Solutions to Problems and Projects for Chapter 5

5.1. The slope is $\frac{-3-2}{2-(-6)} = \frac{-5}{8}$. An equation for the line (in point-slope form) is $y + 3 = -\frac{5}{8}(x - 2)$.

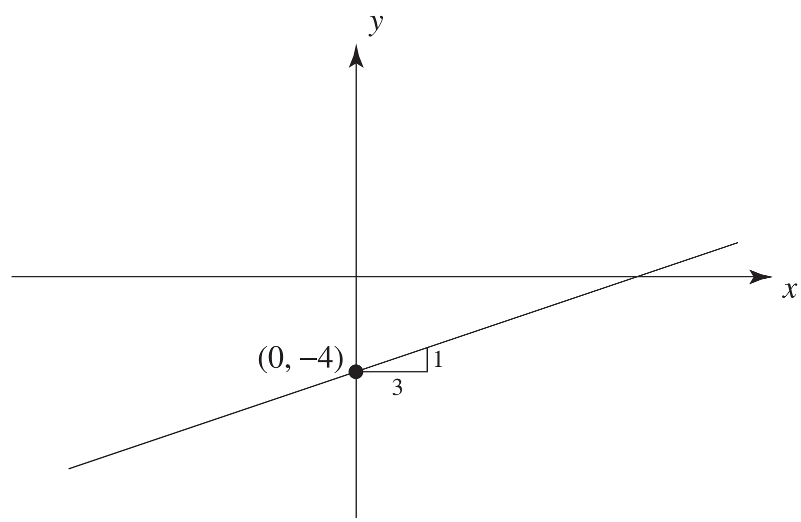
5.2. The slope of this line is $m = \frac{7-2}{-2-5} = -\frac{5}{7}$. Therefore an equation (in point slope form) for the line is $y - 7 = -\frac{5}{7}(x - (-2))$ or $y - 7 = -\frac{5}{7}(x + 2)$.

5.3. The slope-intercept form of the equation is $y = -3x + 4$.

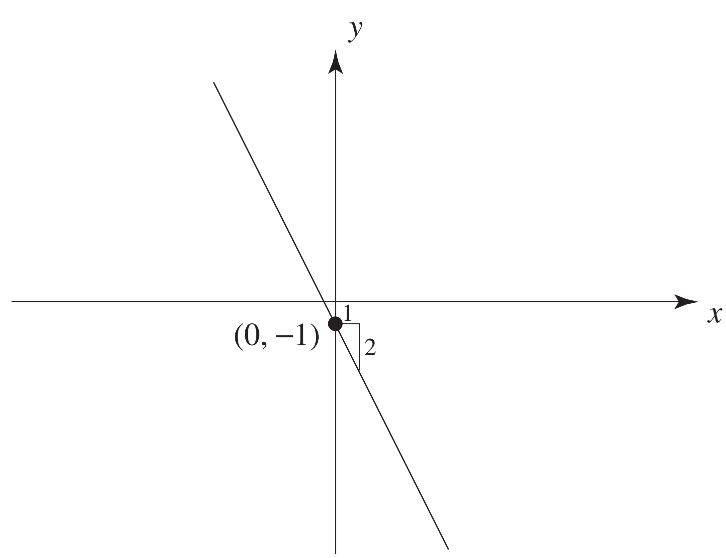
5.4. An equation of the line (in point-slope form) is $y-(-2) = \frac{1}{2}(x-3)$ or $y+2 = \frac{1}{2}(x-3)$).

5.5. The equation $y + 7 = -3(x + 6)$ is an equation of the line in point-slope form.

5.6. The figure below includes the y-intercept of the line and determines its slope.



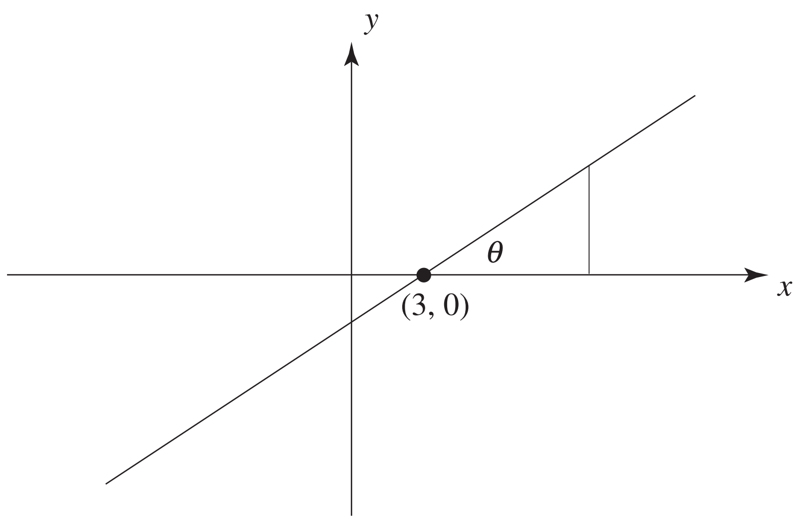
5.7. The figure below includes the y-intercept of the line and determines its slope.



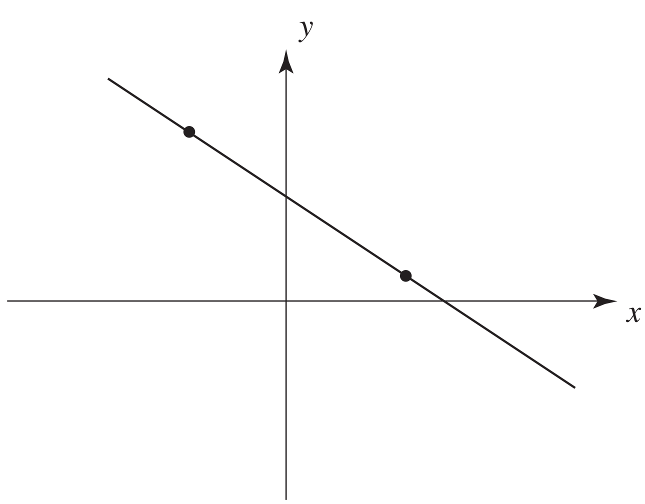
5.8. We need to rewrite the equation in the form $y = mx + b$. Solving for y, we get $7y = -2x - 2$ and $y = -\frac{2}{7}x - \frac{2}{7}$. It follows that the slope and the y-intercepts are both $-\frac{2}{7}$.

5.9. We need to write the equation in the form $y = mx + b$. Since $5y = -3x - 2$ and hence $y = -\frac{3}{5}x - \frac{2}{5}$, it follows that the slope of the line is $-\frac{3}{5}$ and its y-intercept is $\frac{2}{5}$.

5.10. Let a line through the point $(3, 0)$ make an angle θ with the positive x-axis and assume that $0 \leq \theta < \frac{\pi}{2}$. A look at the figure tells us that the slope m of the line is $m = \tan\theta$. So $y = (\tan\theta)(x - 3)$ is an equation of the line. For $\theta = 45°, 30°$, and $25°$, the slopes m are $m = 1, m = \frac{1}{\sqrt{3}} \approx 0.577$, and $m \approx 0.466$, respectively.

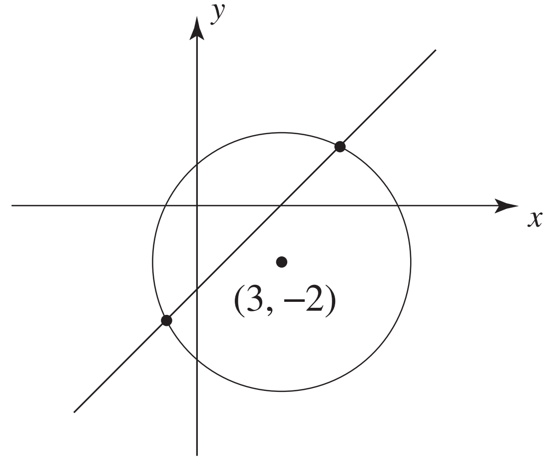


5.11. The slope of the line is $\frac{7-1}{-4-5} = -\frac{6}{9} = -\frac{2}{3}$. The equation $y - 1 = \frac{(7-1)}{(-4-5)}(x - 5)$ is in two-point form, $y - 1 = -\frac{2}{3}(x - 5)$ is in point-slope form, and $y = -\frac{2}{3}x + \frac{10}{3} + 1 = -\frac{2}{3}x + \frac{13}{3}$, so that $y = -\frac{2}{3}x + \frac{13}{3}$ is in point-intercept form. The line is sketched below.



5.12. The two triangular figures are not triangles. The upper boundary of the large dark triangle in each figure has slope $\frac{3}{8} = 0.375$ and the upper boundary of the smaller triangle in each figure has slope $\frac{2}{5} = 0.4$. It follows that the figure at the bottom initially rises faster from left to right than the figure at the top. This advantage in rise gains the figure at the bottom the additional square.

5.13. The circle—it has radius $\sqrt{20} \approx 4.47$—and the line are both sketched below. By substituting $x - 3 = y$ into $(x-3)^2 + (y+2)^2 = 20$, we get $y^2 + y^2 + 4y + 4 = 20$. So $2y^2 + 4y - 16 = 0$, and therefore $y^2 + 2y - 8 = 0$. By recognizing that this polynomial factors as $(y-2)(y+4)$ or by applying the quadratic formula, $y = \frac{-2 \pm \sqrt{4-(4)(1)(-8)}}{2} = \frac{-2 \pm \sqrt{36}}{2}$, we get $y = 2$ or $y = -4$. So the points of intersection are $(-1,-4)$ and $(5,2)$.



5.14. An equation of the circle is $(x-2)^2 + (y-3)^2 = 25$ and an equation of the line is $y - 3 = \frac{1}{2}(x-2)$. A substitution tells us that $(x-2)^2 + (\frac{1}{2}(x-2))^2 = 25$. So $\frac{5}{4}(x-2)^2 = 25$ and hence $(x-2)^2 = 20$. Therefore $x - 2 = \pm\sqrt{20} = \pm 2\sqrt{5}$ and $x = 2 \pm 2\sqrt{5}$. For these two values of x, $\frac{1}{2}(x-2) = \pm\sqrt{5}$, so that $y = 3 \pm \sqrt{5}$. It follows that the two points of intersection are $\big(2(1-\sqrt{5}), -\sqrt{5}+3\big)$ and $\big(2(1+\sqrt{5}), \sqrt{5}+3\big)$.

5.15. Figure 5.42 guides the argument. Let's start by assuming that the lines L_1 and L_2 are perpendicular. So $\alpha + \frac{\pi}{2} + \beta' = \pi$ and hence $\alpha + \beta' = \frac{\pi}{2}$. Also $\alpha + \beta + \frac{\pi}{2} = \pi$, so that $\alpha + \beta = \frac{\pi}{2}$. It follows that $\beta = \beta'$. We now see that $m_2 = \frac{y_2}{x_2} = -\frac{y_2}{-x_2} = -\tan\beta' = -\tan\beta = -\frac{x_1}{y_1} = -\frac{1}{\frac{y_1}{x_1}} = -\frac{1}{m_1}$. Suppose conversely that $m_2 = -\frac{1}{m_1}$. Then from above $\tan\beta' = \tan\beta$ and hence $\beta' = \beta$. Since $\alpha + \beta = \frac{\pi}{2}$, it follows that $\alpha + \beta' = \frac{\pi}{2}$. But this means that the angle γ between the line L_1 and L_2 must be $\frac{\pi}{2}$, so that the lines L_1 and L_2 are perpendicular.

