}{36}x^{\frac{9}{7}} + C$ for $f(x) = 3x^5 + \frac{1}{4}x^{\frac{2}{7}}$.

iv. $F(x) = \frac{6}{5}x^5 - \frac{3}{8} \cdot \frac{3}{8}x^{\frac{8}{3}} + C = \frac{6}{5}x^5 - \frac{9}{64}x^{\frac{8}{3}} + C$ for $f(x) = 6x^4 - \frac{3}{8}x^{\frac{5}{3}}$.

6.3. Newton's area function for a simple function $f(x)$ is an antiderivative $A(x)$ of the function satisfying $A(0) = 0$. Since any two antiderivatives differ by a constant, we get

i. $A(x) = \frac{1}{3}x^3 + C$ with $C = 0$ (since $A(0) = 0$).

ii. $A(x) = \frac{3}{4}x^{\frac{4}{3}} + C$ with $C = 0$ (since $A(0) = 0$).

iii. $A(x) = \frac{2}{7}x^{\frac{7}{2}} + C$ with $C = 0$ (since $A(0) = 0$).

6.4. These problems can be solved by finding Newton's area function $A(x)$ for each of the functions $f(x)$.

i. For $f(x) = 2x^2$, the area function is $A(x) = \frac{2}{3}x^3$. The required area under the graph is $A(8) - A(4) = \frac{2}{3}8^3 - \frac{2}{3}4^3 = \frac{2}{3}[(2 \cdot 4)^3 - 4^3] = \frac{2}{3}[(2^3 - 1)4^3] = \frac{14}{3} \cdot 4^3 = 298\frac{26}{3}$.

ii. For $f(x) = 5x^3$, the area function is $A(x) = \frac{5}{4}x^4$. The area is $A(4) - A(1) = \frac{5}{4}4^4 - \frac{5}{4} = \frac{5}{4}(4^4 - 1) = \frac{1275}{4} = 318\frac{3}{4}$.

iii. For $f(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$, the area function is $A(x) = 2x^{\frac{3}{2}}$. The area is $A(9) - A(4) = 2(9^{\frac{3}{2}} - 4^{\frac{3}{2}}) = 2(3^3 - 2^3) = 38$.

iv. For $f(x) = 4x^2 + 2x^{\frac{1}{3}}$, the area function $A(x) = \frac{4}{3}x^3 + \frac{3}{2}x^{\frac{4}{3}}$. So the area is $A(8) - A(1) = (\frac{4}{3}8^3 + \frac{3}{2}8^{\frac{4}{3}}) - (\frac{4}{3} + \frac{3}{2}) = \frac{4}{3}(8^3 - 1) + \frac{3}{2}(2^4 - 1) = \frac{4223}{6} = 703\frac{5}{6}$.

6.5. *First Problem.* Raise both sides of $y = f(x) = x^{\frac{2}{3}}$ to the 3rd power to get $y^3 = x^2$. See Figure 6.5. Because $Q = (x + \Delta x, y + \Delta y)$ is on the graph,

$$(y + \Delta y)^3 = (x + \Delta x)^2.$$

To multiply out the 3 factors

$$(y + \Delta y)^3 = (y + \Delta y)(y + \Delta y)(y + \Delta y),$$

the y and Δy from any one of the three groups $(y + \Delta y)$ must be multiplied by the y and Δy from each of the other two. The product $y \cdot y \cdot y = y^3$ is one term that arises in this way. Fixing any Δy and multiplying it by the y from each of the other two groups gives the product $y^2\Delta y$. Since three different Δy's can be picked to do this, $y^2\Delta y$ will occur three times. The result of the multiplication will therefore be

$$(y + \Delta y)^3 \;\; = \;\; y^3 + 3y^2\Delta y \;\; + \;\; \text{more terms.}$$

Each of the additional terms contain at least two factors of Δy because terms containing no or one Δy are already accounted for. (The fact that the missing terms are $3y(\Delta y)^2$ and $(\Delta y)^3$ is not relevant to our computation.) Since $(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$, we get

$$y^3 + 3y^2\Delta y + \text{terms with } (\Delta y)^2 \text{ as factor } \;\; = \;\; x^2 + 2x\Delta x + (\Delta x)^2.$$

Because $P = (x, y)$ is on the graph, $y^3 = x^2$. Therefore after a subtraction,

$$3y^2\Delta y + \;\text{ terms with } (\Delta y)^2 \text{ as factor } \;\; = \;\; 2x\Delta x + (\Delta x)^2.$$

Now divide both sides of this equation by Δx to get

$$3y^2\tfrac{\Delta y}{\Delta x} \; + \; \text{ terms with } \Delta y \cdot \tfrac{\Delta y}{\Delta x} \text{ as factor } \;\; = \;\; 2x + \Delta x.$$

Next, push Δx to zero. Since Δy also goes to 0 in the process (see Figure 6.5) and because $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$, all terms with $\Delta y \cdot \frac{\Delta y}{\Delta x}$ as factor go to zero and

$$3y^2 f'(x) = 2x.$$

Recalling that $y = x^{\frac{2}{3}}$, we get by simple algebra that $f'(x) = \frac{2}{3}x \cdot y^{-2} = \frac{2}{3}x \cdot x^{-\frac{4}{3}} = \frac{2}{3}x^{-\frac{1}{3}}$.

Second Problem. Raise both sides of $y = f(x) = x^{\frac{4}{3}}$ to the 3rd power to get $y^3 = x^4$. Because $Q = (x + \Delta x, y + \Delta y)$ is on the graph,

$$(y + \Delta y)^3 = (x + \Delta x)^4.$$

We already dealt with $(y + \Delta y)^3$ above, so we'll focus on $(x + \Delta x)^4$. To multiply out the four factors

$$(x + \Delta x)(x + \Delta x)(x + \Delta x)(x + \Delta x),$$

the x and Δx from any one of the four groups $(x + \Delta x)$ must be multiplied by the x and Δx from each of the other three. The product $x \cdot x \cdot x \cdot x = x^4$ is one term that arises in this way. Fixing any Δx and multiplying it by the xs from each of the other three groups gives the product $x^3 \Delta x$. Since four different Δx's can be picked to do this, $x^3 \Delta x$ occurs four times. The result of the multiplication is

$$(x + \Delta x)^4 \;=\; x^4 + 4x^3 \Delta x \;+\; \text{more terms.}$$

Each of these additional terms contains at least two factors of Δx (because the terms that contain no or one Δx are accounted for). Using what we already know about $(y + \Delta y)^3$ from the solution of the first problem, we have

$$y^3 + 3y^2 \Delta y + \text{terms with } (\Delta y)^2 \text{ as factor} \;=\; x^4 + 4x^3 \Delta x + \text{terms with } (\Delta x)^2 \text{ as factor.}$$

Because $P = (x, y)$ is on the graph, $y^3 = x^4$. Therefore after a subtraction,

$$3y^2 \Delta y + \text{terms with } (\Delta y)^2 \text{ as factor} \;=\; 4x^3 \Delta x + \text{terms with } (\Delta x)^2 \text{ as factor.}$$

Now divide both sides of this equation by Δx to get

$$3y^2 \tfrac{\Delta y}{\Delta x} \;+\; \text{terms with } \Delta y \cdot \tfrac{\Delta y}{\Delta x} \text{ as factor} \;=\; 4x^3 + \text{terms with } \Delta x \text{ as factor.}$$

Next, push Δx to zero. Since Δy also goes to 0 in the process (see Figure 6.5) and $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$, it follows that all terms with $\Delta y \cdot \frac{\Delta y}{\Delta x}$ as factor go to zero and that

$$3y^2 f'(x) = 4x^3.$$

Since $y = x^{\frac{4}{3}}$, we get that $f'(x) = \frac{4}{3}x^3 \cdot y^{-2} = \frac{4}{3}x^3 \cdot x^{-\frac{8}{3}} = \frac{4}{3}x^{\frac{1}{3}}$.

6.6. As suggested, we'll carry six decimal places and then round off to three. Since the power series $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \ldots$ converges for all x with $|x| < 1$, it converges for $0 \le x \le \frac{3}{4}$. We'll begin our computations with the approximation $\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - x^5$ for $0 \le x \le \frac{3}{4}$, and see what pattern emerges. Since $F(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$ is an antiderivative of $1 - x + x^2 - x^3 + x^4 - x^5$, we get that

$$\begin{aligned}
\int_0^{\frac{3}{4}} \tfrac{1}{1+x}\, dx &\approx \int_0^{\frac{3}{4}} (1 - x + x^2 - x^3 + x^4 - x^5)\, dx = F(x)\Big|_0^{\frac{3}{4}} = F(\tfrac{3}{4}) \\
&= 0.75 - \tfrac{1}{2}(0.75)^2 + \tfrac{1}{3}(0.75)^3 - \tfrac{1}{4}(0.75)^4 + \tfrac{1}{5}(0.75)^5 - \tfrac{1}{6}(0.75)^6 \\
&\approx 0.750000 - 0.281250 + 0.140625 - 0.079102 + 0.047461 - 0.029663 \\
&\approx 0.548071.
\end{aligned}$$

To achieve an accuracy of three decimal places—according to our rule of thumb—we'll need to continue to compute the terms $\frac{1}{7}(0.75)^7, -\frac{1}{8}(0.75)^8, \frac{1}{9}(0.75)^9, \ldots$ and include them in the approximation until they round to 0 (with regard to three decimal places) and can be ignored. These computations are

$\frac{1}{7}(0.75)^7 \approx 0.019069$, $\frac{1}{8}(0.75)^8 \approx 0.012514$, $\frac{1}{9}(0.75)^9 \approx 0.008343$,
$\frac{1}{10}(0.75)^{10} \approx 0.005631$, $\frac{1}{11}(0.75)^{11} \approx 0.003840$, $\frac{1}{12}(0.75)^{12} \approx 0.002640$,
$\frac{1}{13}(0.75)^{13} \approx 0.001827$, $\frac{1}{14}(0.75)^{14} \approx 0.001273$, $\frac{1}{15}(0.75)^{15} \approx 0.000891$,
$\frac{1}{16}(0.75)^{16} \approx 0.000626$.

Since the next term $\frac{1}{17}(0.75)^{17} \approx 0.000442$ rounds to zero, the rule of thumb tells us that

$$\int_0^{\frac{3}{4}} \tfrac{1}{1+x}\,dx \approx 0.548071 + 0.019069 - 0.012514 + 0.008343 - 0.005631 + 0.003840 \\ - 0.002640 + 0.001827 - 0.001273 + 0.000891 - 0.000626 \approx 0.559357.$$

The actual value is $\ln 1.75 \approx 0.559616$, where ln is the natural log function (developed in Section 7.11).

Let's turn to the second integral. For any x satisfying $|x| < 1$, $|x^2| < 1$ as well. So we can substitute x^2 for x to get the power series $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$ for all x with $|x| < 1$. Proceeding as before, we'll start with the approximation, $\frac{1}{1+x^2} \approx 1 - x^2 + x^4 - x^6 + x^8 - x^{10}$ and study the pattern that emerges. With $F(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11}$ we get

$$\int_0^{\frac{3}{4}} \tfrac{1}{1+x^2}\,dx \approx \int_0^{\frac{3}{4}} (1 - x^2 + x^4 - x^6 + x^8 - x^{10})\,dx = F(x)\Big|_0^{\frac{3}{4}} = F(\tfrac{3}{4})$$
$$= 0.75 - \tfrac{1}{3}(0.75)^3 + \tfrac{1}{5}(0.75)^5 - \tfrac{1}{7}(0.75)^7 + \tfrac{1}{9}(0.75)^9 - \tfrac{1}{11}(0.75)^{11}$$
$$\approx 0.750000 - 0.140625 + 0.047461 - 0.019069 + 0.008343 - 0.003840 \approx 0.642270.$$

Continue with the terms $\frac{1}{13}(0.75)^{13} \approx 0.001827$ and $\frac{1}{15}(0.75)^{15} \approx 0.000891$. As in the solution of the previous integral, $\frac{1}{17}(0.75)^{17} \approx 0.000442$ so that the process stops. Adjusting the initial approximation, we get

$$\int_0^{\frac{3}{4}} \tfrac{1}{1+x^2}\,dx \approx 0.642270 + 0.001827 - 0.000891 \approx 0.643206.$$

The actual value is 0.643501 is a consequence of a property of the inverse tangent function (studied in Section 9.9.1).