5.16. Along with $P = (2,4)$, let $Q = (2 + \Delta x, 4 + \Delta y)$ be a point on the parabola. Since $4 + \Delta y = (2 + \Delta x)^2$. It follows that $4 + \Delta y = 4 + 4\Delta x + (\Delta x)^2$. Therefore $\Delta y = \Delta x\big(4 + \Delta x\big)$. So $\frac{\Delta y}{\Delta x} = 4 + \Delta x$, and hence $m_P = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 4$.

5.17. In addition to $P = (2,8)$ we'll let $Q = (2 + \Delta x, 8 + \Delta y)$ be a point on the curve. So $8 + \Delta y = (2 + \Delta x)^3$. It follows that $8 + \Delta y = 2^3 + 3{\cdot}2^2\Delta x + 3{\cdot}2(\Delta x)^2 + (\Delta x)^3$.

Therefore $\Delta y = \Delta x\big(12 + 6\Delta x + (\Delta x)^2\big)$. So $\frac{\Delta y}{\Delta x} = 12 + 6\Delta x + (\Delta x)^2$, and hence $m_P = \lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = 12$.

5.18. Since both $P = (x, y)$ and $Q = (x + \Delta x, y + \Delta y)$ are on the curve, $y = \frac{1}{x^2}$ and $y + \Delta y = \frac{1}{(x+\Delta x)^2}$. So $\Delta y = \frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}$. After taking common denominators, $\Delta y = \frac{x^2-(x+\Delta x)^2}{(x+\Delta x)^2x^2} = \frac{-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2} = \frac{\Delta x(-2x-\Delta x)}{(x+\Delta x)^2x^2}$. So $\frac{\Delta y}{\Delta x} = \frac{-2x-\Delta x}{(x+\Delta x)^2x^2}$, and therefore $m_P = \lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{-2x}{x^2x^2} = \frac{-2}{x^3}$.

5.19. Along with $P = (x, y)$, let $Q = (x + \Delta x, y + \Delta y)$ be on the graph. Since $(y + \Delta y)^2 = 2(x + \Delta x) + 7$, we see that $y^2 + 2(\Delta y)y + (\Delta y)^2 = 2x + 2\Delta x + 7$, and hence that $2(\Delta y)y + (\Delta y)^2 = 2\Delta x$. So $\Delta y(2y + \Delta y) = 2\Delta x$ and therefore $\frac{\Delta y}{\Delta x} = \frac{2}{2y+\Delta y}$. When Δx is pushed to zero, Δy goes to zero as well (see Figure 5.7), so that $m_P = \lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{1}{y}$. At the point $(1, -3)$ on the curve, the slope of the tangent line is $\frac{1}{-3}$, so that $y + 3 = -\frac{1}{3}(x - 1)$ is an equation of the tangent.

5.20. In addition to $P = (x, y)$, we'll let $Q = (x + \Delta x, y + \Delta y)$ be on the graph of $x = y^3$. So $x + \Delta x = (y + \Delta y)^3 = y^3 + 3y^2\Delta y + 3y(\Delta y)^2 + (\Delta y)^3$. It follows that $\Delta x = 3y^2\Delta y + 3y(\Delta y)^2 + (\Delta y)^3 = (\Delta y)(3y^2 + 3y(\Delta y) + (\Delta y)^2)$. Notice next that

$$\frac{\Delta y}{\Delta x}(3y^2 + 3y(\Delta y) + (\Delta y)^2) = 1.$$

Since Δy goes to zero when Δx is pushed to zero, we get—by taking $\lim\limits_{\Delta x\to 0}$ of both sides—that $m_P(3y^2) = 1$ and hence that $m_P = \frac{1}{3y^2}$. With a slightly different algebraic move, the fact that $\lim\limits_{\Delta x\to 0} \frac{\Delta}{\Delta x}\Delta y = m_P \cdot 0 = 0$ would have been useful.

5.21. Let the point $P = (x, y)$ as well as the point $Q = (x + \Delta x, y + \Delta y)$ be on the ellipse. Since $\frac{(x+\Delta x)^2}{5^2} + \frac{(y+\Delta y)^2}{4^2} = 1$ we get $\frac{x^2+2x\Delta x+(\Delta x)^2}{5^2} + \frac{y^2+2y\Delta y+(\Delta y)^2}{4^2} = 1$. It follows that $\frac{x^2}{5^2} + \frac{y^2}{4^2} + \frac{2x\Delta x+(\Delta x)^2}{5^2} + \frac{2y\Delta y+(\Delta y)^2}{4^2} = 1$ and therefore that $\frac{2x\Delta x+(\Delta x)^2}{5^2} + \frac{2y\Delta y+(\Delta y)^2}{4^2} = 0$. After a little algebra, $\frac{\Delta y(2y+\Delta y)}{4^2} = -\frac{\Delta x(2x+\Delta x)}{5^2}$. Therefore $\frac{\Delta y}{\Delta x}\frac{(2y+\Delta y)}{4^2} = -\frac{(2x+\Delta x)}{5^2}$ and hence $\frac{\Delta y}{\Delta x} = -\frac{4^2(2x+\Delta x)}{5^2(2y+\Delta y)}$. So $m_P = \lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = -\frac{4^2}{5^2}\frac{x}{y}$.

5.22. Letting $Q = (x + \Delta x, y + \Delta y)$ be on the curve we get $(y + \Delta y)^3 = 3(x + \Delta x)^2 + 7$. Therefore $y^3 + 3y^2\Delta y + 3y(\Delta y)^2 + (\Delta y)^3 = 3x^2 + 6x\Delta x + (\Delta x)^2 + 7$. Since $P = (x, y)$ is on the curve, its coordinates satisfy $y^3 = 3x^2 + 7$, so that $3y^2\Delta y + 3y(\Delta y)^2 + (\Delta y)^3 = 6x\Delta x + (\Delta x)^2$. After factoring out the Δy and Δx, we get $\Delta y(3y^2 + 3y\Delta y + (\Delta y)^2) = \Delta x(6x + \Delta x)$. Therefore $\frac{\Delta y}{\Delta x} = \frac{6x+\Delta x}{3y^2+3y\Delta y+(\Delta y)^2}$. By pushing Δx to zero, $\lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{6x}{3y^2} = \frac{2x}{y^2}$. Raising both sides of $y^3 = 3x^2 + 7$ to the $\frac{2}{3}$ power, we get $y^2 = (3x^2 + 7)^{\frac{2}{3}}$. So we are done.

5.23. With $Q = (x_0 + \Delta x, y_0 + \Delta y)$ also on the circle, $(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 = r^2$. It follows that

$$x_0^2 + 2x_0(\Delta x) + (\Delta x)^2 + y_0^2 + 2y_0(\Delta y) + (\Delta y)^2 = r^2.$$

So $2x_0(\Delta x)+(\Delta x)^2+2y_0(\Delta y)+(\Delta y)^2 = 0$ and $2y_0(\Delta y)+(\Delta y)^2 = -(2x_0(\Delta x)+(\Delta x)^2)$. Factoring out a Δy from the left side and a Δx from the right, gives us $\Delta y(2y_0+\Delta y) = -\Delta x(2x_0 + \Delta x)$. Therefore $\frac{\Delta y}{\Delta x} = -\frac{2x_0+\Delta x}{2y_0+\Delta y}$, and hence $m_P = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{2x_0}{2y_0} = -\frac{x_0}{y_0}$.

i. Continue to let (x_0, y_0) be a point on the circle with $y_0 \neq 0$ and consider the radius from $(0, 0)$ to (x_0, y_0). Suppose first that $x_0 = 0$. Then the radius is vertical and the slope $-\frac{x_0}{y_0}$ of the tangent at the point is zero. So the radius and the tangent are perpendicular. Suppose that $x_0 \neq 0$. Then the slope of the radius to from $(0, 0)$ to (x_0, y_0) is $\frac{y_0-0}{x_0-0} = \frac{y_0}{x_0}$. Since the tangent at (x_0, y_0) has slope $-\frac{x_0}{y_0}$, the radius and the tangent are perpendicular by the conclusion of Problem 5.15.

ii. Let (x_0, y_0) be a point of tangency. Since the slope of the tangent at this point is $-\frac{1}{3}$, we know that $-\frac{x_0}{y_0} = -\frac{1}{3}$. So $y_0 = 3x_0$. Since (x_0, y_0) is also on the circle, it follows that $x_0^2 + (3x_0)^2 = x_0^2 + y_0^2 = 1$. So $10x_0^2 = 1$ and hence $x_0 = \pm\frac{1}{\sqrt{10}}$ and $(x_0, y_0) = \big(\pm\frac{1}{\sqrt{10}}, \pm\frac{3}{\sqrt{10}}\big)$. Turn to the line $y = -\frac{1}{3}x + b$. Taking $\big(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\big)$ as the point of tangency, we get $\frac{3}{\sqrt{10}} = -\frac{1}{3}\frac{1}{\sqrt{10}} + b$. So

$$b = \tfrac{3}{\sqrt{10}} + \tfrac{1}{3}\tfrac{1}{\sqrt{10}} = \tfrac{1}{\sqrt{10}}\big(3 + \tfrac{1}{3}\big) = \tfrac{1}{\sqrt{10}}\big(\tfrac{10}{3}\big) = \tfrac{\sqrt{10}}{3}.$$

Using $\big(-\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\big)$ as the point of tangency, we get $-\frac{3}{\sqrt{10}} = \frac{1}{3}\frac{1}{\sqrt{10}} + b$. So $b = -\frac{3}{\sqrt{10}} - \frac{1}{3}\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{3}$.

5.24. Solving $y^2 = x$ for y, we get $y = \pm\sqrt{x}$ as the two possibilities for y. Since $y = f_+(x) = \sqrt{x}$ is positive, the graph of this function is the upper part of the parabola of Figure 5.13a. Similarly, the graph of $y = f_+(x) = \sqrt{x}$ is the lower part of the parabola.

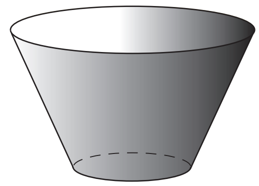
If the squares of two numbers are equal, the numbers are either equal or one is the negative of the other. It follows that for any two numbers x and y that satisfy $y^2 = x^2$, either $y = x$ or $y = -x$. So any point (x, y) on the graph of $y^2 = x^2$ must lie on one of the two lines $y = x$ or $y = -x$. This provides the graph of $y^2 = x^2$ of Figure 5.13b. Let (x, y) be a point on the graph. If $y \geq 0$, then we must have $y = |x|$ and if y is negative, then $y = -|x|$.

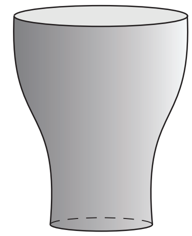
5.25. Solving $x^2 + y^2 = 9$ for y we get $y^2 = 9 - x^2$ and hence $y = \pm\sqrt{9 - x^2}$. Define $f_+(x)$ and $f_-(x)$ by $f_+(x) = \sqrt{9 - x^2}$ and $f_-(x) = -\sqrt{9 - x^2}$. Both functions have domain $-3 \leq x \leq 3$. The graph of $f_+(x)$ is the upper half of the circle (including the points $(\pm 3, 0)$) and the graph of $f_-(x)$ is the lower half (again including the points $(\pm 3, 0)$).

5.26. It is clear that the height of the water level is increasing in all cases. So the issue is the rate of this increase and the fact that this rate is provided by the slope of the graph. Whether this rate is larger or smaller at a given time depends on the cross sectional area of the drinking glass or vase at that time. The smaller the cross sectional area, the greater the rate of increase of the height of the water level, the larger the cross sectional area, the smaller this rate of increase.

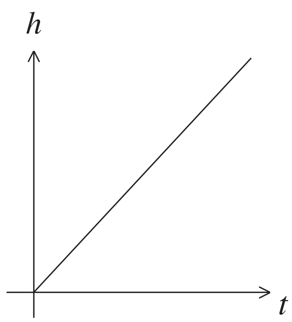
The graphs of the height function for the three glasses

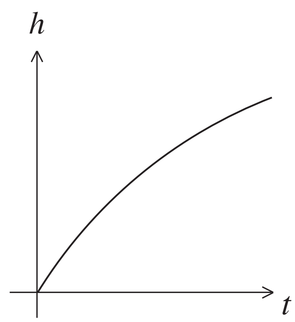

(a)

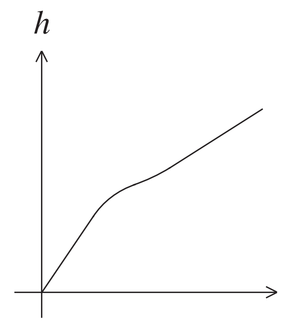

(b)


(c)

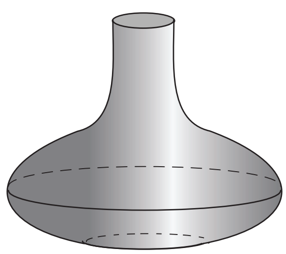
are sketched below







The graphs for those for the three vases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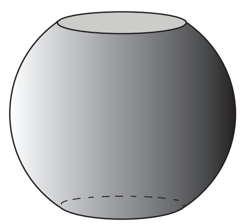
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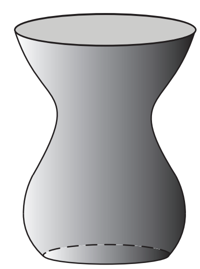
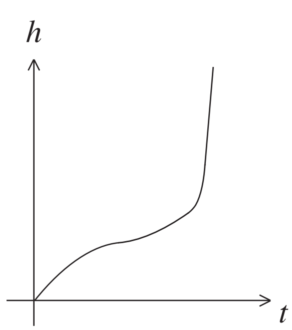
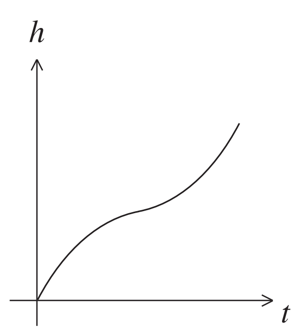
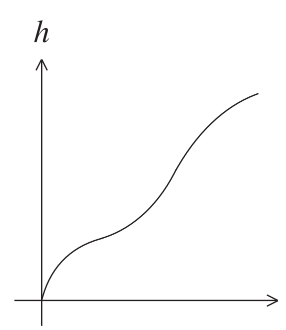

(f)

are depicted here.







5.27. Since

$$\frac{(x+\Delta x)^3-x^3}{\Delta x} = \frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x} = \frac{\Delta x(3x^2+3x\Delta x+(\Delta x)^2)}{\Delta x} = 3x^2+3x\Delta x+(\Delta x)^2,$$

it is clear that $f'(x) = \lim\limits_{\Delta x\to 0} \frac{(x+\Delta x)^3-x^3}{\Delta x} = 3x^2$.

5.28. Since $h(x) = \frac{1}{x^2}$, we get

$$\frac{h(x+\Delta x)-h(x)}{\Delta x} = \frac{1}{\Delta x}\Big(\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}\Big) = \frac{1}{\Delta x}\frac{x^2-(x+\Delta x)^2}{(x+\Delta x)^2x^2} = \frac{1}{\Delta x}\frac{-2x(\Delta x)-(\Delta x)^2}{(x+\Delta x)^2x^2} = \frac{-2x-\Delta x}{(x+\Delta x)^2x^2}.$$

Therefore $h'(x) = \lim\limits_{\Delta x\to 0}\frac{h(x+\Delta x)-h(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{-2x-\Delta x}{(x+\Delta x)^2x^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$.

5.29. **i.** Since $f'(x) = 3x^2$, the slope of the tangent at $(2,8)$ is $f'(2) = 12$.

ii. Since $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, the slope is $g'(-3) = \frac{1}{3}(-3)^{-\frac{2}{3}} = \frac{1}{3(-3)^{\frac{2}{3}}} = \frac{1}{3\cdot 9^{\frac{1}{3}}} \approx 0.16$.

iii. The derivative of $f(x) = \frac{1}{x} = x^{-1}$ is $f'(x) = -x^{-2} = -\frac{1}{x^2}$. Therefore the slope is $f'(-\frac{1}{3}) = -\frac{1}{\frac{1}{9}} = -9$.

iv. Since $f'(x) = -2x^{-3} = -\frac{2}{x^3}$, the slope is $f'(-2) = -\frac{2}{(-2)^3} = \frac{1}{4}$.

5.30. **i.** $f'(x) = 0$

ii. $\frac{dy}{dx} = 4$

iii. $f'(x) = 14x - 5$

iv. $\frac{dy}{dx} = \frac{2}{3}x^{-\frac{2}{3}} + 3\pi x^2$

v. $g'(x) = -3x^{-2} + 3$

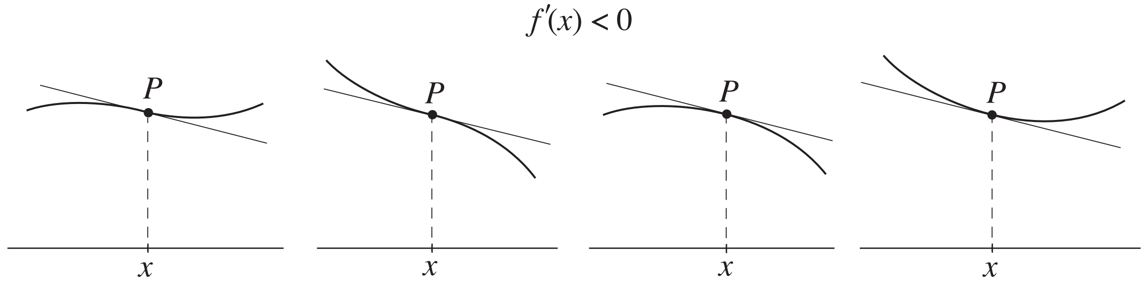
vi. $f'(x) = 6x^2 + 3 + 2x^{-3}$

vii. $h'(x) = 2x^{-\frac{1}{2}} - 5x^{-2}$

5.31. Since a tangent line is involved, we'll need the derivative $f'(x) = 10x - 6x^2$ of the function. It provides the slope of the tangent to the graph at the point $P = (x,y)$. If we don't change notation as suggested, we get—by substituting into the point-slope form of the equation of a line—the unhelpful and confusing $y - y = f'(x)(x-x) = (10x-6x^2)(x-x)$ for the tangent. After changing notation to $P = (x_0, y_0)$, we get $f'(x_0) = 10x_0 - 6x_0^2$ and hence $y - y_0 = f'(x_0)(x-x_0)$ or $y - y_0 = (10x_0 - 6x_0^2)(x-x_0)$ for the equation of the tangent.

5.32. Solving the equations $y = x^2$ and $y = 3x-4$ for x, we get $x^2 = 3x-4$ and hence $x^2-3x+4 = 0$. By the quadratic formula, $x = \frac{3\pm\sqrt{9-4\cdot 4}}{2} = \frac{3\pm\sqrt{-7}}{2}$. It follows that there is no solution, and hence that the parabola and the line cannot intersect. The slope intercept form of any line with slope 3 is $y = 3x+b$. It will touch the parabola at a single point if it intersects the parabola at a single point. This means that $x^2 = 3x+b$ needs to have a single solution. Since the solutions of $x^2-3x-b = 0$ are $x = \frac{3\pm\sqrt{9+4b}}{2}$, this means that $4b = -9$. So the point of "first touch" is $\big(\frac{3}{2}, \frac{9}{4}\big)$ (corresponding to $b = -\frac{9}{4} = -2\frac{1}{4}$).

5.33. The four "generic" possibilities are sketched below.



5.34. Squaring both sides of $y = \sqrt{5^2 - x^2}$, we get $y^2 = 5^2 - x^2$ and hence $x^2 + y^2 = 5^2$. Example 5.7 informs us that the slope of the tangent at any point (x, y) on this circle is $-\frac{x}{y}$. Since in the current situation, $y = \sqrt{5^2 - x^2}$, it follows that $f'(x) = -\frac{x}{\sqrt{5^2-x^2}}$. Problem 5.22 considers the curve defined by $y^3 = 3x^2 + 7$ and shows that the slope of the tangent at any point $P = (x, y)$ on the curve is $m_P = \frac{2x}{y^2}$. Since $y = (3x^2 + 7)^{\frac{1}{3}}$ is equivalent to $y^3 = 3x^2 + 7$ and $y^2 = (3x^2 + 7)^{\frac{2}{3}}$, it follows that $f'(x) = \frac{2x}{y^2} = \frac{2x}{(3x^2+7)^{\frac{2}{3}}}$.

5.35. Taking antiderivatives term by term and adding the required constant C gives us the antiderivatives $F(x) = x - \frac{3}{4}x^4 + 4x^{\frac{1}{2}} + C$ for $f(x)$, $G(x) = \frac{1}{3}x^{-1} + 6x^{\frac{4}{3}} + C$ for $g(x)$, and $H(x) = -4x - 3x^{-1} + \frac{14}{3}x^{\frac{3}{2}} + C$ for $h(x)$.

5.36. By proceeding as in Problem 5.34 or by applying the conclusion of Problem 5.23, we see that the slope of the tangent at any point (x, y) on the circle $x^2 + y^2 = 1$ with $y \neq 0$, is $-\frac{x}{y}$. Now consider the function $y = H(x) = \sqrt{1 - x^2}$ and notice that the graph of $H(x)$ is the upper half of this circle. It follows that $H'(x) = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$. Let $F(x) = -H(x)$ and observe that $F'(x) = \frac{x}{\sqrt{1-x^2}}$. So the antiderivatives of $f(x) = \frac{x}{\sqrt{1-x^2}}$ have the form $-\sqrt{1 - x^2} + C$.