6.7. Multiplying the power series $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \ldots$ by $x^{\frac{1}{2}}$ gives the series $\frac{x^{\frac{1}{2}}}{1+x} = x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}} + x^{\frac{9}{2}} - \ldots$. It converge for all x with $0 \le x < 1$, because $x^{\frac{1}{2}}(1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \ldots)$ converges for $0 \le x < 1$. At $x = 1$, the sum bounces back and forth between 0 and 1, so that the series does not converge at $x = 1$. We'll use the approximation $\frac{x^{\frac{1}{2}}}{1+x} \approx x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}} + x^{\frac{9}{2}}$. With the antiderivative $F(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} - \frac{2}{9}x^{\frac{9}{2}} + \frac{2}{11}x^{\frac{11}{2}}$, we get

$$\int_0^{\frac{1}{2}} \tfrac{x^{\frac{1}{2}}}{1+x}\,dx \approx F(x)\Big|_0^{\frac{1}{2}} = F(\tfrac{1}{2}) = \tfrac{2}{3}(\tfrac{1}{\sqrt{2}})^3 - \tfrac{2}{5}(\tfrac{1}{\sqrt{2}})^5 + \tfrac{2}{7}(\tfrac{1}{\sqrt{2}})^7 - \tfrac{2}{9}(\tfrac{1}{\sqrt{2}})^9 + \tfrac{2}{11}(\tfrac{1}{\sqrt{2}})^{11}$$
$$\approx 0.235702 - 0.070711 + 0.025254 - 0.009821 + 0.004018 \approx 0.184442.$$

Computing additional terms, we get

$$\tfrac{2}{13}(\tfrac{1}{\sqrt{2}})^{13} \approx 0.001700, \tfrac{2}{15}(\tfrac{1}{\sqrt{2}})^{15} \approx 0.000737, \tfrac{2}{17}(\tfrac{1}{\sqrt{2}})^{17} \approx 0.000325.$$

Since this last term rounds to 0 to three decimal places, our rule of thumb tells us that

$$0.184442 - 0.001700 + 0.000737 = 0.183479$$

is an approximation of the integral that should be accurate up to three decimal places. The actual value—accurate to 6-decimal places—is 0.183254 (a computation also involves the inverse tangent function).

The next several exercises study specific examples of the *binomial series*,

$$(1+x)^r = 1 + rx + \tfrac{r(r-1)}{2!}x^2 + \tfrac{r(r-1)(r-2)}{3!}x^3 + \cdots + \tfrac{r(r-1)(r-2)\cdots(r-(k-1))}{k!}x^k + \dots .$$

6.8. The binomial series for $(1+x)^3$ is equal to

$$(1+x)^3 = 1 + 3x + \tfrac{3(3-1)}{2!}x^2 + \tfrac{3(3-1)(3-2)}{3!}x^3 = 1 + 3x + 3x^2 + x^3.$$

The series stops there because the coefficient $\frac{3(3-1)(3-2)(3-3)}{3!}$ of the term x^4 is equal to zero. Since they contain $(3-3)$ as a factor, all the subsequent coefficients are zero as well. Notice that in this case the binomial series is equal to the multiplied out form of $(1+x)^3$. Since this equality holds for all x, this binomial series converges for all x.

The binomial series for $(1+x)^4$ is equal to

$$(1+x)^4 = 1 + 4x + \tfrac{4(4-1)}{2!}x^2 + \tfrac{4(4-1)(4-2)}{3!}x^3 + \tfrac{4(4-1)(4-2)(4-3)}{4!}x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

All subsequent terms of this series contain the factor $(4-4)$ and are zero.

These two examples illustrate what happens for any positive integer r. For $k = r+1$, the coefficient $\binom{r}{k}$ of x^k is zero because the factor $r-(k-1) = r+1-k$ is zero. The same is true for any $k > r+1$. In this case $k-i = r+1$ for some $i > 0$, so that the factor $r-(k-(i+1)) < r-(k-1)$ is zero. The series for $(1+x)^r$ is the multiplied-out version of the polynomial $(1+x)^r$. Since there are only a finite number of terms, it converges for all x.

6.9. For any k, the numbers $1, 2, 3, \dots, k-2, k-1$ count the number of factors of the numerator of the coefficient

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!}$$

except the first r. So k is the number of terms in the numerator of the kth coefficient of the binomial series. In the case $r = -1$,

$$\binom{-1}{k} = \frac{(-1)(-1-1)(-1-2)\cdots(-1-(k-1))}{k!} = \frac{(-1)(-2)(-3)\cdots(-k)}{k!} = \frac{(-1)^k \cdot k!}{k!} = (-1)^k.$$

It follows that this coefficient is equal to 1 if k is even and -1 if k is odd. So when $r = -1$ the binomial series is

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots .$$

6.10. For $r = \frac{1}{2}$ we get the binomial series

$$(1+x)^{\frac{1}{2}} = 1 + \binom{\frac{1}{2}}{1} x + \binom{\frac{1}{2}}{2} x^2 + \binom{\frac{1}{2}}{3} x^3 + \binom{\frac{1}{2}}{4} x^4 + \binom{\frac{1}{2}}{5} x^5 + \binom{\frac{1}{2}}{6} x^6 + \dots .$$

The first few coefficients are

$$\binom{\frac{1}{2}}{1} = \frac{1}{2},\ \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} = -\frac{1}{8},\ \binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} = \frac{\frac{3}{8}}{3!} = \frac{1}{16},$$

$$\binom{\frac{1}{2}}{4} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!} = \frac{\frac{3}{8}\cdot\frac{-5}{2}}{4!} = -\frac{5}{128},\ \binom{\frac{1}{2}}{5} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)(\frac{1}{2}-4)}{5!} = \frac{\frac{3}{8}\cdot\frac{-5}{2}\cdot\frac{-7}{2}}{5!} = \frac{7}{256}$$

$$\binom{\frac{1}{2}}{6} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)(\frac{1}{2}-4)(\frac{1}{2}-5)}{6!} = \frac{\frac{3}{8}\cdot\frac{-5}{2}\cdot\frac{-7}{2}\cdot\frac{-9}{2}}{6!} = -\frac{7}{64(2^2\cdot 4)} = -\frac{3\cdot 7}{1024} - \frac{21}{1024}.$$

So the start of this binomial series is

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 - \tfrac{21}{1024}x^6 + \dots .$$

As a consequence, for any x with $0 \le x \le \frac{1}{2}$,

$$\sqrt{1+x} \approx 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 - \tfrac{21}{1024}x^6.$$

It follows that

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt{1+x}\,dx &\approx \int_0^{\frac{1}{2}} \left(1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 - \tfrac{21}{1024}x^6\right) dx \\
&\approx \left(x + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3 + \tfrac{1}{64}x^4 - \tfrac{1}{128}x^5 + \tfrac{7}{6\cdot 256}x^6 - \tfrac{3}{1024}x^7\right)\Big|_0^{\frac{1}{2}} \\
&\approx 0.500000 + 0.062500 - 0.005208 + 0.000977 - 0.000244 + \cdots \approx 0.558269.
\end{aligned}$$

This approximation made use of our rule of thumb. Since 0.000244 rounds to 0 with three decimal place accuracy, this term was ignored. The approximation is accurate up to three decimal places. (The value of the integral is 0.558078 with an accuracy of 6 decimal places.)

The same strategy applied to $\displaystyle\int_0^5 \sqrt{1+x}\,dx$ gives us

$$\begin{aligned}
\int_0^5 \sqrt{1+x}\,dx &\approx \left(x + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3 + \tfrac{1}{64}x^4 - \tfrac{1}{128}x^5 + \tfrac{7}{6\cdot 256}x^6 - \tfrac{3}{1024}x^7\right)\Big|_0^5 \\
&\approx 5 + \tfrac{1}{4}5^2 - \tfrac{1}{24}5^3 + \tfrac{1}{64}5^4 - \tfrac{1}{128}5^5 + \tfrac{7}{6\cdot 256}5^6 - \tfrac{3}{1024}5^7 \\
&\approx 5 + 6.25 - 5.21 + 9.77 - 24.41 + 71.21 - 288.88.
\end{aligned}$$

Notice that the numbers get larger and larger as their coefficients alternate. So this does not deliver an approximation of the integral. The basic problem is that the binomial series for $\sqrt{1+x}$ does not converge for $x = 5$. (A substitution—see Section 9.7.1—can be used to show that the integral is approximately equal to 9.13.)

We turn to the exploration of moving points. Rather than sketched, the motion of the points is explicitly described.

6.11. **i.** For $p(t) = 2t - 5$ with $t \geq 0$, the velocity is $v(t) = p'(t) = 2$ and the acceleration is $a(t) = v'(t) = 0$. So the point starts at $p(0) = -5$ on the coordinate axis and moves with the constant speed of 2 in the positive direction.

ii. For $p(t) = 2t^2 + 2t + 12$ with $t \geq -10$, the velocity is $v(t) = 4t + 2$ and the acceleration is $a(t) = v'(t) = 4$. So the point starts at $p(-10) = 192$ on the coordinate axis and moves with a velocity of $v(-10) = -38$ at that time. The point moves to the left over the time $-10 \leq t - \frac{1}{2}$, stops at time $t = -\frac{1}{2}$, and then moves to the right for $t > -\frac{1}{2}$. Regarding the point to have a mass of 1, it is pushed to the right with a constant force of 4 units.

iii. For $p(t) = t^3 - 4t^2 - 21t$ with $t \geq -6$, the velocity is $v(t) = p'(t) = 3t^2 - 8t - 21$ and the acceleration is $a(t) = v'(t) = 6t - 8$. By the quadratic formula, $v(t) = 0$ when $t = \frac{8 \pm \sqrt{64-(4)(3)(-21)}}{6} = \frac{8 \pm \sqrt{64+252}}{6} = \frac{8 \pm 2\sqrt{79}}{6} = \frac{4 \pm \sqrt{79}}{3}$, so when $t \approx -1.63$ or $t \approx 4.30$. The point starts at $p(-6) = -224$ on the coordinate axis. Since $v(-6) = 135$ it moves to the right until it stops at time $t = \frac{4-\sqrt{79}}{3}$. From that time on it moves to the left (for example, $v(-1) = -10$) until it stops again at time $t = \frac{4+\sqrt{79}}{3}$. From then on, it moves to the right again (for instance, $v(5) = 14$) with greater and greater velocity.

iv. For $p(t) = \frac{3}{t} = 3t^{-1}$ with $t \geq 1$, the velocity is $v(t) = p'(t) = -3t^{-2} = \frac{-3}{t^2}$ and the acceleration is $a(t) = v'(t) = 6t^{-3} = \frac{6}{t^3}$. So the point starts at $p(1) = 3$ on the coordinate axis moving with a speed of 3 to the left. As t increases it continues to move to the left with diminishing speed getting closer and closer to the origin in the process. Taking the points mass to be 1, note that the force on the point always acts to the right but with a magnitude that decreases to 0 as time goes on.