To find an antiderivative of $g(x) = \frac{x}{(3x^2+7)^{\frac{2}{3}}}$, let $G(x) = \frac{1}{2}(3x^2 + 7)^{\frac{1}{3}} + C$ and refer to Problem 5.34 for the fact that $G'(x) = \frac{1}{2}\frac{d}{dx}(3x^2 + 7)^{\frac{1}{3}} = \frac{1}{2}\frac{2x}{(3x^2+7)^{\frac{2}{3}}} = \frac{x}{(3x^2+7)^{\frac{2}{3}}}$.

5.37. Let the sides of the rectangle be x and y. Since $A = xy$ we know that $y = Ax^{-1}$ and hence that $p = 2x + 2y = 2x + 2Ax^{-1}$. So $p'(x) = 2 - 2Ax^{-2} = \frac{2(x^2-A)}{x^2}$. It follows that $p'(x) = 0$ for $x = \sqrt{A}$. Notice that $p'(x) < 0$ for $0 < x < \sqrt{A}$ and that $p'(x) > 0$ for $x > \sqrt{A}$. Therefore $p(x)$ achieves its minimum value at $x = \sqrt{A}$. The corresponding y is $y = \frac{A}{\sqrt{A}} = \sqrt{A}$. So for a given area A, the rectangle that has the smallest perimeter is a square.

5.38. Let x and y be the sides of the rectangle. By the Pythagorean theorem $d^2 = x^2 + y^2$, so that $y = \sqrt{d^2 - x^2}$. It follows that the area of the rectangle is given by the function $A(x) = x(\sqrt{d^2 - x^2})$. The hint tells us that we need to find the x for which the function $f(x) = x^2(d^2 - x^2) = d^2x^2 - x^4$ has its maximum value. Since $f'(x) = 2d^2x - 4x^3 =$

$2x(d^2 - 2x^2)$, we see that $f'(x) = 0$ for $x = 0$ or $x = \frac{d}{\sqrt{2}}$. Notice that $f'(x) > 0$ for $x < \frac{d}{\sqrt{2}}$ and $f'(x) < 0$ for $x > \frac{d}{\sqrt{2}}$. It follows that the function $f(x)$ is increasing for $x < \frac{d}{\sqrt{2}}$ and decreasing for $x > \frac{d}{\sqrt{2}}$. So $x = \frac{d}{\sqrt{2}}$ provides the maximum value of $f(x)$ and hence also of $A(x)$. For this x, $y = \sqrt{d^2 - \frac{d^2}{2}} = \frac{d}{\sqrt{2}}$. So as Kepler had asserted, of all the rectangles with a fixed diameter d, the square has the largest area.

5.39. The area of the inscribed rectangle is $x \cdot g(x) = x\frac{1}{x} = 1$. Therefore any such rectangle has area equal to 1.

5.40. Let (x, x^2+1) be any point on the parabola. The distance between it and $(3, 1)$ is given by the function $d(x) = \sqrt{(x-3)^2 + ((x^2+1)-1)^2} = \sqrt{x^2 - 6x + 9 + x^4}$. The task is to find the x for which $d(x)$ is a minimum. This is also the x for which the function $D(x) = d(x)^2 = x^2 - 6x + 9 + x^4$ is a minimum. The function $D(x)$ is easier to deal with than $d(x)$ so we'll focus on it. The derivative of $D(x)$ is $D'(x) = 4x^3 + 2x - 6$. Since $D'(1) = 4 + 2 - 6 = 0$, the term $(x - 1)$ divides $D'(x) = 4x^3 + 2x - 6$. A polynomial division confirms that $D'(x) = (x - 1)(4x^2 + 4x + 6)$. The quadratic formula applied to $4x^2 + 4x + 6 = 0$, tells us that $4x^2 + 4x + 6$ is never 0. So it is always positive. Returning to $D'(x) = (x-1)(4x^2+4x+6)$, we now know that $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$. Therefore the functions $D(x)$ and also $d(x)$ reach their minimum values when $x = 1$. We have verified that the point on the parabola that is closest to $(3, 1)$ is the point $(1, 2)$.

5.41. Let $L = x + y$ be the length of the segment S. Since $xy = 225$, $y = \frac{225}{x}$, so that $L = x + \frac{225}{x} = x + 225x^{-1}$ is now a function of x. To find the minimal length that S can have, we need to deal with $L'(x) = 1 - 225x^{-2} = \frac{x^2-225}{x^2}$. Observe that $L'(x) = 0$ when $x^2 = 225$ or $x = 15$, that $L'(x) < 0$ for $x < 15$ and that $L'(x) > 0$ for $x > 15$. It follows that $x = 15$ provides the shortest possible length for the segment S. With $x = 15$, $y = \frac{225}{15} = 15$, so that this length is $15 + 15 = 30$ units.

5.42. This time $x + y = 1200$. Let $p = x^2y^2$ be the product of the squares of the two pieces. After substituting $y = 1200 - x$, we get $p = x^2(1200 - x)^2 = 1200^2x^2 - 2400x^3 + x^4$. After factoring out the term $4x$ from the derivative $p'(x) = 2(1200^2)x - 3(2400)x^2 + 4x^3$, we get $p'(x) = 4x\big(x^2 - 3(600)x + 2(600^2)\big)$. By the quadratic formula, $p'(x)$ has the three roots

$$x = 0 \text{ and } x = \frac{3(600) \pm \sqrt{3^2(600)^2 - 4 \cdot 2(600^2)}}{2} = \frac{3(600) \pm 600}{2} = 600 \text{ or } 1200.$$

It follows that $p'(x) = x(x - 600)(x - 1200)$, and hence that $p'(x) > 0$ for $0 < x < 600$ and $p'(x) < 0$ for $600 < x < 1200$. Therefore $p(x)$ is as large as possible when $x = 600$. So $y = 600$ also, and the largest value that the product x^2y^2 can have is $600^2 \cdot 600^2 = 129{,}600{,}000{,}000$.

5.43. A look at Figure 5.47 tells us that the volume of the cylinder is $V(x) = \pi(3-x)^2x = \pi(9x - 6x^2 + x^3)$ and that $0 \le x \le 3$. The derivative is $V'(x) = \pi(3x^2 - 12x + 9) = 3\pi(x^2 - 4x + 3) = 3\pi(x-1)(x-3)$. So $V'(x) = 0$ for $x = 1$ or $x = 3$. Since $V'(x) = 3\pi(x-1)(x-3)$, it follows that $V'(x) > 0$ for $0 \le x < 1$ and that $V'(x) < 0$ for $1 < x < 3$. So $x = 1$ provides the maximum value for the volume. This maximum volume is 4π.

5.44. For a point $P = (x, y)$ on the line $y = -\frac{1}{2}x+5$ the distance between (x, y) and $(-4, 3)$ is

$$\begin{aligned}\sqrt{(x+4)^2 + (y-3)^2} &= \sqrt{(x+4)^2 + (-\tfrac{1}{2}x + 5 - 3)^2} = \sqrt{(x+4)^2 + (-\tfrac{1}{2}x + 2)^2} \\ &= \sqrt{(x^2 + 8x + 16) + (\tfrac{1}{4}x^2 - 2x + 4)} = \sqrt{\tfrac{5}{4}x^2 + 6x + 20}.\end{aligned}$$

The square of the distance is $f(x) = \frac{5}{4}x^2 + 6x + 20$. Notice that the x that provides the minimum distance, also provides the minimum of $f(x)$. After differentiating, $f'(x) = \frac{5}{2}x + 6$. The minimum value of $f(x)$ occurs for $\frac{5}{2}x + 6 = 0$, or at $x = -\frac{12}{5}$. Since $f(\frac{-12}{5}) = \frac{5}{4} \cdot \frac{(-12)^2}{5^2} + 6 \cdot \frac{-12}{5} + 20 = \frac{36}{5} - \frac{72}{5} + \frac{100}{5} = \frac{64}{5}$, it follows that the distance from $(-4, 3)$ to the line $y = -\frac{1}{2}x + 5$ is $\sqrt{\frac{64}{5}} = \frac{8}{\sqrt{5}}$.

5.45. We turn to Toricelli's solution of the problem posed by Fermat and explain the construction of the point P. We'll let ΔABC be a triangle and assume that all of its three angles are less than 120°.

i. As Toricelli had done, complete the sides of the triangle ΔABC to equilateral triangles $\Delta ABL, \Delta ACM$, and ΔBCN. In each case the triangle that falls outside the given ΔABC is taken. Then the centers of the circles that each of the three sets of points ABL, ACM, and BCN determine are constructed by making use of two sides and applying the conclusion of Problem 1.9. You will see that these three circles intersect at a single point P. This is the point P that minimizes the sum $PA + PB + PC$. It turns out that P is also the intersection of the segments AN, BM, and CL.

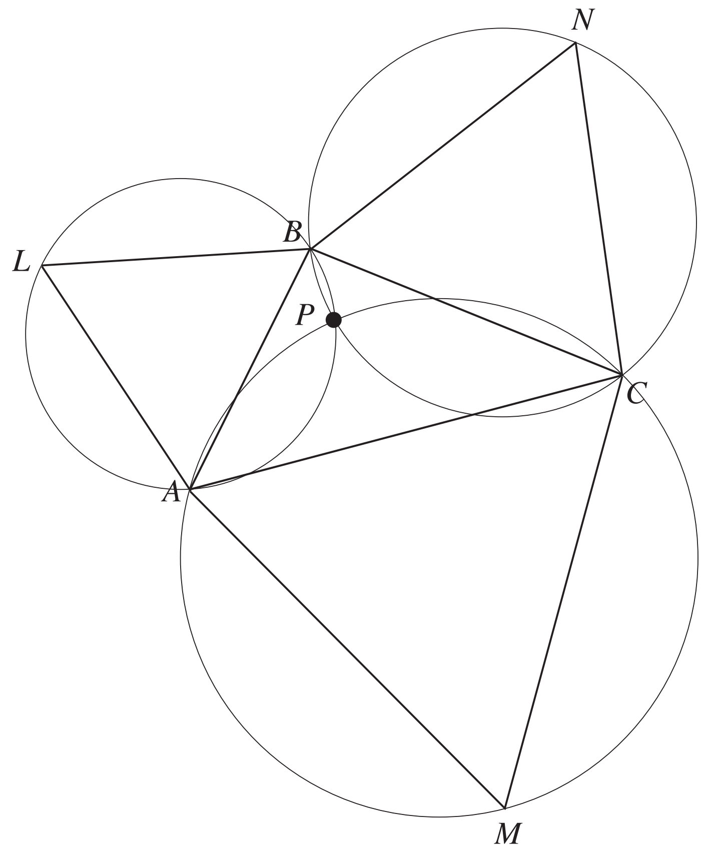
ii. To reformulate the problem (as simply as possible), start by choosing an xy-coordinate system so that $A = (0, 0), B = (b, 0)$ and $C = (c, d)$. Let $P = (x, y)$. The distances PA, PB, and PC are equal to

$$PA = \sqrt{x^2 + y^2}, PB = \sqrt{(x-b)^2 + y^2}, \text{and } PC = \sqrt{(x-c)^2 + (y-d)^2},$$

respectively. The reformulation of the question of Fermat is this: For what x and y does the sum

$$\sqrt{x^2 + y^2} + \sqrt{(x-b)^2 + y^2} + \sqrt{(x-c)^2 + (y-d)^2}$$

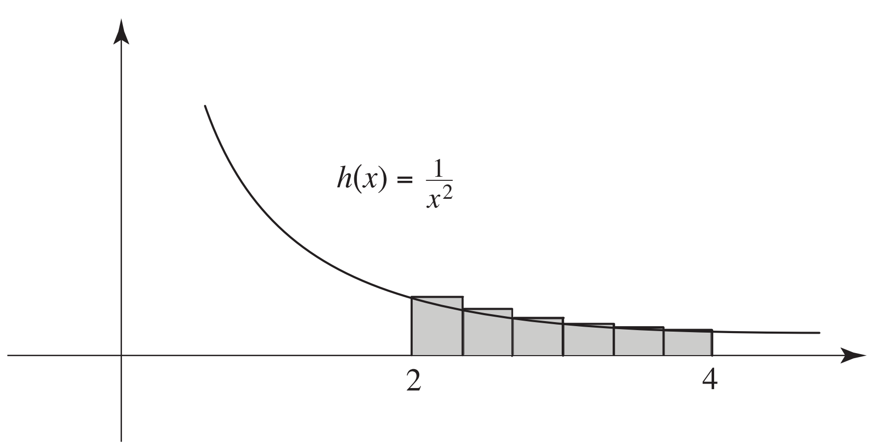
have its minimum value? Since both x and y are "free to move," this is a function of the two variables x and y. So it does not fit into the context of Chapter 5.



The article J. Krarup and S. Vajda, "Toricelli's geometrical solution to a problem of Fermat," *IMA Journal of Mathematics Applied in Business & Industry* (1997) 8, 215-224, discusses several solutions and interesting historical connections. It also explores the situation where one of the angles of the triangle is greater than or equal to 120°.

5.46. The sum of the terms $f(x) \cdot dx$ with $f(x) = \frac{1}{x^2}$, $2 \leq x \leq 4$, and $dx = \frac{1}{3}$ is

$$f(2) \cdot \tfrac{1}{3} + f(\tfrac{7}{3}) \cdot \tfrac{1}{3} + f(\tfrac{8}{3}) \cdot \tfrac{1}{3} + f(3) \cdot \tfrac{1}{3} + f(\tfrac{10}{3}) \cdot \tfrac{1}{3} + f(\tfrac{11}{3}) \cdot \tfrac{1}{3} = (\tfrac{1}{4} + \tfrac{9}{49} + \tfrac{9}{64} + \tfrac{1}{9} + \tfrac{9}{100} + \tfrac{9}{121})\tfrac{1}{3}.$$



Using a calculator we get the approximation of 0.28 for the sum of the areas of the six rectangles. Since $F(x) = -x^{-1}$ is an antiderivative of $f(x) = \frac{1}{x^2}$, the fundamental theorem of calculus tells us that $\int_2^4 \frac{1}{x^2}\,dx = -\frac{1}{4} - (-\frac{1}{2}) = 0.25$. As the figure illustrates, the sum of the areas of the rectangles is a little larger than the area under the graph (and over $2 \le x \le 4$).

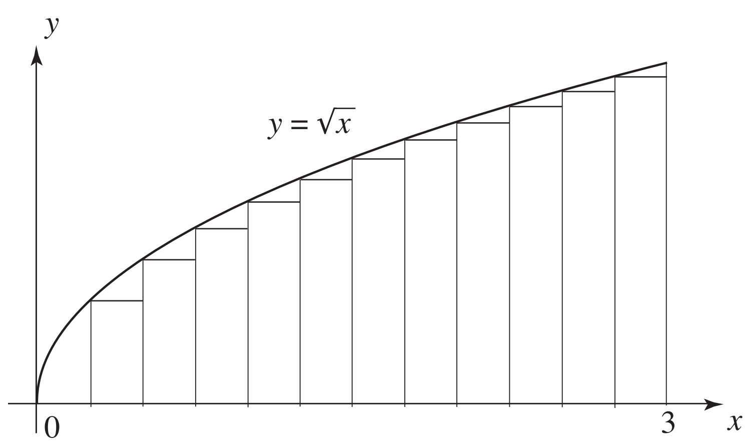
5.47. For the function $y = f(x) = \sqrt{x}$ over the interval from 0 to 3 with the points

$$0 < \tfrac{1}{4} < \tfrac{1}{2} < \tfrac{3}{4} < 1 < \tfrac{5}{4} < \tfrac{3}{2} < \tfrac{7}{4} < 2 < \tfrac{9}{4} < \tfrac{5}{2} < \tfrac{11}{4} < 3$$

the dx is $dx = \frac{1}{4}$. The sum of the terms $f(x) \cdot dx$ is

$$\begin{aligned} &\sqrt{0} \cdot \tfrac{1}{4} + \sqrt{\tfrac{1}{4}} \cdot \tfrac{1}{4} + \sqrt{\tfrac{1}{2}} \cdot \tfrac{1}{4} + \sqrt{\tfrac{3}{4}} \cdot \tfrac{1}{4} + \sqrt{1} \cdot \tfrac{1}{4} + \sqrt{\tfrac{5}{4}} \cdot \tfrac{1}{4} \\ &\qquad + \sqrt{\tfrac{3}{2}} \cdot \tfrac{1}{4} + \sqrt{\tfrac{7}{4}} \cdot \tfrac{1}{4} + \sqrt{2} \cdot \tfrac{1}{4} + \sqrt{\tfrac{9}{4}} \cdot \tfrac{1}{4} + \sqrt{\tfrac{5}{2}} \cdot \tfrac{1}{4} + \sqrt{\tfrac{11}{4}} \cdot \tfrac{1}{4} \\ &= \big(0 + \tfrac{1}{2} + \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{3}}{2} + 1 + \tfrac{\sqrt{5}}{2} + \tfrac{\sqrt{6}}{2} + \tfrac{\sqrt{7}}{2} + \sqrt{2} + \tfrac{3}{2} + \tfrac{\sqrt{10}}{2} + \tfrac{\sqrt{11}}{2}\big)\tfrac{1}{4} \\ &\approx 3.22. \end{aligned}$$

The graph of the function and the corresponding rectangles under the graph are drawn



in the figure below. The fundamental theorem of calculus tells us that

$$\int_0^3 \sqrt{x}\,dx = \tfrac{2}{3}x^{\frac{3}{2}}\Big|_0^3 = \tfrac{2}{3}3^{\frac{3}{2}} = \tfrac{2}{3}3\sqrt{3} = 2\sqrt{3} \approx 3.46.$$

The numbers confirm what the figure shows, namely that the area under the graph is a bit larger than the corresponding sum of the areas of the rectangles.

5.48. The expectation is that of these two approximations the one with the smaller dx will be better. The sum of the terms $g(x) \cdot dx$ with $g(x) = 9 - x^2$, $0 \le x \le 1$, and $dx = 0.2$ is

$$
\begin{aligned}
g(0) \cdot 0.2 + g(0.2) \cdot 0.2 + g(0.4) \cdot 0.2 + g(0.6) \cdot 0.2 + g(0.8) \cdot 0.2 \\
&= (9 + 8.96 + 8.84 + 8.64 + 8.36)(0.2) \\
&= 8.76.
\end{aligned}
$$

For the tighter set of points with $dx = 0.1$, the sum of the $g(x) \cdot dx$ is

$$
\begin{aligned}
g(0) \cdot 0.1 + g(0.1) \cdot 0.1 + g(0.2) \cdot 0.1 + g(0.3) \cdot 0.1 + g(0.4) \cdot 0.1 + g(0.5) \cdot 0.1 \\
+g(0.6) \cdot 0.1 + g(0.7) \cdot 0.1 + g(0.8) \cdot 0.1 + g(0.9) \cdot 0.1 \\
= \big(9 + (9 - 0.1^2) + (9 - 0.2^2) + (9 - 0.3^2) + (9 - 0.4^2) + (9 - 0.5^2) \\
+(9 - 0.6^2) + (9 - 0.7^2) + (9 - 0.8^2) + (9 - 0.9^2)\big)(0.1) \\
= 87.15(0.1) = 8.715.
\end{aligned}
$$

Since $G(x) = 9x - \frac{1}{3}x^3$ is an antiderivative of $g(x) = 9 - x^2$, the exact area under the graph over $0 \le x \le 1$ is equal to $G(1) - G(0) = 9 - \frac{1}{3} = 8\frac{2}{3} \approx 8.67$.

5.49. For the function $f(x) = 16 - x^2$ with $-2 \le x \le 2$ and the points

$$-2 < -1.5 < -1 < -0.5 < 0 < 0.5 < 1 < 1.5 < 2$$

the sum of the areas $f(x) \cdot dx$ of all the rectangles that these points determine is

$$
\begin{aligned}
(16 - (-2)^2)(0.5) + (16 - (-1.5)^2)(0.5) + (16 - (-1)^2)(0.5) + (16 - (-0.5)^2)(0.5) \\
+(16 - (0)^2)(0.5) + (16 - (0.5)^2)(0.5) + (16 - (1)^2)(0.5) + (16 - (1.5)^2)(0.5) \\
= 58.5.
\end{aligned}
$$

This is an approximation of the area $\int_{-2}^{2} (16 - x^2)\, dx$ under the graph of $f(x) = 16 - x^2$ over $-2 \le x \le 2$. The precise value is

$$\int_{-2}^{2} (16 - x^2)\, dx = (16x - \tfrac{1}{3}x^3)\Big|_{-2}^{2} = (32 - \tfrac{8}{3}) - (-32 + \tfrac{8}{3}) = 64 - \tfrac{16}{3} \approx 58.67.$$

5.50. With the function $y = f(x) = \sqrt{4 - x^2}$ over the interval $0 \le x \le 2$ and the points

$$0 < 0.2 < 0.4 < 0.6 < 0.8 < 1 < 1.2 < 1.4 < 1.6 < 1.8 < 2$$

the corresponding sum of the areas $f(x) \cdot dx$ of the rectangles is

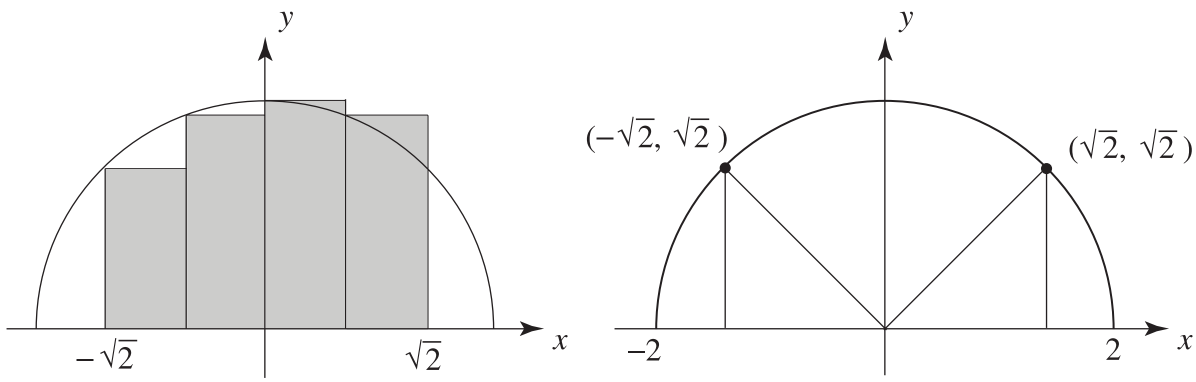
$$
\begin{aligned}
(\sqrt{4 - 0^2})0.2 + (\sqrt{4 - 0.2^2})0.2 + (\sqrt{4 - 0.4^2})0.2 + (\sqrt{4 - 0.6^2})0.2 + (\sqrt{4 - 0.8^2})0.2 \\
+(\sqrt{4 - 1^2})0.2 + (\sqrt{4 - 1.2^2})0.2 + (\sqrt{4 - 1.4^2})0.2 + (\sqrt{4 - 1.6^2})0.2 + (\sqrt{4 - 1.8^2})0.2 \\
\approx 3.30.
\end{aligned}
$$

Since the graph of the function $y = f(x) = \sqrt{4 - x^2}$ over the interval $0 \le x \le 2$ is a quarter circle of radius 2, 3.30 is an approximation of the area of a quarter circle of radius 2. The exact value of this area is $\frac{1}{4}4\pi \approx 3.14$.