6.12. **i.** The velocity function $v(t)$ is an antiderivative of $a(t) = 6t - 12$, so that $v(t) = 3t^2 - 12t + C$. Since $v(0) = 0$, $v(t) = 3t^2 - 12t$. Since $p(t)$ is an antiderivative of $v(t)$, $p(t) = t^3 - 6t^2 + C$. Since $p(0) = 0$, $p(t) = t^3 - 6t^2$. The equation $v(t) = 3t(t-4)$ describes the motion. During the time $0 < t < 4$, the velocity is negative and the point moves from the origin to the left. It stops at time $t = 4$ at the point $p(4) = 64 - 96 = -32$. From time $t > 4$ onward the velocity is positive and the point moves to the right with ever increasing speed.

ii. A diagram of the motion over the time interval $[0, 7]$ similar to Figure 6.12 is easily drawn.

6.13. Since $a(t) = 2t - 6$ and the velocity $v(t)$ is an antiderivative of $a(t)$, $v(t) = t^2 - 6t + C$. Since $v(0) = 5$, $v(t) = t^2 - 6t + 5$. Notice that $v(t) = (t-1)(t-5)$. Since $p(t)$ is an antiderivative of $v(t)$ and $p(0) = 6, p(t) = \frac{1}{3}t^3 - 3t^2 + 5t + 6$. Combining the information that $v(t)$ and $p(t)$ provide, we see that: for $0 \le t < 1$, $v(t)$ is positive so that the point, starting at $p(0) = 6$, moves to the right; it stops at $t = 1$ at the point $p(1) = \frac{1}{3} - 3 + 5 + 6 = 8\frac{1}{3}$; during $0 < t < 5$, $v(t)$ is negative, so that the point moves to the left; it stops again at $t = 5$ at the point $p(5) = \frac{1}{3}5^3 - 3 \cdot 5^2 + 5^2 + 6 = -2\frac{1}{3}$; finally for $t > 5$, $v(t)$ is positive again and the point moves to the right with ever increasing speed.

6.14. **i.** If $x(t) = 2$ and $y(t) = 5$ for all $t \ge 0$, the point remains at $(2, 5)$ in the xy-plane and does not move.

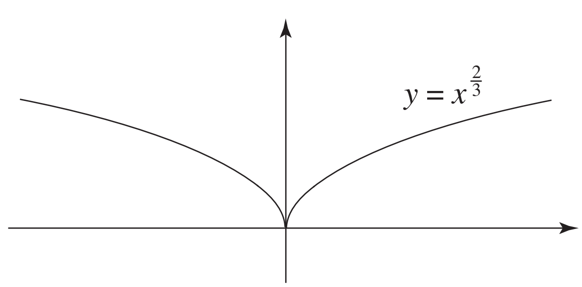
ii. In the situation $x(t) = t$ and $y(t) = t^2$ notice that $x(t)^2 = t^2 = y(t)$. So the point moves on the parabola $y = x^2$ for the entire time $t \ge -2$. We can draw the following conclusions from the fact that $x'(t) = 1$ and $y'(t) = 2t$. From its starting point $(x(-2), y(-2)) = (-2, 4)$ on the parabola, the point moves with a constant horizontal speed of 1 unit in the direction of the positive x-axis. The speed of the point along the parabola is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1 + 4t^2}$. So while its speed in the horizontal direction is constant, its speed along the parabola decreases initially until time $t = 0$ when its speed is 1. Thereafter the speed along the parabola increases. After only a short time t it is approximately equal to $2t$.

iii. In the situation $x(t) = t^{\frac{1}{3}}$ and $y(t) = t^{\frac{2}{3}}$ it is again the case that $y(t) = x(t)^2$ so that the point also moves on the parabola $y = x^2$ for $t \ge -8$. Since $-8^{\frac{1}{3}} = -2$, it also starts at $(-2, 4)$. Since $x'(t) = \frac{1}{3}t^{-\frac{2}{3}} = \frac{1}{3t^{\frac{2}{3}}}$ and $y'(t) = \frac{2}{3}t^{-\frac{1}{3}} = \frac{2}{3t^{\frac{1}{3}}}$ the speed of the point along the parabola is equal to

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\tfrac{1}{9}t^{-\frac{4}{3}} + \tfrac{4}{9}t^{-\frac{2}{3}}} = \tfrac{1}{3}\sqrt{\tfrac{1}{t^{\frac{4}{3}}} + \tfrac{4}{t^{\frac{2}{3}}}}.$$

Notice that the speed of the point increases quickly during $-8 \le t < 0$ before it reaches infinite speed at the origin at time $t = 0$. Thereafter, the point continues to move to the left initially at a great speed, but as t increases the point slows down. Even though the speed of the point goes to zero with advancing t, a look at the equations $x(t) = t^{\frac{1}{3}}$ and $y(t) = t^{\frac{2}{3}}$ tells us that no matter how far out on the parabola a point is, the moving point will pass it eventually.

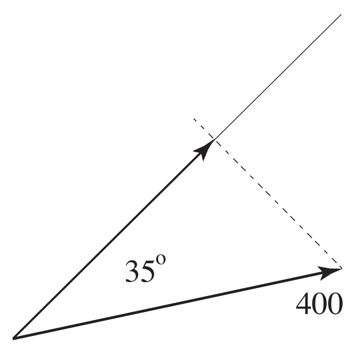
iv. Since $x(t) = t^3$ and $y(t) = t^2$, $y(t)^3 = x(t)^2$ so that $y(t) \ge 0$ and $y(t) = x(t)^{\frac{2}{3}}$. Hence the point moves on the curve depicted below. It starts at time $t = -1$ at



$(x(-1), y(-1)) = (-1, 1)$ and moves to the right with increasing t. Since $x'(t) = 3t^2$ and $y'(t) = 2t$, $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9t^4 + 4t^2}$. It follows that the initial speed of the point is $\sqrt{13}$, that it then slows to stop at the origin at time $t = 0$, and that thereafter it speeds up to greater and greater speeds with increasing t.

v. Since $x(t) = t$ and $y(t) = f(t)$ for $t \geq b$ the point moves along the graph of the function $y = f(x)$. It starts its motion at the point $(b, f(b))$. Since $x'(t) = 1$ its speed in the horizontal direction is constant. The speed of the point along the graph is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1 + f'(t)^2}$ for any $t \geq b$. It follows that its speed is at its minimum of 1 whenever $f'(t) = 0$. Since $x(t) = t$, this happens whenever the moving point encounters a horizontal tangent of the graph $y = f(x)$.

6.15. The component of this force in the indicated direction is drawn into the diagram. Its magnitude is $400 \cos 35° \approx 327.66$ N.



6.16. The resultant of the two horizontal components of the vectors of Figure 6.40a is the vector from $(0, 0)$ to $(-3, 0)$ obtained by adding the x-coordinates 1 and -4 and the resultant of the two vertical components is the vector from $(0, 0)$ to $(0, 1)$ obtained by adding the y-coordinates 4 and -3. By the parallelogram law, the resultant of these two resultants is the vector from $(0, 0)$ to $(-3, 1)$ depicted in Figure 6.40b. (If the two

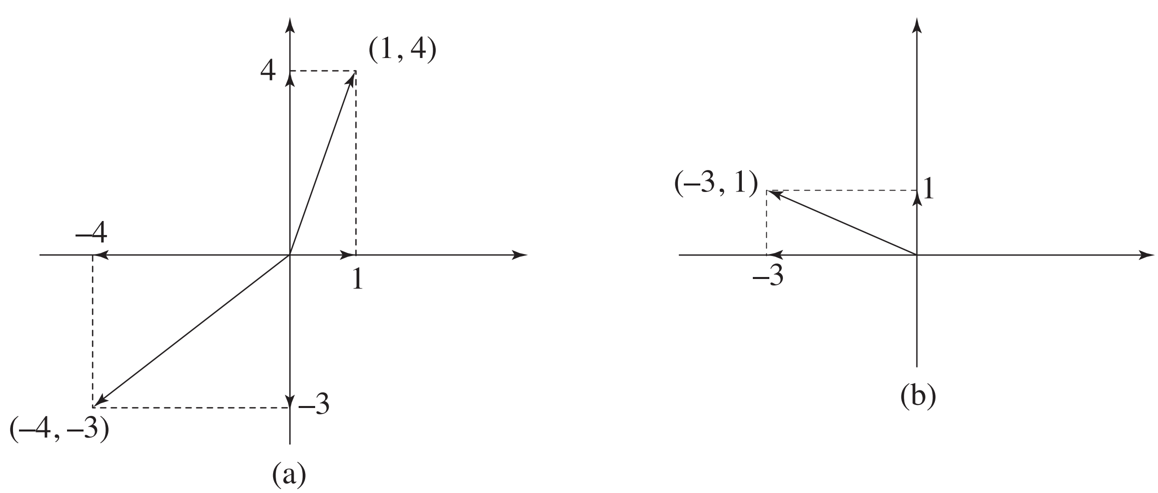


Figure 6.40

end points $(1,4)$ and $(-4,-3)$ of the vectors in Figure 6.40a are placed accurately, then the resultant of Figure 6.40b is also obtained by applying the parallelogram law to the two vectors. Is this the case?)

6.17. Refer to Figure 6.41a. The resultant of the horizontal components of the two vectors is the vector from $(0,0)$ to $(1,0)$. The resultant of the vertical components of the two vectors is the vector from $(0,0)$ to $(0,4)$. The resultant of the two resultants is the vector from $(0,0)$ to $(1,4)$. (What about the accuracy question of Problem 6.16 here?)

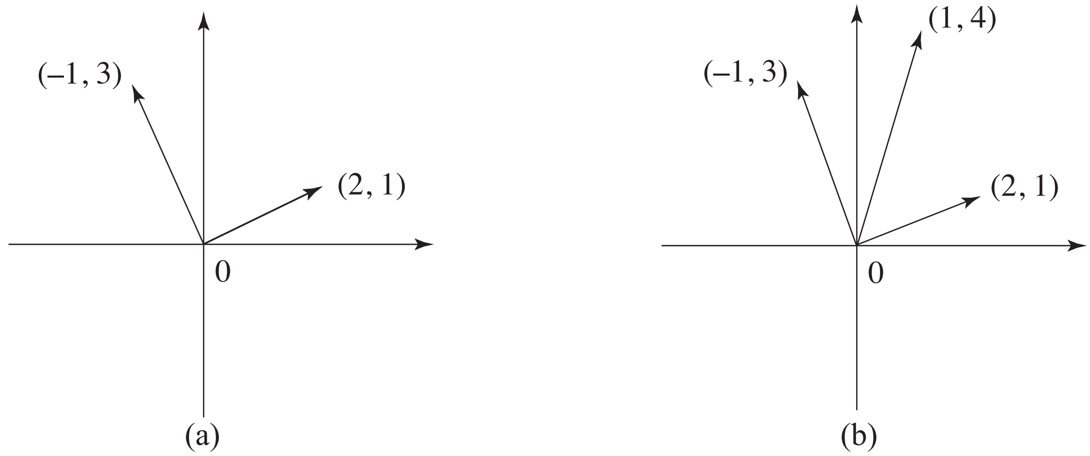
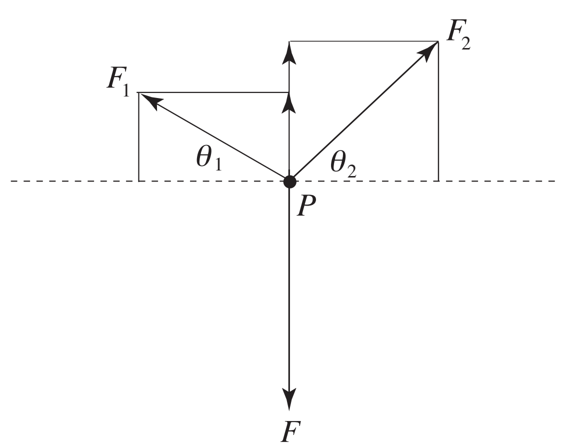