5.51. The sum of the terms $f(x) \cdot dx$ that corresponds to the given situation is

$$\begin{aligned} &f(-\sqrt{2}) \cdot \tfrac{1}{2}\sqrt{2} + f(-\tfrac{1}{2}\sqrt{2}) \cdot \tfrac{1}{2}\sqrt{2} + f(0) \cdot \tfrac{1}{2}\sqrt{2} + f(\tfrac{1}{2}\sqrt{2}) \cdot \tfrac{1}{2}\sqrt{2} \\ &= \left(\sqrt{2} + \sqrt{3\tfrac{1}{2}} + 2 + \sqrt{3\tfrac{1}{2}}\right)\left(\tfrac{1}{2}\sqrt{2}\right) \\ &= 5.06. \end{aligned}$$

The graph of $y = f(x)$ is the upper half of the circle $x^2 + y^2 = 4$ of radius 2. Notice



that the points $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$ are on the graph. The segments from the origin to these two points lie on the lines $y = -x$ and $y = x$, respectively. They are therefore perpendicular to each other. So the area in question consists of a quarter of a circle of radius 2 together with two triangles of base $\sqrt{2}$ and height $\sqrt{2}$. This area is equal to $\frac{1}{4} \cdot 4\pi + 2 = \pi + 2 \approx 5.14$. This is also the value of $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4 - x^2}\, dx$.

5.52. A look at

$$3^2 \cdot \tfrac{1}{1000} + (3 + \tfrac{1}{1000})^2 \cdot \tfrac{1}{1000} + (3 + \tfrac{2}{1000})^2 \cdot \tfrac{1}{1000} + (3 + \tfrac{3}{1000})^2 \cdot \tfrac{1}{1000} + \cdots + (5 + \tfrac{999}{1000})^2 \cdot \tfrac{1}{1000}$$

tells us that with $dx = \frac{1}{1000}$, $f(x) = x^2$, and the interval $3 \le x \le 6$, this is a sum of terms of the form $f(x_i) \cdot dx$ that approximates the integral $\int_3^6 x^2\, dx$. Since $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$, this integral is equal to $\frac{1}{3}(6^3 - 3^3) = \frac{1}{3}(216 - 27) = \frac{189}{3} = 63$. Therefore this number approximates the sum.

5.53. The sum

$$\sqrt{4} \cdot \tfrac{1}{10{,}000} + \sqrt{4 + \tfrac{1}{10{,}000}} \cdot \tfrac{1}{10{,}000} + \sqrt{4 + \tfrac{2}{10{,}000}} \cdot \tfrac{1}{10{,}000} + \cdots + \sqrt{7 + \tfrac{9{,}999}{10{,}000}} \cdot \tfrac{1}{10{,}000}$$

is gotten by taking $dx = \frac{1}{10{,}000}$, $f(x) = \sqrt{x}$, the interval $4 \le x \le 8$, and forming the sum of the corresponding $f(x_i) \cdot dx$. So this sum is closely approximated by

$$\int_4^8 x^{\frac{1}{2}}\, dx = \tfrac{2}{3}x^{\frac{3}{2}}\Big|_4^8 = \tfrac{2}{3}(8^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \tfrac{2}{3}((2\sqrt{2})^3 - 8) = \tfrac{2}{3}((8 \cdot 2\sqrt{2} - 8) = \tfrac{16}{3}(2\sqrt{2} - 1) \approx 9.75.$$

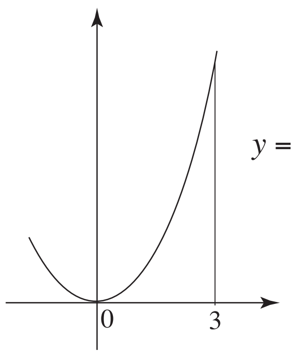
5.54. The antiderivatives of the functions $y = x^2, y = \frac{1}{x^2} = x^{-2}$, and $y = x^{\frac{1}{2}}$ are $y = \frac{1}{3}x^3, y = -x^{-1}$, and $y = \frac{2}{3}x^{\frac{3}{2}}$, respectively. Three applications of the fundamental theorem of calculus tell us that

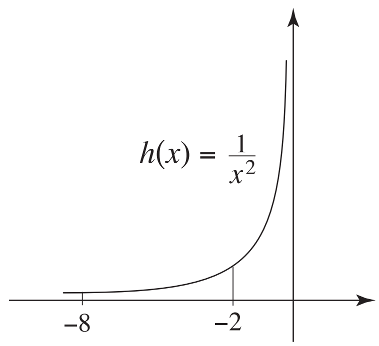
$$\int_0^3 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^3 = \tfrac{1}{3}(27-0) = 9,$$

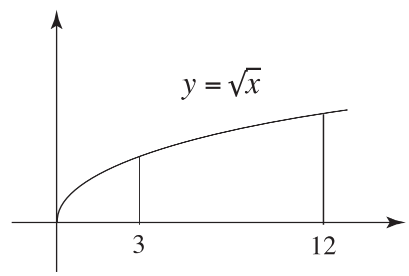
$$\int_{-8}^{-2} \frac{1}{x^2}\,dx = -x^{-1}\Big|_{-8}^{-2} = -\tfrac{1}{-2} - (-\tfrac{1}{-8}) = \tfrac{1}{2} - \tfrac{1}{8} = \tfrac{3}{8}, \quad \text{and}$$

$$\int_3^{12} \sqrt{x}\,dx = \tfrac{2}{3}x^{\frac{3}{2}}\Big|_3^{12} = \tfrac{2}{3}\big(12\cdot\sqrt{12} - 3\cdot\sqrt{3}\big) = \tfrac{2}{3}\big(12\cdot 2\cdot\sqrt{3} - 3\cdot\sqrt{3}\big) = 14\sqrt{3}.$$

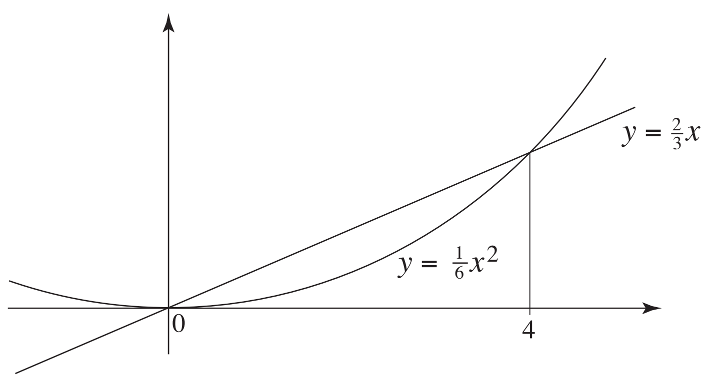
The areas that the integrals represent are sketched below.







5.55. Let's determine the points of intersection of the parabola $y = \frac{1}{6}x^2$ and the line $y = \frac{2}{3}x$. Set $\frac{1}{6}x^2 = \frac{2}{3}x$. Note that $x = 0$ is one solution and that $\frac{1}{6}x = \frac{2}{3}$ determines the other. So the second solution is $x = 4$. The area under the triangle (over the x-axis) is $\frac{1}{2}(4)(\frac{8}{3}) = \frac{16}{3}$. The area under the parabola and over the interval $0 \le x \le 4$ is equal to



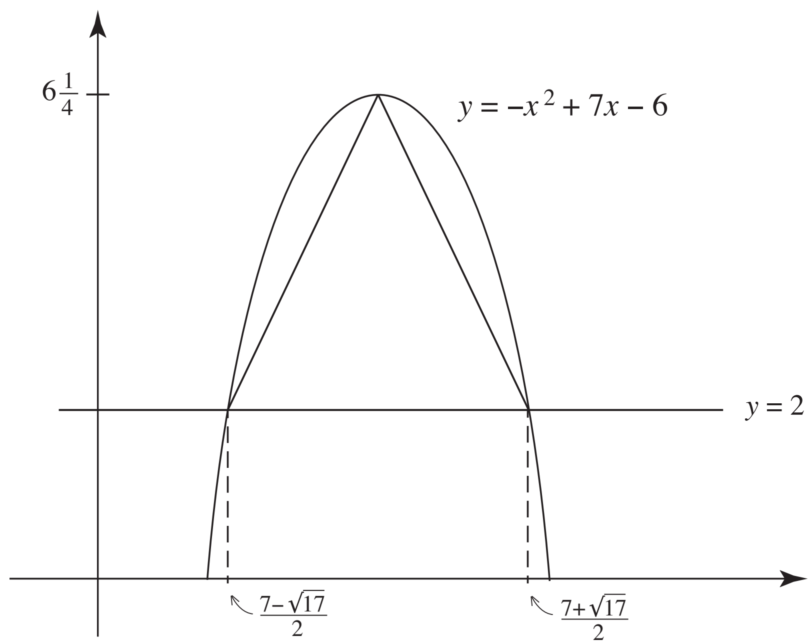
$\int_0^4 \frac{1}{6}x^2\,dx = \frac{1}{18}x^3\Big|_0^4 = \frac{64}{18} = \frac{32}{9}$. It follows that the area in question is $\frac{48}{9} - \frac{32}{9} = \frac{16}{9}$.

5.56. By an application of the quadratic formula, the parabola $y = -3x^2 + 2x + 1$ crosses the x-axis at $x = \frac{-2\pm\sqrt{4-4(-3)(1)}}{-6} = \frac{-2\pm\sqrt{16}}{-6} = \frac{-2\pm 4}{-6} = -\frac{1}{3}, 1$. So the parabolic section obtained by cutting the parabola $y = -3x^2 + 2x + 1$ with the x-axis has area

$$\int_{-\frac{1}{3}}^{1} \bigl(-3x^2+2x+1\bigr)\,dx = (-x^3+x^2+x)\Big|_{-\frac{1}{3}}^{1} = 1-\left(\tfrac{1}{27}+\tfrac{1}{9}-\tfrac{1}{3}\right) = 1+\tfrac{5}{27} = 1\tfrac{5}{27}.$$

The area can also be computed by using Archimedes's theorem of Section 2.4. Since $\frac{dy}{dx} = -6x+2$, we know that the parabola has a horizontal tangent for $x = \frac{1}{3}$. The corresponding y-coordinate is $-3\bigl(\frac{1}{3}\bigr)^2 + \frac{2}{3} + 1 = \frac{4}{3}$. Therefore $\bigl(\frac{1}{3}, \frac{4}{3}\bigr)$ is the vertex of the parabolic section. With regard to Archimedes's theorem, the relevant triangle has area $\frac{1}{2}(1+\frac{1}{3})(\frac{4}{3}) = \frac{1}{2}(\frac{4^2}{3^2}) = \frac{8}{9}$. So the area of the parabolic section is $\frac{4}{3}\cdot\frac{8}{9} = \frac{32}{27} = 1\frac{5}{27}$.