Figure 6.41

6.18. Since the vectors are in equilibrium, the magnitude of the total upward force is equal to the downward force. The vertical components of F_1 and F_2 are $F_1 \sin\theta_1$ and $F_2 \sin\theta_2$ respectively, so that $F_1 \sin\theta_1 + F_2 \sin\theta_2 = F$. The equilibrium condition also implies that the magnitude of the horizontal component of F_1 is equal to the magnitude of the



horizontal component of F_2. Therefore $F_1 \cos\theta_1 = F_2 \cos\theta_2$.

i. If F_1 has a magnitude of 10 pounds and if the angles θ_1 and θ_2 are 30° and 60°, respectively, then $F_1 \cos\theta_1 = 10 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3}$ pounds. Since $F_2 \cos\theta_2 = F_2 \cdot \frac{1}{2}$, we obtain that $F_2 = 10\sqrt{3}$ pounds. Since $F = F_1 \sin\theta_1 + F_2 \sin\theta_2$, we see that $F = 10 \cdot \frac{1}{2} + 10\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 5 + 15 = 20$ pounds.

ii. Let the mass of the object attached at the point P be 2 kg and let then F represent the object's weight. So $F = 2 \cdot 9.81 = 19.62$ N. With $\theta_1 = 30°$ and $\theta_2 = 45°$, we get the equations $F_1 \sin 30° + F_2 \sin 45° = 19.62$ N and $F_1 \cos 30° = F_2 \cos 45°$ N. After putting in the values for sine and cosine, we get $\frac{1}{2}F_1 + \frac{\sqrt{2}}{2}F_2 = 19.62$ N and $\frac{\sqrt{3}}{2}F_1 = F_2\frac{\sqrt{2}}{2}$ N. Using $\frac{\sqrt{3}}{2} \approx 0.87$ and $\frac{\sqrt{2}}{2} \approx 0.71$, we get $0.87F_1 \approx 0.71F_2$ and $0.5F_1 + 0.71F_2 \approx 19.62$ N. So $0.5F_1 + 0.87F_1 \approx 1.37F_1 \approx 19.62$ N, and hence $F_1 \approx 14.32$ N and $F_2 \approx \frac{0.87}{0.71}F_1 \approx (1.23)(14.32) \approx 17.61$ N.

6.19. **i.** By the law of sines and Figure 6.43a, $\frac{\sin 125°}{115} = \frac{\sin 25°}{F_1} = \frac{\sin 30°}{F_2}$. Putting in the values, we get $\frac{0.82}{115} \approx \frac{0.42}{F_1} \approx \frac{0.50}{F_2}$. So $F_1 \approx \frac{0.42 \cdot 115}{0.82} \approx 58.90$ and $F_2 \approx \frac{0.50 \cdot 115}{0.82} \approx 70.12$.

ii. By the law of cosines and Figure 6.43b, we get that $F^2 = 35^2 + 49^2 - 2(35)(49)\cos 115°$. With $\cos 115° \approx -0.42$, we get $F^2 \approx 35^2 + 49^2 + 2(35)(49)(0.42)$. So the magnitude F satisfies $F^2 \approx 5066.6$, so that $F \approx 71.18$.

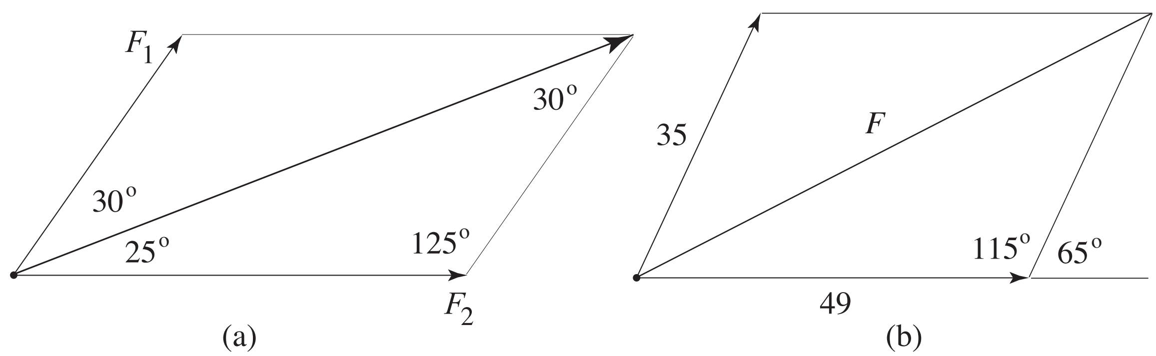


Figure 6.43

6.20. We'll put Newton's toss of the apple into the context of Section 6.7 and Figure 6.22.

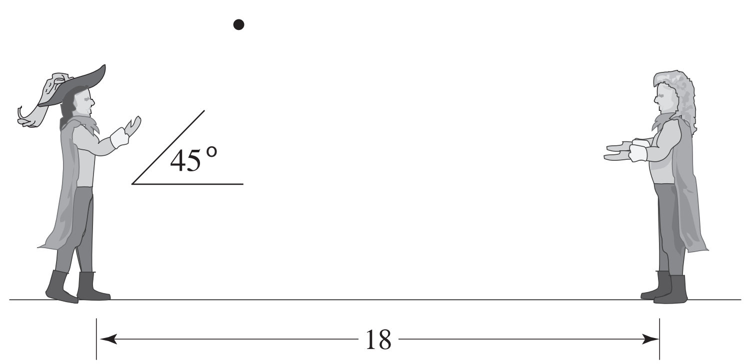


Figure 6.44

So $y_0 = 3$ feet, $\varphi_0 = 45°$, and $v_0 = 25$ feet/sec.

i. To find the time t, turn to equation (6a), set $x(t) = (v_0 \cos\varphi_0)t = 18$ and solve for t. So $(25\cos 45°)t = 25 \cdot \frac{\sqrt{2}}{2} t = 18$, and hence $t = \frac{18}{25} \cdot \frac{2}{\sqrt{2}} \approx 1.02$ sec. Evaluating $y(t) = -\frac{g}{2}t^2 + (v_0 \sin\varphi_0)t + y_0$ of (6b) at $t \approx 1.02$ we get the height $y(1.02) \approx -\frac{32}{2}(1.02)^2 + 25\frac{\sqrt{2}}{2}(1.02) + 3 \approx 4.38$ feet of the apple at that time. So—assuming Newton is accurate in terms of the direction of his toss—the apple will reach the location of Halley almost exactly 1 second after it is released and it will be close to 4.4 feet above the ground at that time.

ii. The range formula (6f) $R = \frac{v_0}{g}\cos\varphi_0\big(v_0 \sin\varphi_0 + \sqrt{v_0^2 \sin^2\varphi_0 + 2gy_0}\big)$ with $g = 32$ feet/sec^2 tells us that the apple will hit the ground about

$$\tfrac{25}{32}\tfrac{\sqrt{2}}{2}\big(25\tfrac{\sqrt{2}}{2} + \sqrt{25^2 \cdot 0.5 + 2(32)(3)}\big) \approx 22.17 \text{ feet}$$

from where Newton is standing. By formula (6e), the time of impact is

$$t_{\text{imp}} = \frac{v_0 \sin\varphi_0 + \sqrt{v_0^2 \sin^2\varphi_0 + 2gy_0}}{g} \approx \frac{25 \cdot \frac{\sqrt{2}}{2} + \sqrt{25^2 \cdot 0.5 + 2(32)(3)}}{32} \approx 1.25 \text{ sec.}$$

iii. By formula (6i), the speed of the apple at impact is equal to

$$\sqrt{x'(t_{\text{imp}})^2 + y'(t_{\text{imp}})^2} = \sqrt{v_0^2 + g^2t^2 - 2g(v_0 \sin\varphi_0)t}$$
$$\approx \sqrt{25^2 + (32^2)(1.25^2) - 64(25 \cdot \tfrac{\sqrt{2}}{2})1.25} \approx 28.47 \text{ feet/sec.}$$

6.21. The initial data are $y_0 = 5$ feet, $\varphi_0 = 20°$, and $v_0 = 40$ feet/sec. The maximal height reached by the apple is given by (6d) $\frac{1}{2g}v_0^2 \sin^2\varphi_0 + y_0$. Inserting the data into this expression, we get $\frac{1}{32}40^2 \sin^2 20° + 5 \approx 7.92$ feet. Since Hooke is 35 feet away, the time that the apple arrives at the position held by Hooke, is gotten by solving $x(t) = (v_0 \cos\varphi_0)t = 35t$ for t. So $(40\cos 20°)t = 35$ and hence $t \approx \frac{35}{37.59} \approx 0.93$ seconds. The height of the apple at this time is

$$y(t) = \tfrac{-g}{2}t^2 + (v_0 \sin\varphi_0)t + y_0 \approx -16(0.93)^2 + (40\sin 20°)(0.93) + 5 \approx 3.88 \text{ feet.}$$

That Hooke was at least 4 feet tall is likely, so that the ball would have hit him. Since

$$x'(t) = v_0 \cos\varphi_0 = 40\cos 20 \;\text{ and }\; y'(t) = -gt + v_0 \sin\varphi_0 = -32t + 40\sin 20°,$$

the speed at time $t = 0.93$ is equal to

$$\sqrt{x'(0.93)^2 + y'(0.93)^2} \approx \sqrt{(40\cos 20°)^2 + (-(32)(0.93) + 40\sin 20°)^2} \approx 40.88 \text{ ft/sec.}$$

Hooke was small in stature, but it is safe to assume that he was not a midget (and taller than 3.9 feet). Hooke and Newton quarreled over scientific matters for years (the nature of gravity, for instance). Newton closed one of his friendlier letters to Hooke with "if I have seen further it has only been by standing on the shoulders of giants." This has been widely interpreted as a nasty allusion to Hooke's smallish stature. But one of Hooke's friends describes him as having "midling stature, something crooked, pale faced ...".

6.22. In this problem $x(t)$ and $y(t)$ are the coordinates of the bottom point of the basketball. The relevant data is $y_0 = 8$ feet, $v_0 = 22$ feet per second, and $\varphi_0 = 45°$. Let the shot be released at time $t = 0$. Since the rim of the basket is 10 feet above the floor, the key is to find the time t for which $y(t) = 10$ feet on the ball's descent. So the first question is: What t gives us $y(t) = 10$? Taking $y(t) = 10$ in formula (6b) and solving for t, we get

$$10 = -\tfrac{g}{2}t^2 + (v_0 \sin\,\varphi_0)t + y_0 \;= -16t^2 + 22 \cdot \tfrac{\sqrt{2}}{2}t + 8 = -16t^2 + 11\sqrt{2}t + 8$$

and hence by applying the quadratic formula to $16t^2 - 11\sqrt{2}t + 2 = 0$, that

$$t \;=\; \tfrac{11\sqrt{2}\pm\sqrt{242-4\cdot 16\cdot 2}}{32} \;=\; \tfrac{11\sqrt{2}\pm\sqrt{114}}{32} \;\approx\; \tfrac{15.56\pm 10.68}{32} \;\approx\; 0.152 \;\text{ or } 0.820 \;\text{ seconds.}$$

Since the ball is on its descent, $t = 0.82$ seconds is the time of interest. The x-coordinate of the bottom of the ball at that time determines the distance of a successful shot from the basket. Inserting $t = 0.82$ into formula (6a) $x(t) = (v_0 \cos\varphi_0)t$, tells us that

$$x(0.820) \approx 22 \cdot \tfrac{\sqrt{2}}{2} \cdot 0.820 \approx 12.76 \text{ feet.}$$

To maximize the likelihood of scoring, the player should take his jump shot about $12\frac{3}{4}$ feet from the basket.