5.57. The x-coordinates of the points of intersection of the parabola and the line are the solutions of the equation $-x^2+7x-6 = 2$. So $-x^2+7x-8 = 0$, and therefore $x = \frac{-7\pm\sqrt{49-4(-1)(-8)}}{-2} = \frac{-7\pm\sqrt{17}}{-2} = \frac{7\pm\sqrt{17}}{2}$. A calculator shows that $\frac{7-\sqrt{17}}{2} \approx 1.44$ and



and $\frac{7+\sqrt{17}}{2} \approx 5.56$. Since the cut is parallel to the x-axis, the vertex of the parabola is obtained by setting the derivative $-2x+7$ equal to 0 and solving for x. So the vertex has x-coordinate $\frac{7}{2}$. The corresponding y-coordinate is $y = -\frac{49}{4}+7\cdot\frac{7}{2}-6 = \frac{-49+98-24}{4} = \frac{25}{4}$. In reference to Archimedes theorem, the relevant triangle has height $\frac{25}{4}-2 = \frac{17}{4}$ and base $\bigl(\frac{7+\sqrt{17}}{2} - \frac{7-\sqrt{17}}{2}\bigr) = \sqrt{17}$. It follows that the triangle has area $\frac{1}{2}\sqrt{17}(\frac{17}{4}) = \frac{17}{8}\cdot\sqrt{17}$. Therefore the area of the parabolic section is equal to $\frac{4}{3}\cdot\frac{17}{8}\cdot\sqrt{17} = \frac{17}{6}\cdot\sqrt{17} \approx 11.68$. A look at the figure tells us that using calculus, the area of the parabolic section is

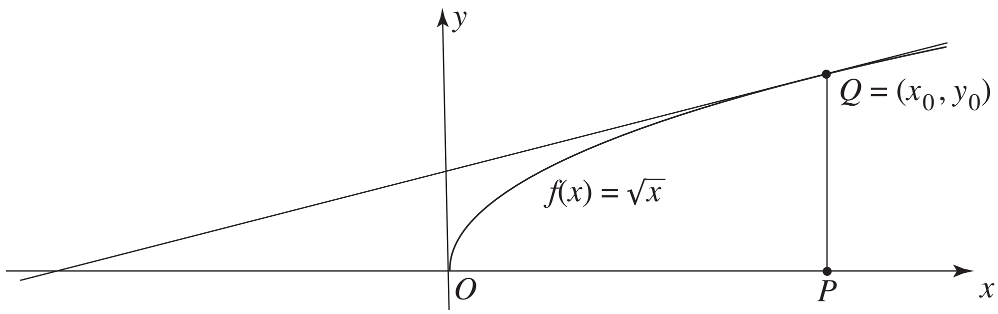
$$\int_{\frac{7-\sqrt{17}}{2}}^{\frac{7+\sqrt{17}}{2}} \bigl(-x^2+7x-6\bigr)\,dx - 2(\sqrt{17}).$$

Since $-\frac{1}{3}x^3 + \frac{7}{2}x^2 - 6x$ is an antiderivative of the integrand, the integral is equal to

$$\int_{\frac{7-\sqrt{17}}{2}}^{\frac{7+\sqrt{17}}{2}} \bigl(-x^2+7x-6\bigr)\,dx = (-\tfrac{1}{3}x^3+\tfrac{7}{2}x^2-6x)\Big|_{\frac{7-\sqrt{17}}{2}}^{\frac{7+\sqrt{17}}{2}} = x(-\tfrac{1}{3}x^2+\tfrac{7}{2}x-6)\Big|_{\frac{7-\sqrt{17}}{2}}^{\frac{7+\sqrt{17}}{2}}.$$

The conclusion—after the remaining arithmetic (it's a bit tedious) is done—that the area is the same $\frac{17}{6} \cdot \sqrt{17}$ that Archimedes's theorem gave us. The approach using Archimedes's theorem is clearly simpler.

5.58. The graph of the function $f(x) = \sqrt{x}$ with $x \geq 0$ is shown below along with any point $Q(x_0, y_0)$ on the graph. So $y_0 = (x_0)^{\frac{1}{2}}$. The point $P = (x_0, 0)$ lies below Q on the



x-axis. The area A under the graph of $f(x) = \sqrt{x}$ and over the interval $0 \leq x \leq x_0$, is

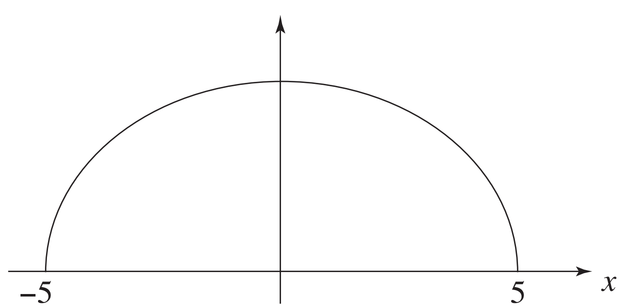
$$A = \int_0^{x_0} x^{\frac{1}{2}}\, dx = \tfrac{2}{3} x^{\frac{3}{2}} \Big|_0^{x_0} = \tfrac{2}{3}(x_0)^{\frac{3}{2}}.$$

To compute the area B, we need to find where the tangent to the graph at Q intersects the x-axis. Since $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, the slope of the tangent is $f'(x_0) = \frac{1}{2}(x_0)^{-\frac{1}{2}}$. So the equation of the tangent is $y - y_0 = \frac{1}{2}(x_0)^{-\frac{1}{2}}(x - x_0)$. We need to set $y = 0$ and solve $-(x_0)^{\frac{1}{2}} = \frac{1}{2}(x_0)^{-\frac{1}{2}}(x - x_0)$ for x. Doing this, we get $-2x_0 = x - x_0$, and hence $x = x_0$. It follows that $B = \frac{1}{2}(2x_0)(x_0)^{\frac{1}{2}} = (x_0)^{\frac{3}{2}}$ and therefore that $A = \frac{2}{3}B$.

5.59. The volume formula of Section 5.9 tells us that this volume is equal to

$$\pi \int_0^4 (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx = \tfrac{\pi}{2} x^2 \Big|_0^4 = 8\pi.$$

5.60. The upper half of the ellipse $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ is depicted in the figure below. Solving this equation for y^2, we get $\frac{y^2}{4^2} = 1 - \frac{x^2}{5^2} = \frac{5^2 - x^2}{5^2}$ and hence $y^2 = \frac{4^2}{5^2}(5^2 - x^2)$. It follows that



the curve shown in the figure is the graph of the function $y = \frac{4}{5}\sqrt{5^2 - x^2}$.

i. The area of the upper half of the ellipse is $\int_{-5}^{5} \frac{4}{5}\sqrt{5^2 - x^2}\, dx$.

ii. The volume of the solid is $\pi \int_{-5}^{5} (\frac{4}{5}\sqrt{5^2 - x^2})^2\, dx = \frac{4^2\pi}{5^2} \int_{-5}^{5} (5^2 - x^2)\, dx$.

5.61. Since $x^2 + y^2 = 4$ is the equation of the circle with radius 2 and center the origin, the graph of $y = \sqrt{4 - x^2}$ is the upper half of this circle, the integral is the area under this graph and over $0 \le x \le 2$. Therefore

$$\int_0^2 \sqrt{4 - x^2}\, dx = \tfrac{1}{4}\pi \cdot 2^2 = \pi.$$

Since $x^2 + y^2 = a^2$ is a circle of radius a (a is assumed to be positive), and $y = \sqrt{a^2 - x^2}$ is the upper half of this circle, it follows that $\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{1}{4}\pi a^2$.

5.62. Notice first that $\int_0^5 \frac{2}{5}\sqrt{5^2 - x^2}\, dx = \frac{2}{5}\int_0^5 \sqrt{5^2 - x^2}\, dx$. By a conclusion of Problem 5.61 $\int_0^5 \sqrt{5^2 - x^2}\, dx = \frac{1}{4}(\pi \cdot 5^2)$, so that $\int_0^5 \frac{2}{5}\sqrt{5^2 - x^2}\, dx = \frac{2}{5} \cdot \frac{1}{4}\pi 5^2 = \frac{1}{4}(5 \cdot 2)\pi$. This is the area of one-quarter of the ellipse with semimajor axis 5 and semiminor axis 2.

5.63. This problem generalizes the conclusion of the previous one. We know that the graph of the function $f(x) = \sqrt{a^2 - x^2}$ with $-a \le x \le a$ is the upper half of a circle of radius a. Since $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$ is the area under this semicircle, $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\pi a^2$. Solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y^2 we get

$$\tfrac{y^2}{b^2} = 1 - \tfrac{x^2}{a^2} = \tfrac{a^2 - x^2}{a^2} = \tfrac{1}{a^2}(a^2 - x^2)$$

so that $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$. It follows that the graph of the function $g(x) = \frac{b}{a}\sqrt{a^2 - x^2}$ is the upper half of this ellipse. We can conclude that the area of the upper half of the ellipse with semimajor axis a and semiminor axis b is

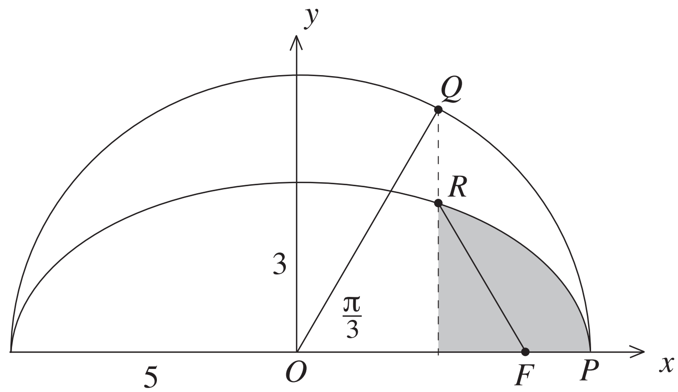
$$\int_{-a}^{a} \tfrac{b}{a}\sqrt{a^2 - x^2}\, dx = \tfrac{b}{a}\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \tfrac{b}{a} \cdot \tfrac{1}{2}\pi a^2 = \tfrac{1}{2}ab\pi.$$

Therefore the area of the full ellipse is $ab\pi$.