6.23. The relevant data about Newton's pet parakeet—more accurately its dropping—are $v_0 = 6$ meters/sec, $\varphi_0 = 30°$, and $y_0 = 15$ meters. Inserting this into expression (6d) $\frac{1}{2g}v_0^2 \sin^2\varphi + y_0$, we get that $\frac{1}{2(9.80)}6^2 \sin^2 30° + 15 \approx 15.46$ meters is the maximal height reached by the dropping. To determine when the dropping might splatter against Newton's house, we'll set $x(t) = (v_0 \cos\varphi_0)t = 9$ and solve for t. Doing so we get $t = \frac{9}{6\cdot\frac{\sqrt{3}}{2}} \approx 1.73$ seconds. Using the formula $y(t) = \frac{-g}{2}t^2 + (v_0 \sin\varphi_0)t + y_0$ of (6b), we see that the height of the dropping at this time is $\frac{-9.8}{2}(1.73)^2 + 6 \cdot \frac{1}{2}(1.73) + 15 \approx 5.52$ meters. So the dropping will hit the house. Using formula (6i) we get that the speed of the dropping at the time of impact $t = 1.73$ is

$$\sqrt{x'(1.73)^2 + y'(1.73)^2} \approx \sqrt{(6\cos 30)^2 + (-(9.80)(1.73) + 6\sin 30)^2} \approx 14.89 \text{ m/sec.}$$

6.24. In this problem $y_0 = 1.5$ meters and $\varphi_0 = 70°$. We are looking for v_0. Formula (6c) asserts that any point (x, y) on the trajectory of the arrow satisfies

$$y = \Big(\frac{-g}{2v_0^2\cos^2\varphi_0}\Big)x^2 + (\tan\;\varphi_0)x + y_0.$$

We are given that the point (25, 55) is on the trajectory. If we plug all of our information into this equation, v_0 should emerge. Doing this, we get

$$25 = \Big(\tfrac{-9.8}{2v_0^2\cos^2 70°}\Big)55^2 + (\tan 70°)55 + 1.5 \approx -\tfrac{41.89}{v_0^2}\,55^2 + (2.75)55 + 1.5.$$

Therefore, $\frac{41.89}{v_0^2}\,55^2 = -25 + 151.25 + 1.5 = 127.75$. So $v_0^2 \approx \frac{(41.89)(55^2)}{127.75}$ and hence $v_0 \approx 31.5$ meters per second.

6.25. Here $y_0 = 6$ feet and $v_0 = 120$ feet per second and we are looking for φ_0. Let's start by repeating the strategy used in Problem 6.24. Using the data we have, the fact that the point $(240, 62)$ lies on the trajectory, and plugging what we know into

$$y = \Big(\frac{-g}{2v_0^2\cos^2\varphi_0}\Big)x^2 + (\tan\varphi_0), x + y_0\,,$$

we get $62 = \big(\frac{-32}{2(120^2)\cos^2\varphi_0}\big)240^2 + (\tan\varphi_0)240 + 6$. Using the fact that $\frac{1}{\cos\varphi_0} = \sec\varphi_0$, we can rewrite this as $62 = -16(4)(\sec^2\varphi_0) + 240(\tan\varphi_0) + 6$. Because $\sec^2\varphi_0 = 1+\tan^2\varphi_0$, we get $62 = -16(4)(\tan^2\varphi_0) + 240(\tan\varphi_0) - 16(4) + 6$ and therefore

$$64(\tan^2\varphi_0) - 240(\tan\varphi_0) + 120 = 0\,.$$

By the quadratic formula, $\tan\varphi_0 = \frac{240\pm\sqrt{240^2-4(64)(120)}}{2\cdot 64}$. Because $240 = 16\cdot 15$, notice that $16^2 = 256$ is a factor of both terms under the radical. Therefore,

$$\tan\varphi_0 = \tfrac{240\pm 16\sqrt{15^2-120}}{2\cdot 64} = \tfrac{240\pm 16\sqrt{105}}{2\cdot 64} = \tfrac{15\pm\sqrt{105}}{8} \approx 0.594 \text{ or } 3.16.$$

By pushing "inverse tan" on your calculator, you will get $\varphi_0 \approx 30.7°$ or $\varphi_0 \approx 72.4°$. Are there two different angles with which the arrow can be shot off so as to hit the target? Intuitively, if the arrow has the steeper trajectory, it will gain altitude earlier and will then descend towards the cauldron as the archer intends. To see more convincingly that this is the case, consider the steeper trajectory $\varphi_0 \approx 72.4°$ and refer to the discussion that develops equation (6d). Note that the arrow will reach its maximal height at time $t_1 = \frac{v_0\sin\varphi_0}{g} \approx \frac{(120)(0.953)}{32} \approx 3.57$ seconds. Thereafter, it will descend. By one of the equations in (6a), the arrow will reach its target $(240, 62)$ at time $t = \frac{240}{v_0\cos\varphi_0} \approx \frac{240}{(120)(0.302)} \approx 6.62$ seconds. So the flaming arrow will hit the target on its descent. With the flatter trajectory the arrow will approach the cauldron from below and hit against its base.

6.26. **i.** We'll apply the range formula (6f) of Section 6.7

$$R = \tfrac{v_0^2}{2g}\sin(2\varphi_0) + \tfrac{v_0}{g}\sqrt{\tfrac{v_0^2}{4}\sin^2(2\varphi_0) + 2gy_0\cos^2\varphi_0},$$

with $\varphi_0 = 0, y_0 = 3.6$ feet and $v_0 = 1439$, to get

$$\begin{aligned} R &= \tfrac{(1439)^2}{2\cdot 32}\sin 0° + \tfrac{1439}{32}\sqrt{\tfrac{(1439)^2}{4}\sin 0° + 2(32)(3.6)(\cos^2 0°)} \\ &\approx 0 + 44.97\sqrt{0+230.4} \approx 682.60 \text{ feet.} \end{aligned}$$

For the cannonball propelled by a 6-pounder field gun with 1.25 pounds of powder and an angle of elevation of $\varphi_0 = 0$, Table 6.1 provides a range of 315 yards or $3(315) = 945$ feet.

So the observed range of the actual shot is considerable greater than the range predicted by the theory, even though the former occurred against air resistance and the latter assumes no air resistance. The inescapable conclusion is that there is a problem with the data in *The Artillerist's Manual*. Since the range would seem to be easy to measure, it is likely that the angle of departure and/or the muzzle velocity are inaccurate. Suppose, for instance, that the angle of departure and the muzzle velocity were in fact $0.2°$ and 1480 feet per second, instead of $0°$ and 1439 feet per second. The predicted range under those assumptions is

$$\begin{aligned} R &= \tfrac{1480^2}{2(32)} \sin 0.4° + \tfrac{1480}{32}\sqrt{\tfrac{1480^2}{4}(\sin^2 0.4°) + 2(32)(3.6)(\cos^2 0.2°)} \\ &\approx 238.93 + 46.25\sqrt{26.69 + 230.40} \approx 980 \text{ feet.} \end{aligned}$$

Now, as expected, the predicted range exceeds the actual range.

ii. Let's try $\varphi_0 = 1°$ next (while keeping $y_0 = 3.6$ and $v_0 = 1439$). Inserting these values into the range formula, we obtain

$$\begin{aligned} R &= \tfrac{(1439)^2}{2\cdot 32} \sin 2° + \tfrac{1439}{32}\sqrt{\tfrac{(1439)^2}{4} \sin^2 2° + 2(32)(3.6)(\cos^2 1°)} \\ &\approx 1129.17 + 44.97\sqrt{630.52 + 230.33} \\ &\approx 1129.17 + (44.97)(29.68) \\ &\approx 2448.57 \text{ feet.} \end{aligned}$$

The corresponding observed value from Table 6.1 is 674 yards = 2022 feet. So the theoretical prediction exceeds the observed value by about 20%.

iii. Let's try $\varphi_0 = 5°$ next (again keeping $y_0 = 3.6$ and $v_0 = 1439$). Plugging these values into the range formula, we obtain

$$\begin{aligned} R &= \tfrac{(1439)^2}{2\cdot 32} \sin 10° + \tfrac{1439}{32}\sqrt{\tfrac{(1439)^2}{4} \sin^2 10° + 2(32)(3.6)(\cos^2 5°)} \\ &\approx 5{,}618.39 + 44.97\sqrt{15{,}609.97 + 228.65} \\ &\approx 5{,}618.39 + (44.97)(125.85) \\ &\approx 11{,}277.78 \text{ feet.} \end{aligned}$$

The corresponding observed value from Table 6.1 is 1523 yards = 4569 feet. So the theoretical prediction far exceeds the observed value by a factor of about $2\frac{1}{2}$.

6.27. This is another application of the range formula

$$R = \tfrac{v_0^2}{2g} \sin(2\varphi_0) + \tfrac{v_0}{g}\sqrt{\tfrac{v_0^2}{4} \sin^2(2\varphi_0) + 2gy_0 \cos^2 \varphi_0},$$

this time with $v_0 = 1486$ feet and φ_0 equal to $0°, 1°$, and $5°$. We will assume that the muzzle of the 12-pdr. field gun is the same $y_0 = 3.6$ feet from the ground as that of the 6-pdr. field gun. Taking $\varphi_0 = 0°$ we get

$$R = \tfrac{v_0}{g}\sqrt{2gy_0} = \tfrac{1486}{32}\sqrt{(64)(3.6)} \approx (46.44)(15.18) \approx 704.87 \text{ feet.}$$

With $\varphi_0 = 1°$ we get

$$\begin{aligned} R &= \tfrac{(1486)^2}{64}\sin 2° + \tfrac{1486}{32}\sqrt{\tfrac{(1486)^2}{4}\sin^2 2° + 64(3.6)(\cos^2 1°)} \\ &\approx 1204.14 + 46.44\sqrt{672.38 + 230.33} \approx 2599.39\,\text{feet}. \end{aligned}$$

Finally with $\varphi_0 = 5°$ we get

$$\begin{aligned} R &= \tfrac{(1486)^2}{64}\sin 10° + \tfrac{1486}{32}\sqrt{\tfrac{(1486)^2}{4}\sin^2 10° + 64(3.6)(\cos^2 5°)} \\ &\approx 5991.39 + 46.44\sqrt{16646.31 + 228.65} \approx 12{,}023.95 \text{ feet}. \end{aligned}$$

The observed ranges from Table 6.1 are respectively, 347 yards, 662 yards, and 1663 yards, or 1041 feet, 1986 feet, and 4,989 feet. The large discrepancies between the theoretical and observed distances are again explained by inaccuracies of the data (in the first case) and most certainly by air resistance (in the other two).

In sum, the only valid conclusion that can be drawn from the results of Problems 6.26 and 6.27 is that in situations involving large velocities, air resistance plays an overwhelming role and the formulas derived in Section 6.7 are practically useless.

6.28. With an xy-coordinate plane given, we saw in Section 4.4 that an ellipse with focal axis the x-axis, center the origin, semimajor axis a, and semiminor axis b is the graph of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It was also shown in Section 4.4 that the two focal points are the points $(-c, 0)$ and $(c, 0)$ where $c = \sqrt{a^2 - b^2}$. Its definition tells us that the latus rectum L is the distance between the two points on the ellipse that have x-coordinate equal to c. To find these points, set $x = c$ in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and solve for y. Doing this, we get $\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1$ and hence $\frac{y^2}{b^2} = 1 - \frac{c^2}{a^2} = \frac{a^2-c^2}{a^2} = \frac{b^2}{a^2}$. So

$$y^2 = \tfrac{b^4}{a^2} \quad\text{and hence}\quad y = \pm\tfrac{b^2}{a}.$$

Therefore the two points in question are $\left(c, \frac{b^2}{a}\right)$ and $\left(c, -\frac{b^2}{a}\right)$. The distance between them is the difference $\frac{b^2}{a} - (-\frac{b^2}{a}) = \frac{2b^2}{a}$ between their y-coordinates.

6.29. Let $P_1 = (x_1, y_1)$ be any point on the ellipse with $y_1 \neq 0$ (and hence $x_1 \neq \pm a$). So $b^2x_1^2 + a^2y_1^2 = a^2b^2$. Let $Q_1 = (x_1 + \Delta x, y_1 + \Delta y)$ be any point on the ellipse different from P_1. Since the coordinates of Q_1 satisfy the equation $b^2x^2 + a^2y^2 = a^2b^2$, we get

$$\begin{aligned} a^2b^2 &= b^2(x_1 + \Delta x)^2 + a^2(y_1 + \Delta y)^2 \\ &= b^2\big(x_1^2 + 2x_1(\Delta x) + (\Delta x)^2\big) + a^2\big(y_1^2 + 2y_1(\Delta y) + (\Delta y)^2\big) \\ &= b^2x_1^2 + 2b^2x_1(\Delta x) + b^2(\Delta x)^2 + a^2y_1^2 + 2a^2y_1(\Delta y) + a^2(\Delta y)^2 \\ &= a^2b^2 + 2b^2x_1(\Delta x) + b^2(\Delta x)^2 + 2a^2y_1(\Delta y) + a^2(\Delta y)^2. \end{aligned}$$

It follows that $(\Delta x)(2b^2x_1 + b^2(\Delta x)) + (\Delta y)(2a^2y_1 + a^2(\Delta y)) = 0$ and hence that $(\Delta y)(2a^2y_1 + a^2(\Delta y)) = -(\Delta x)(2b^2x_1 + b^2(\Delta x))$. So

$$\frac{\Delta y}{\Delta x} = -\frac{2b^2x_1+b^2(\Delta x)}{2a^2y_1+a^2(\Delta y)}.$$

Since Δy goes to zero when Δx does it follows that $\lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = -\frac{2b^2x_1}{2a^2y_1} = -\frac{b^2x_1}{a^2y_1}$. This is the slope of the tangent line to the ellipse at the point $P_1 = (x_1, y_1)$.

6.30. The equation of the line that P_1 and O determine is has slope $\frac{y_1-0}{x_1-0} = \frac{y_1}{x_1}$ and y-intercept 0. So the equation of this line is $y = \frac{y_1}{x_1}x$. Since the coordinates of the point $(-x_1, -y_1)$ satisfies the equation of both the line and the ellipse, it follows that $P_2 = (-x_1, -y_1)$. If x_1 is equal to a or $-a$, then this is also true for $-x_1$. So P_1 and P_2 are the points $(-a, 0)$ and $(a, 0)$. In this case the tangent lines at P_1 and P_2 are both vertical and therefore parallel to each other. So suppose that $x_1 \neq \pm a$. This means that the conclusion of Problem 6.29 applies to both P_1 and P_2. It follows that the slope of the tangent line to the ellipse at P_1 is $-\frac{b^2x_1}{a^2y_1}$ and that the slope of the tangent line to the ellipse at P_2 is $-\frac{b^2(-x_1)}{a^2(-y_1)} = -\frac{b^2x_1}{a^2y_1}$. Since they have the same slope, the two tangent lines are parallel. [The conclusion of this problem has historical significance. Newton's proof of the fact that $\lim\limits_{Q\to P} \frac{QR}{QT^2} = \frac{1}{L}$ in the elliptical case makes use it.]

With regard to the study "Newton's Test Case: The Orbit of the Moon" all the details are included in the text.

6.31. Newton's theory predicts that Kepler's third law holds for the satellites of Jupiter. So we need to check whether $\frac{a^3}{T^2}$, where a is the semimajor axis and T the period of the elliptical orbit, is the same for the four satellites of Jupiter that Newton knew about. Let's check:

$$\text{Satellite 1: } \frac{5.578^3}{42.48^2} \approx \frac{173.55}{1804.55} \approx 0.096$$
$$\text{Satellite 2: } \frac{8.876^3}{85.30^2} \approx \frac{699.28}{7974.49} \approx 0.096$$
$$\text{Satellite 3: } \frac{14.159^3}{171.99^2} \approx \frac{2838.56}{29580.56} \approx 0.096$$
$$\text{Satellite 4: } \frac{24.903^3}{402.09^2} \approx \frac{15443.83}{161676.37} \approx 0.096$$

So the prediction is confirmed.

6.32. As in the previous problem, the concern is the verification of Kepler's third law, namely the equality $\frac{a^3}{T^2}$ for the five satellites of Saturn that were known in Newton's time. As noted in the Errata for this chapter, the first distance listed should be $1\frac{19}{20}$ and not $1\frac{19}{2}$. Since

$$\text{Satellite 1: } \frac{1.95^3}{45.31^2} \approx \frac{3.375}{2053.00} \approx 0.00361$$
$$\text{Satellite 2: } \frac{2.5^3}{65.69^2} \approx \frac{15.625}{4315.18} \approx 0.00362$$
$$\text{Satellite 3: } \frac{3.5^3}{108.42^2} \approx \frac{2838.56}{29584} \approx 0.00365$$
$$\text{Satellite 4: } \frac{8^3}{372.69^2} \approx \frac{15443.83}{161676.37} \approx 0.00369$$
$$\text{Satellite 5: } \frac{24^3}{1903.8^2} \approx \frac{15443.83}{161676.37} \approx 0.00381$$

this is approximately so. The variation in the numbers is explained by the fact that the data for Saturn's satellites available at the time (especially the the distances involved) lacked sufficient accuracy.

In his analysis of the satellites of Jupiter and Saturn, Newton assumes that the semimajor axes of their orbits are equal (at least approximately) to their maximal distances from Jupiter or Saturn. The potential problem with this assumption is that if the astronomical eccentricity of the orbit is large, then the difference between the semimajor axis and the maximal distance is large. Why? However we know today, see

https://nssdc.gsfc.nasa.gov/planetary/factsheet/joviansatfact.html
https://nssdc.gsfc.nasa.gov/planetary/factsheet/saturniansatfact.html

that the orbits of the satellites of Jupiter and Saturn referred to above are all close to being circles. The eccentricities of Jupiter range from 0.002 to 0.009. (Their average distances from Jupiter range from 670,000 to 1,890,000 kilometers.) The eccentricities of the five moons of Saturn lie between 0 and 0.029 and their distances from Saturn range from 295,000 km to 1,222,000 km. One last question. Given that he had determined G, could Cavendish have deduced estimates for the masses of Jupiter and Saturn from Newton's data?

We turn next to the speculation of Newton about the possibility of determining the magnitude of the gravitational force and in particular the gravitational constant G. So that Newton's two spheres are "in spaces void of resistance" let's assume that they float, side by side, somewhere in outer space, isolated from all other gravitational forces.

6.33. With $r = 6.371 \times 10^6$ meters or 6.371×10^8 cm, we get the estimate

$$\tfrac{4}{3}\pi r^3 \approx \tfrac{4}{3}\pi\, 6.371^3 \times 10^{24} \text{ cm}^3 \approx 1.083 \times 10^{27} \text{ cm}^3$$

for the volume of the Earth. Since 1 kg = 1000 gm, the Earth's average density in CGS is $\frac{6.00\times10^{27}}{1.083\times10^{27}} \approx 5.54$ gm/cm^3. The value 5.514 gm/cm^3 is more accurate.

6.34. Since 1 foot is equal to 30.48 cm, each of Newton's spheres has a radius of $r = 15.24$ cm and hence a volume of $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 15.24^3 \approx 14.83 \times 10^3$ cm^3. Multiplying this by the density of 5.514 gm/cm^3, we get that each of the spheres "of like nature to the Earth" has a mass of about 81.77×10^4 gm.

6.35. Since $G = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$ in MKS, it follows that

$$G = 6.67 \times 10^{-11} \, \tfrac{100^3 \text{ cm}^3}{1000 \text{ gm}\cdot\text{s}^2} = 6.67 \times 10^{-8} \, \tfrac{\text{cm}^3}{\text{gm}\cdot\text{s}^2}$$

in CGS. Note that the force between the spheres is greatest when the spheres are closest, in other words touch each other, and weakest at the start of the motion. When the spheres touch, the centers of the spheres are a distance of 30.48 cm apart. Therefore by Newton's law of universal gravitation, the maximal gravitational force between the two spheres is

$$G\,\frac{(81.77)(81.77)\times 10^8}{(30.48)^2} \approx 48.00 \text{ dynes.}$$

Since 1 inch is equal to 2.54 cm, $\frac{1}{4}$ inch is equal to 0.635 cm. So when the spheres are 0.635 cm apart, their centers are $30.48 + 0.635 \approx 31.12$ cm apart. In this position the gravitational force between the spheres is at its minimum of

$$G\frac{(81.77)(81.77)\times 10^8}{(31.12)^2} \approx 46.05 \text{ dynes.}$$

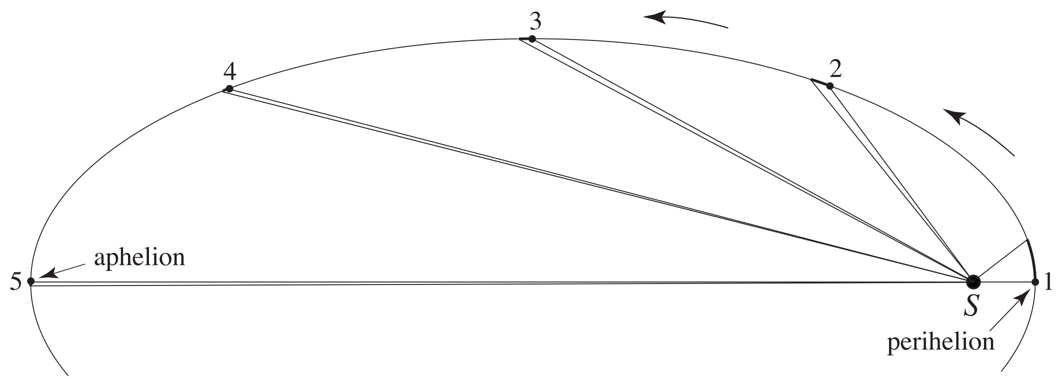
6.36. Since 1 inch is equal to 2.54 cm, $\frac{1}{4}$ inch is equal to 0.635 cm. So the a in Figure 6.47 is $a = 0.3175$ cm. Let the minimal gravitational force of $F = 46.05$ dynes act on the sphere on the right. How long will it take for this force to move this sphere from rest at $a = 0.3175$ cm to the origin? Since the same force acts on the sphere on the left, the sphere on the left will reach the origin at the same time. So this is the time it takes for the minimum gravitational force to move the spheres from rest until they touch. As was observed in the solution of the previous problem, only when the motion begins is the gravitational force equal to the minimal force. Later, when the spheres touch the force equal to its maximum. It follows that the time that we will compute is somewhat more than the actual time.