5.64. The semicircle of radius 5 along with the upper half of the ellipse with semimajor axis 5 and semiminor axis 3 depicted in Figure 5.50 is repeated below.

i. Let (x_0, y_0) be the coordinates of Q. Since the radius of the circle is 5, it follows that $x_0 = 5\cos\frac{\pi}{3} = \frac{5}{2}$ and that $y_0 = 5\sin\frac{\pi}{3} = 5\frac{\sqrt{3}}{2}$. So $Q = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.

ii. Since the equation of the ellipse is $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$, we find by solving for y, that $y = \frac{3}{5}\sqrt{5^2 - x^2}$ is the equation of the upper half of the ellipse. The equation of the upper half of the circle is $y = \sqrt{5^2 - x^2}$.



iii. The area of the circular sector POQ is equal to $\frac{1}{2}(5^2)\frac{\pi}{3} = \frac{1}{2}\frac{5^2}{3}\pi$ by applying the formula for the area of a circular sector developed in Section 2.2. By subtracting the area of the right triangle with hypotenuse OQ, we get that the area of the section of the circle bounded by the vertical line through Q and a part of the segment OP is $\frac{1}{2}\frac{5^2}{3}\pi - \frac{1}{2}\frac{5}{2}5\frac{\sqrt{3}}{2} = \frac{1}{2}\frac{5^2}{3}\pi - \frac{1}{2}\frac{5^2}{2}\frac{\sqrt{3}}{2} = \frac{5^2}{2}(\frac{1}{3}\pi - \frac{\sqrt{3}}{4})$. We know from part ii that for a given x the y-coordinate of the point on the ellipse is $\frac{3}{5}$ times the y-coordinate of the circle. This implies that $R = \big(\frac{5}{2}, \frac{3}{5}\cdot\frac{5\sqrt{3}}{2}\big) = \big(\frac{5}{2}, \frac{3\sqrt{3}}{2}\big)$. It also means that Cavalieri's principle can be applied to show that the shaded area is equal to $\frac{3}{5}\big(\frac{5^2}{2}(\frac{1}{3}\pi - \frac{\sqrt{3}}{4})\big) = \frac{5}{2}\big(\pi - \frac{3\sqrt{3}}{4}\big)$.

iv. From the discussion in Section 4.4 we know that the distance from the center O of the ellipse to F is $c = \sqrt{5^2 - 3^3} = 4$. Since $R = \big(\frac{5}{2}, \frac{3\sqrt{3}}{2}\big)$, it follows that the right triangle with hypotenuse RF has area $\frac{1}{2}(4 - \frac{5}{2})\frac{3\sqrt{3}}{2} = \frac{3}{2}\frac{3\sqrt{3}}{4}$. So the area of the elliptical sector FPR is $\frac{5}{2}\big(\pi - \frac{3\sqrt{3}}{4}\big) - \frac{3}{2}\frac{3\sqrt{3}}{4} = \frac{5}{2}\pi - (\frac{5}{2} + \frac{3}{2})\frac{3\sqrt{3}}{4} = \frac{5}{2}\pi - 3\sqrt{3}$.

A look back at Section 3.5 tells us that areas of elliptical sectors such as FPR are an important aspect of Kepler's second law. In fact the computation of the area of such elliptical sectors is essential to the quantitative understanding of the motion of the planets. Section 10.4 will discuss these connections in detail.

5.65. Let $y = F(x)$ be antiderivatives of the function $y = f(x)$. Since $\frac{d}{dx}(cF(x)) = c\frac{d}{dx}F(x) = c \cdot f(x)$, the function $cF(x)$ is an antiderivative of $cf(x)$. So

$$\int_a^b cf(x)\,dx = cF(b) - cF(a) = c(F(b) - F(a)) = c\int_a^b f(x)\,dx.$$

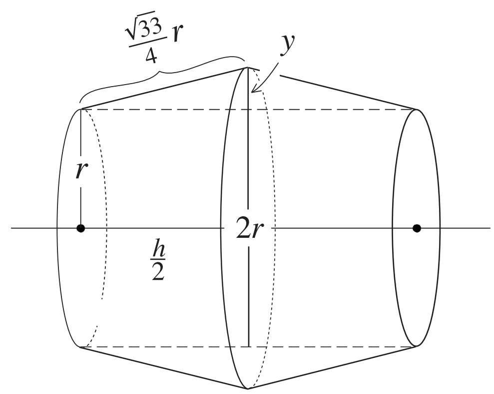
If $y = G(x)$ is an antiderivative of $y = g(x)$, then $\frac{d}{dx}(F(x)+G(x)) = \frac{d}{dx}F(x)+\frac{d}{dx}G(x) = f(x) + g(x)$. So $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$, and hence

$$\int_a^b \big(f(x) + g(x)\big)\,dx = F(b) + G(b) - (F(a) + G(a))$$

$$= (F(b) - F(a)) + (G(b) - G(a)) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

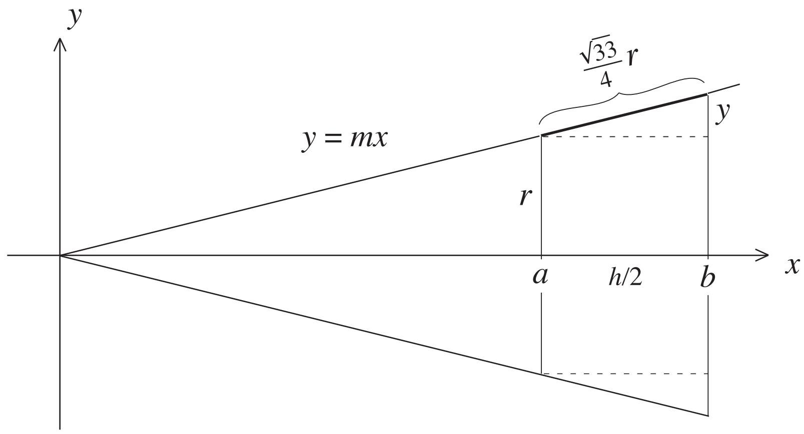
5.66. Turn to Kepler's model of the Austrian barrel as depicted in Figure 5.39a. To model the Austrian barrel, Kepler started with a cylinder that has height to diameter ratio $\frac{h}{2r} = \sqrt{2}$. So Kepler relied on a "Kepler cylinder." He then specified the length of the slanting segment of the model of the barrel to be $\frac{3}{2}r$. This resulted in a model with a rather wide girth (the ample vertical circular cross section at the center of the figure). The idea behind Problem 5.66 is to reduce this girth by taking the length of the slanting segment to be $\frac{4}{3}r$ while retaining the central cylinder that Kepler had used. A look at Figure 5.39a tells us that the length of such a slanting segment has to exceed $\frac{h}{2} = \sqrt{2}\,r$. However, since $\frac{16}{9} < 2$ and hence $\frac{4}{3} < \sqrt{2}$, this means that the choice $\frac{4}{3}r$ in the text's formulation of Problem 5.66 is too small. We'll correct this by letting the length of the slanting segment be $\frac{\sqrt{33}}{4}r$ instead. Since $\frac{33}{16} > 2$, we see that $\frac{\sqrt{33}}{4}r > \sqrt{2}r$ as required. Also, since $\frac{33}{16} < \frac{9}{4}$, the segment's length of $\frac{\sqrt{33}}{4}r$ is less than the $\frac{3}{2}r$ that Kepler had taken.

The model of the barrel that this slanting segment determines is smaller around the middle than Kepler's model of the Austrian barrel. It is depicted below. Recall from Section 5.5 that the wine merchants' $0.6s^3$ rule for measuring the volume of a barrel is closely tied to the volume of a Kepler cylinder. The fact that this new model of a



barrel is closer to being a Kepler cylinder suggests that the merchants' method for assessing volume should be more accurate for it than for Kepler's model of the Austrian barrel.

The volume of this "leaner" Austrian barrel can be calculated in the same way that the volume of Kepler's version was calculated in Section 5.9. In reference to the figure above, the height h of the barrel and the radius r of its circular cross sections at the two ends continue to satisfy the condition $\frac{h}{2r} = \sqrt{2}$ or $\frac{h}{2} = \sqrt{2}r$. As already specified, the length of the slanting segment of each cutoff cone is now $\frac{\sqrt{33}}{4}r$. To determine the volume V of this barrel design turn to the figure below. Let $y = mx$ be the line through the origin with slope m, and consider the segment of this line over an interval $0 < a \leq x \leq b$. The first important question is: For what m, a, and b does this segment



when revolved around the x-axis, generate the left half of the barrel shape depicted in the earlier figure?

Since h is the height of the barrel, $b-a = \frac{h}{2} = \sqrt{2}\,r$. So by the Pythagorean theorem $\left(\frac{\sqrt{33}}{4}r\right)^2 = y^2 + (\sqrt{2}r)^2$ where y is the length of the short vertical segment in the figure. Therefore $y^2 = (\frac{33}{16} - 2)r^2 = \frac{1}{16}r^2$. So $y = \frac{1}{4}r$ and hence $m = \frac{\frac{1}{4}r}{\sqrt{2}r} = \frac{1}{4\sqrt{2}}$. Since $m = \frac{r}{a}$, we have determined that

$$a = \tfrac{r}{m} = 4\sqrt{2}\,r \quad \text{and} \quad b = a + \sqrt{2}\,r = 5\sqrt{2}\,r.$$

Feeding these data into the volume of revolution formula of Section 5.9 and multiplying by 2 (to pick up both cutoff cones), we get that the volume V of this revised model of the Austrian barrel satisfies

$$\begin{aligned} V &= 2\pi\int_a^b (mx)^2 dx = 2\pi m^2\left[\tfrac{x^3}{3}\Big|_{4\sqrt{2}r}^{5\sqrt{2}r}\right] = \tfrac{2\pi m^2}{3}\left[125\cdot 2\sqrt{2}\,r^3 - 64\cdot 2\sqrt{2}\,r^3\right] \\ &= \tfrac{2\pi m^2}{3}122\sqrt{2}\,r^3 = \tfrac{2\pi}{3}\tfrac{1}{32}122\sqrt{2}\,r^3 = \tfrac{61}{24}\sqrt{2}\,\pi r^3. \end{aligned}$$

As expected, this volume is less than the volume $V = \frac{19}{6}\sqrt{2}\pi r^3$ of Kepler's model of the Austrian barrel.

From the figure above we see that the diagonal length s that the wine merchant's assessment is based on satisfies

$$s^2 = (2r+y)^2 + \left(\tfrac{h}{2}\right)^2 = (2r + \tfrac{1}{4}r)^2 + (\sqrt{2}r)^2 = \left(\tfrac{9}{4}r\right)^2 + 2r^2 = (2 + \tfrac{81}{16})r^2 = \tfrac{113}{16}r^2.$$

So $r = \frac{4}{\sqrt{113}}s$ and hence $r^3 = \frac{64}{113\sqrt{113}}s^3$. Therefore the volume of the revised model of the Austrian barrel in terms of the measure s is equal to

$$V = \tfrac{61}{24}\sqrt{2}\pi r^3 = \tfrac{61}{24}\sqrt{2}\pi\cdot\tfrac{64}{113\sqrt{113}}s^3 = \tfrac{61}{3}\sqrt{2}\pi\cdot\tfrac{8}{113\sqrt{113}}s^3 \approx 0.60s^3.$$

So the wine merchant's assessment $V_{\text{rule}} = 0.6s^3$ of the volume of this barrel provides a very close estimate of the actual volume of the barrel (closer than the one it provides for Kepler's model, because for it $V \approx 0.59s^3$).