Since the mass of the sphere is $m = 81.77 \times 10^4$ grams, the acceleration of the sphere is $\frac{F}{m} = \frac{46.05}{81.77\times 10^4} \approx 56.32 \times 10^{-6}$ cm/sec^2. Let the force begin to act at time $t = 0$ and let $t > 0$ be the elapsed time thereafter. So $v(t) \approx 56.32t \times 10^{-6}$ cm/sec and the distance the sphere is moved by the force is $x(t) \approx 28.16t^2 \times 10^{-6}$ cm. Setting $28.16t^2 \times 10^{-6} = 0.3175$ or $28.16t^2 \times 10^{-4} = 31.75$, we get $t^2 \approx \frac{31.75}{28.16} \times 10^4$ sec^2, so that $t \approx 1.06 \times 10^2 = 106$ seconds. The same computation with the maximal force in place of the minimal force results in a time of about one second less. In conclusion, the time it takes for gravity to move the two spheres until they touch is about $1\frac{3}{4}$ minutes.

So Newton got it wrong when he speculated that the spheres would not come together "in less than a month's time." (Why couldn't Newton simply have carried out the above computation?)

The study of the Earth-Moon-Sun system and in particular Airy's computation of the mass of the Moon that the text engages next is complete in its details. So we turn next to a discussion of the speeds of the bodies in the solar system.

Consider a planet (comet, or asteroid) P in its elliptical orbit with the Sun S at a focal point of the ellipse. Let a be the semimajor axis, b the semiminor axis, $c = \sqrt{a^2 - b^2}$, and $\varepsilon = \frac{c}{a}$ the eccentricity of the orbit. Let T be the period of the orbit and κ its Kepler constant.
The figure below shows P in five different locations of its orbit. (The ellipse is drawn much flatter than that of any planetary orbit in order to add transparency to our discussion.) The five locations are labeled from 1 to 5 in the figure. The numbers 1 and 5 denote the perihelion and aphelion positions respectively. Let Δt be a short fixed interval of time and consider the



five short arcs starting from the five points that P traces out during this time. The five arcs and the thin wedges that the segment SP sweeps out in the process are drawn in as well. Since they are all swept out in the same time Δt, these wedges have the same area by Kepler's second law. Since the wedges get longer as P proceeds from perihelion to aphelion, the arcs get shorter and shorter. Since they are all traced out over the same time, this means that the average speed of P over the arcs decreases from one arc to the next. Pushing Δt to zero shortens the five arcs and pushes the average speed to the speed at the initial point of each arc. Since the distance from P to S is shortest at perihelion, P achieves its greatest speed v_{max} at perihelion. Similarly, since the distance from P to S is longest at aphelion, P attains its least speed v_{min} at aphelion.

The next two problems illustrate how Newton would have determined the maximum and minimum speeds of the planet or in fact of any body in the solar system. His Figure 6.30 is the key.

6.37. Figure 6.52 shows the perihelion position of the planet P, a short stretch of the orbit, and the position Q of the planet a short time Δt later. The tangent to the orbit at P

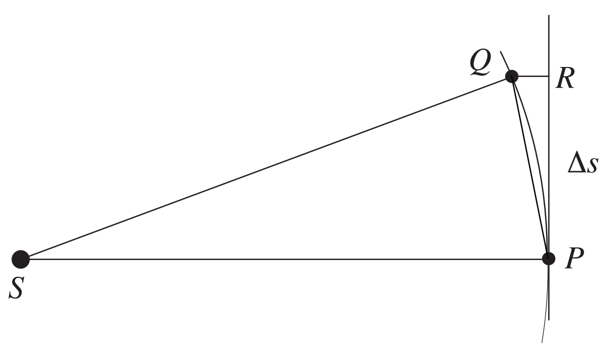


Figure 6.52

is drawn in and R is chosen so that RQ is parallel to SP. The segment PR has length Δs.

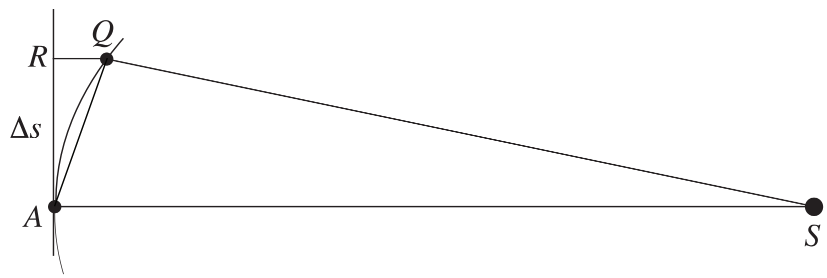
i. The average speed v_{av} of the planet during its motion from P to Q is equal to $\frac{\text{arc } PQ}{\Delta t}$. Because arc $PQ \approx \Delta s$, $v_{\text{av}} \approx \frac{\Delta s}{\Delta t}$.

ii. Since the area of the wedge SPQ is approximately equal to that of the triangle ΔSPQ, we see that $\kappa = \frac{\text{area wedge } SPQ}{\Delta t} \approx \frac{\frac{1}{2}\text{SP}\cdot\Delta s}{\Delta t}$. Since $SP = a - c$ and $\kappa = \frac{ab\pi}{T}$, we get the approximation $\frac{1}{2}(a-c)\frac{\Delta s}{\Delta t} \approx \frac{ab\pi}{T}$. Therefore, $v_{\text{av}} \approx \frac{\Delta s}{\Delta t} \approx \frac{2ab\pi}{(a-c)T}$.

iii. When Δt is pushed to zero, the average speed $\frac{\Delta s}{\Delta t} = v_{\text{av}}$ becomes the speed $v_{\max}$ at perihelion. Secondly, the approximation of the area of the wedge SPQ by the area $\frac{1}{2}(a-c)\cdot\Delta s$ of ΔSPQ becomes tighter and tighter. Therefore, the approximations $\kappa = \frac{\text{area wedge } SPQ}{\Delta t} \approx \frac{\frac{1}{2}(a-c)\cdot\Delta s}{\Delta t} \approx \frac{1}{2}(a-c)v_{\max}$ snap to equalities. Since $\kappa = \frac{ab\pi}{T}$ this leads to the conclusion $v_{\max} = \frac{2ab\pi}{(a-c)T}$.

iv. Since $b = \sqrt{a^2 - c^2}$ and $c = \varepsilon a$, we get $a - c = a(1-\varepsilon)$ and $b = a\sqrt{1-\varepsilon^2}$. So $v_{\max} = \frac{2a^2\sqrt{1-\varepsilon^2}}{a(1-\varepsilon)}\frac{\pi}{T} = \frac{2a(\sqrt{1-\varepsilon})(\sqrt{1+\varepsilon})}{(\sqrt{1-\varepsilon})^2}\frac{\pi}{T} = \frac{2a\pi}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$.

6.38. We have already established that a planet (or comet or asteroid) attains its minimum speed $v_{\min}$ at aphelion. It is a routine matter to modify the solution to Problem 6.37



(by using the figure above and $SA = a + c$ in place of $SP = a - c$) to show that $v_{\min} = \frac{2ab\pi}{(a+c)T} = \frac{2\pi a}{T}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$.

Several of the computations that follow use the fact that 1 au ≈ 149,598,000 km and that

$$1\,\frac{\text{au}}{\text{year}} = \frac{149{,}597{,}892 \text{ km}}{1 \text{ year}} \times \frac{1 \text{ year}}{31{,}558{,}000 \text{ s}} \approx 4.74\,\frac{\text{km}}{\text{s}}.$$

6.39. Simply plug the data of Table 3.3 and the relationship between au/year and km/s into the formulas that Problem 6.37 and 6.38 developed.

6.40. The orbit of the comet Halley has semimajor axis $a = 17.83$ au, eccentricity $\varepsilon = 0.967$, and period T = 75.32 years. Putting these data into the formula $v_{\max} = \frac{2a\pi}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$, we get $v_{\max} = \frac{35.66\pi}{75.32}\sqrt{\frac{1+0.967}{1-0.967}} \approx 11.48$ au/year or $11.48 \cdot 4.74 \approx 54.42$ km/sec. Doing this with $v_{\min} = \frac{2\pi a}{T}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$, provides $v_{\min} \approx \frac{35.66\pi}{75.32}\sqrt{\frac{1-0.967}{1.967}} \approx 0.19$ au/year or 0.91 km/sec.

The next set of problems establishes the fact that there is a "maximum speed limit" on the speed of all the planets, comets, or asteroids that are in elliptical orbit around the Sun.

6.41. The equality $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$ is derived in Section 6.10. Since no orbital parameters appear on the right side of the equation (M is the mass of the Sun), $\frac{a^3}{T^2}$ is the same constant for the planets and any astronomical body in elliptical orbit around the Sun. Since $a = 1$ au and $T = 1$ year for Earth, it follows that in the units au and years, $\frac{a^3}{T^2} = 1$ for Earth and hence for any astronomical body in elliptical orbit around the Sun.

6.42. Consider any object in an elliptical orbit around the Sun. Take distances in the solar system in au and time in years. Let a be the semimajor axis, ε the eccentricity, and T the period of the orbit of the object. By Problem 6.41, $a^3 = T^2$. Hence $T = a^{\frac{3}{2}}$. Since $\varepsilon < 1$, the conclusion of Problem 6.37 implies that the maximum speed $v_{\max}$ of the object in au/year satisfies

$$v_{\max} = \frac{2a\pi}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} = \frac{2\pi}{a^{\frac{1}{2}}}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} < \frac{2\pi \cdot \sqrt{2}}{\sqrt{a(1-\varepsilon)}}.$$

Notice that $a(1-\varepsilon) = a - a\varepsilon = a - c$ is the perihelion distance of the orbiting object in au.

6.43. Inserting the smallest perihelion distance of $a(1-\varepsilon) = 0.005$ au into the term $\frac{2\pi\cdot\sqrt{2}}{\sqrt{a(1-\varepsilon)}}$ derived in Problem 6.42, provides the approximate value 125.66 au/year for the maximum speed of a comet in elliptical orbit around the Sun. Since 1 au/year is equal to 4.74 km/s, this maximum speed corresponds to about 598 km/s.

We'll turn to provide estimates of the maximal speeds of some of the great sungrazing comets.

6.44. The Great Comet of 1680 that Newton tracked attained a perihelion distance of close to 0.006 au. Substituting $a(1-\varepsilon) = 0.006$ au into $\frac{2\pi\cdot\sqrt{2}}{\sqrt{a(1-\varepsilon)}}$ tells us that the Great Comet of 1680 attained a maximum speed of about 114.71 au/year or 544 km/s.

6.45. For the Great Comet of 1843, take $a(1-\varepsilon) = 0.00546$ to be the perihelion distance. So its speed at perihelion was $\frac{2\pi\cdot\sqrt{2}}{\sqrt{a(1-\varepsilon)}} = \frac{2\pi\cdot\sqrt{2}}{\sqrt{0.00546}} \approx 120.25$ au/year or 570 km/sec. For the Great Comet of 1882, $\frac{2\pi\cdot\sqrt{2}}{\sqrt{a(1-\varepsilon)}} = \frac{2\pi\cdot\sqrt{2}}{\sqrt{0.0076}} \approx 101.93$ au/year or 483 km/sec. Finally for the Great Comet of 1965, $\frac{2\pi\cdot\sqrt{2}}{\sqrt{a(1-\varepsilon)}} = \frac{2\pi\cdot\sqrt{2}}{\sqrt{0.00778}} \approx 100.74$ au/year or 478 km/s.

The next group of problems focuses on artificial satellites of Earth. Recall that the Earth is a sphere that is slightly flattened at the poles, so that the distances from the Earth's center to its surface vary slightly. The radius at the equator is 6378 km and that at the poles is 6357 km. The average radius is 6371 km.

6.46. Let a and ε be the semimajor axis and eccentricity of Sputnik's elliptical orbit. The addition of the distance 942 km to Earth's radius of 6371 km gives an aphelion distance of $a + a\varepsilon = 7313$ km for the orbit and adding 230 km to 6371 km gives a perihelion distance of $a - a\varepsilon = 6601$ km. So $2a = (a+\varepsilon a)+(a-\varepsilon a) = 7313+6601 = 13{,}914$ km and hence $a = 6957$ km. Since $2\varepsilon a = (a+\varepsilon a)-(a-\varepsilon a) = 7313-6601 = 712$ km, $a\varepsilon = 356$ and hence $\varepsilon = \frac{356}{6957} \approx 0.051$. Given Sputnik's orbital period of $96{\cdot}60 = 5760$ seconds, we get that Earth's mass in MKS is $M = \frac{4\pi^2 a^3}{GT^2} = \frac{4\pi^2 \cdot 6{,}957{,}000^3}{6.67\times 10^{-11} 5760^2} \approx 6.00 \times 10^{24}$ kg.

6.47. For Explorer 1, the aphelion and perihelion distances of the orbit were $a + a\varepsilon = 6371+2534 = 8905$ km and $a - a\varepsilon = 6371+360 = 6731$ km, respectively. So $2a = 15{,}636$ km and $2a\varepsilon = 8905-6731 = 2174$ km. It follows that $a = 7818$ km and $a\varepsilon = 1087$ km. So $\varepsilon = \frac{1087}{7818} \approx 0.14$. Since the period of Explorer's orbit was $T = (114.9)(60) = 6894$ seconds, we get the estimate $M = \frac{4\pi^2 a^3}{GT^2} = \frac{4\pi^2 \cdot 7{,}818{,}000^3}{6.67\times 10^{-11} \cdot 6894^2} \approx 5.95 \times 10^{24}$ kg for Earth's mass.

We'll turn to the Moon and its orbit around Earth next.

6.48. Since Earth revolves once around its axis every 24 hours and the Moon takes 27.32 day to complete its orbit around Earth, the Earth's rotation has the greater effect on the Moon's observed change of position.

6.49. We'll use Newton's formula $F = G\frac{mM}{r^2}$ of universal gravitation. With the mass of the Sun equal to 2.0×10^{30} kg and the average distance from the Moon to the Sun the same 1.5×10^{8} km $= 1.5\times 10^{11}$ m as that from Earth to the Sun, we get that the gravitational force of the Sun on the Moon is $F = 6.67 \times 10^{-11}\frac{(7.4\times 10^{22})(2.0\times 10^{30})}{(1.5\times 10^{11})^2} \approx 4.39 \times 10^{20}$ N. Taking 6.0×10^{24} kg for Earth's mass, $F = 6.67\times 10^{-11}\frac{(7.4\times 10^{22})(6.0\times 10^{24})}{(3.8\times 10^{8})^2} \approx 2.05\times 10^{20}$ N is the gravitational force of Earth on the Moon. So the gravitational force of the Sun on the Moon is about twice as great as the gravitational force of Earth on the Moon. The best way to understand the Earth-Moon-Sun dynamic is to regard both Earth and Moon to be in orbit around the Sun. As this occurs the Earth's gravitational force pulls the Moon into orbit around it.

6.50. Putting $a = 3.8 \times 10^8, T = (27.32)(24)(60^2)$ seconds, and $\varepsilon = 0.0549$ into the formulas $v_{\max} = \frac{2\pi a}{T}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$ and $v_{\min} = \frac{2\pi a}{T}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$, we get

$$v_{\max} \approx \frac{2\pi(3.8\times 10^8)}{(27.32)(24)(3600)}\sqrt{\frac{1.0549}{0.9451}} \approx 1069 \text{ and } v_{\min} \approx \frac{2\pi(3.8\times 10^8)}{(27.32)(24)(3600)}\sqrt{\frac{0.9451}{1.0549}} \approx 957$$

both in meters/sec. This is consistent with the fact that the average orbital speed of the Moon is 1.023 km/s.

6.51. The formula that provides the answer is $M = \frac{4\pi^2 a^3}{GT^2}$ where a is the semimajor axis of Luna 10's orbit and T is the period. So from all the information supplied, only $a = 2{,}413{,}000$ m and $T = (178.05)(60) = 10{,}683$ seconds (we're using MKS) is relevant.

Inserting this, we get $M = \frac{4\pi^2 a^3}{GT^2} \approx \frac{4\pi^2(2{,}413{,}000)^3}{(6.67\times10^{-11})(10{,}683^2)} \approx 7.29 \times 10^{22}$ kg for the mass of the Moon.

6.52. If g_M is the gravitational acceleration of a mass m falling near the Moon's surface, then the force of gravity on m is $F = mg_M$. On the other hand, by the law of universal gravitation, $F = G\frac{mM}{r_M^2}$ where M is the mass of the Moon and r_M its radius. It follows that $g_M = G\frac{M}{r_M^2} \approx (6.67 \times 10^{-11})\frac{7.4\times10^{22}}{(1{,}740{,}000)^2} \approx 1.63$ m/s^2 in MKS.

The next two examples determine the masses of the asteroids Eros and Eugenia.

6.53. The formula $M = \frac{4\pi^2 a^3}{GT^2}$ is the key. The semimajor axis a of NEAR's orbit is its radius $r = 99.8$ km. The circumference of the orbit $2\pi r = 2\pi(99.8) \approx 627$ km. Since NEARs speed is 4.8 km/hour, its period is $T \approx \frac{627}{4.8} \approx 131$ hours. Inserting the data in MKS into the formula, tells us that the mass of Eros is $M = \frac{4\pi^2 a^3}{GT^2} \approx \frac{4\pi^2(99{,}800^3)}{6.67\times10^{-11}(131\cdot3600)^2} \approx 2.65 \times 10^{15}$ kg.

6.54. Again, $M = \frac{4\pi^2 a^3}{GT^2}$ is the formula to be applied. We'll take Eugenia's moon to have a circular orbit of radius 1130 km and a period of 4.7 days. Converting to MKS, we get that the mass of Eugenia is $M = \frac{4\pi^2 a^3}{GT^2} \approx \frac{4\pi^2(1{,}130{,}000^3)}{6.67\times10^{-11}(4.7\cdot24\cdot3600)^2} \approx 5.18 \times 10^{18}$ kg.

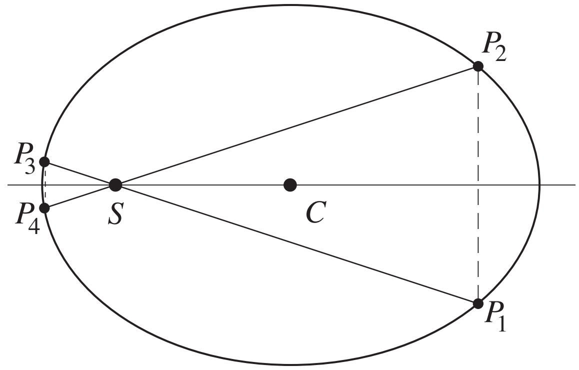
We'll close our discussion about artificial Earth satellites with a look at some of the important information that they have provided.

6.55. Nothing to do here except to go to

http://www.nasa.gov/hubble/ and http://antwrp.gsfc.nasa.gov/apod/

(in the latter case to the Archive) and explore.

6.56. Consider the modified version of Figure 6.54 below. Start with the object in position P_1 and wait for it to move to the position P_2 opposite P_1 with respect to the focal axis. Suppose that the object takes the time t to get from P_1 to P_2. The object continues to positions P_3 and then P_4 as these are determined by continuing the segments from P_1 to S and P_2 and S, respectively, to the ellipse. Since $\angle P_1SP_2 = \angle P_3SP_4$ and the object traces out equal angles in equal times, it follows that the time it takes for the object to



move from P_3 to P_4 is also equal to t. As the object also sweeps out equal areas in equal times, it follows that the elliptical sectors P_1SP_2 and P_3SP_4 have equal areas. But a look at the figure shows that this is possible only if the focal point S of the ellipse coincides with its center C. In terms of the discussion in Section 4.4, this means that $c = 0$ and hence that the eccentricity $\varepsilon = \frac{c}{a}$ of the ellipse is 0. So the ellipse is a circle.

6.57. The Earth's center of mass C is also the focal point of the elliptical orbit of the satellite. If P designates the position of the satellite and Q the point on the equator over which it hovers, then the segment CQP traces out equal angles in equal times because CQ does so. By Kepler's second law, the segment CQP also traces out equal areas in equal times. So by the conclusion of Problem 6.56, the orbit of the satellite must be a circle. When applying the formula $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$ to the orbit of the satellite, a is the radius r of the orbit, $T = 24$ hours, and $M = 6 \times 10^{24}$ kg is the Earth's mass. It follows that $\frac{r^3}{(24 \cdot 3600)^2} = \frac{(6.67\times10^{-11})(6\times10^{24})}{4\pi^2}$ in MKS and that $r^3 \approx 75674 \times 10^{18}$. Hence $r \approx 42.3 \times 10^6$ meters or 42,300 km.

6.58. The study of the GPS satellite system is self contained except for the answers to three questions: "Why does this mean that the radii of their circular orbits have to be the same? What is this common radius equal to? What is the speed of the satellites?" The only fact needed for the answers is that the satellites are in circular orbits with a period of 12 hours. The relevant formula is $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$, where (in the current situation) $a = r$ is the radius of the circular orbit of any of the satellites, $T = 12$ hours is the common period, and $M = 6 \times 10^{24}$ kg is the mass of the Earth. Since the right side of this equation and the period T are the same for all satellites, it follows that all satellites have the same r. Converting the data to MKS we get $\frac{r^3}{(12 \cdot 3600)^2} = \frac{(6.67\times10^{-11})(6\times10^{24})}{4\pi^2}$, so that $r^3 \approx 18{,}918 \times 10^{18}$ and $r \approx 26.65 \times 10^6$ m or 26,650 km. Since the orbit is a circle, its eccentricity is $\varepsilon = 0$, so that by the conclusions of Problems 6.37 and 6.38, $v_{\max} = v_{\min} = \frac{2\pi r}{T} \approx \frac{2\pi \cdot 26650}{12} \approx 13{,}950$ km/hour.

Einstein's theory of general relativity correctly predicts that a clock that is farther from a massive body (or at lower gravitational force) runs more quickly, and a clock close to a massive body (or at greater gravitational force) runs more slowly. The reason for this is that space and time are interconnected and that gravity pulls on the composite four dimensional structure of space-time. When gravity warps space-time it also warps time. This has an impact on the GPS system as all the clocks on the satellites run 45 microseconds per day faster than a clock on Earth. This difference of 0.000045 seconds per day adds up over time. A highly accurate estimate of a location on Earth depends (as we have seen) on highly accurate estimates of the distances of the location to several of the moving satellites at synchronized times. This requires that the relativistic differences in the readings of the clocks need to be taken into account.