Solutions to Problems and Projects for Chapter 2

We'll start with some problems that investigate some basics about parabolas and ellipses.

2.1. **i.** Since V is on the parabola, the distance from V to the directrix is d. Let Q be a point on the parabola such that FQ is parallel to the directrix. So the distance from Q to the directrix is $2d$. Hence the distance from Q to F is equal to $2d$. Hence Q is the point specified in Figure 2.36a.

ii. By Proposition P1, the tangent to the parabola at V is perpendicular to the directrix. An application of Proposition P2 to Figure 2.36a, where the points P and Q take the place of S and C, tells us that $\frac{d}{y} = \frac{(2d)^2}{x^2}$. So $\frac{y}{d} = \frac{x^2}{4d^2}$, and hence $y = \frac{1}{4d}x^2$.

2.2. Consider the vertical axis of the ellipse through O. This is the focal axis of the ellipse. By Proposition E1 the tangents at the upper and lower points of intersection of the ellipse with its focal axis are both perpendicular to the focal axis. By applying Proposition E2 to Figure 2.36b with the point P taking the place of C, we get that $\frac{(a+y)(a-y)}{x^2}$ does not depend on the value of y. Since $x = b$ when $y = 0$, it follows that $\frac{(a+y)(a-y)}{x^2} = \frac{a^2}{b^2}$. Therefore, $a^2 - y^2 = \frac{a^2}{b^2} \cdot x^2$ and hence $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

2.3. Suppose that the mirror is a part of the ellipse of Figure 2.7. Let a light ray emanate from the focal point F_1 and suppose that it strikes the mirror at the point P. The basic property of a light ray striking a mirror is that the angle of incidence is equal to the angle of reflection. Combining this with the assertion of Proposition E1 tells us that the elliptical mirror reflects the light ray from P to F_2. So any light ray that emanates from F_1 and strikes the mirror is reflected back through F_2. Similarly, if a powerful sound source is placed near an elliptical screen in such a way that the sound source is at one focus and a patient's kidney stone at the other, then the sound waves that hit the screen will be deflected to reconverge at the kidney stone. If all goes according to plan, this reconvergence will be powerful enough to shatter the stone.

2.4. The trapezoid of Figure 2.37a is divided into two triangles of height h. One has base a, the other, base b. The sum of their areas is $\frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}(a+b)h$. Another approach involves the observation that the trapezoid consists of a rectangle with base b and height h and (once it is removed), a triangle with base $a - b$ and height h. The sum of the two areas is $bh + \frac{1}{2}(a-b)h = \frac{1}{2}(a+b)$.

2.5. Notice that $\angle DAE + \angle CAB = \frac{\pi}{2}$ and that $\angle DAE + \angle EAB + \angle CAB = \pi$. So $\angle EAB + \frac{\pi}{2} = \pi$ and therefore $\angle EAB = \frac{\pi}{2}$. Using the fact that the quadrilateral $CBED$ consists of three triangles, tells us that its area is equal to $\frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 = ab + \frac{1}{2}c^2$. Since $CBED$ is a trapezoid with parallel lines of lengths a and b that are a distance $a+b$ apart, the area of $CBED$ is also equal to $\frac{1}{2}(a+b)(a+b)$ (using the conclusion of Problem 2.4.). Since this last term is equal to $ab + \frac{1}{2}(a^2+b^2)$, it follows that $a^2+b^2 = c^2$.

The derivation of Heron's formula that Problems 2.6, 2.7, and 2.8 outlines relies on the triangle T sketched in Figure 2.38.

2.6. Let $\alpha = \angle CAB$. Let h be the height of T relative to the base AB and notice that $\sin\alpha = \frac{h}{b}$. Since $AB = c$, the area of T is $\frac{1}{2}ch = \frac{1}{2}bc\sin\alpha$. If $\alpha = \angle CAB$ is obtuse, then by Figure 1.28, the discussion that follows it, and similar triangles, the equality $\sin\alpha = \frac{h}{b}$ still holds. Therefore the conclusion that $\frac{1}{2}bc\sin\alpha$ is the area of T does as well.

2.7. By the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos\alpha$. So $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$. Hence $\sin^2\alpha = 1 - \left[\frac{b^2+c^2-a^2}{2bc}\right]^2 = \frac{(2bc)^2-(b^2+c^2-a^2)^2}{(2bc)^2}$, and therefore, $\sin\alpha = \frac{\sqrt{(2bc)^2-(b^2+c^2-a^2)^2}}{2bc}$. By sliding this into the conclusion of Problem 2.6, we get area $T = \frac{1}{4}\sqrt{(2bc)^2 - (b^2+c^2-a^2)^2}$.

2.8. In reference to the steps,

$$\begin{aligned}
\text{area } T &= \tfrac{1}{4}\sqrt{[2bc + (b^2+c^2-a^2)][2bc - (b^2+c^2-a^2)]} \\
&= \tfrac{1}{4}\sqrt{[(b+c)^2 - a^2)][a^2 - (b-c)^2]} \\
&= \tfrac{1}{4}\sqrt{[(b+c)+a)][(b+c)-a)][a-(b-c)][a+(b-c)]} \\
&= \sqrt{\tfrac{1}{2}[(b+c)+a)]\cdot\tfrac{1}{2}[(b+c)-a)]\cdot\tfrac{1}{2}[a-(b-c)]\cdot\tfrac{1}{2}[a+(b-c)]} \\
&= \sqrt{s(s-a)(s-b)(s-c)},
\end{aligned}$$

the first applies the identity $x^2 - y^2 = (x+y)(x-y)$ with $x = 2bc$ and $y = (b^2+c^2-a^2)$ to the area formula of Problem 2.7; the second is verified by multiplying $(b+c)^2$ and $(b-c)^2$ out; the third step uses $x^2 - y^2 = (x+y)(x-y)$ twice (first with $x = (b+c)$ and $y = a$, and again with $x = a$ and $y = (b-c)$); the fourth step brings $\frac{1}{4}$ inside the radical as $\frac{1}{16} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$; and the fifth uses $s = \frac{1}{2}(a+b+c)$.

We already know that the conclusion of Problem 2.6 does not depend on the assumption that α is acute. A look at the solutions of Problems 2.7 and 2.8 (as described above) shows that neither of them depends on the assumption that α is acute..

2.9. Letting $s = \frac{1}{2}(5+7+10) = 11$ in Heron's formula, we get that the area of the triangle is $\sqrt{11(11-5)(11-7)(11-10)} = \sqrt{11\cdot 6\cdot 4} = \sqrt{264} \approx 16.25$.

2.10. The bisector of the angle at the upper vertex is perpendicular to the base of the triangle. (Why?) So $\sin 60° = \frac{h}{2}$, where the is the height h of the triangle. Since $\sin 60° = \frac{\sqrt{3}}{2}$, $h = \sqrt{3}$ and hence the area of the triangle is $\sqrt{3}$. Since the three angles of the triangle add to $180°$, the areas of the triangular sectors at the vertices add to one-half of the area of a circle of radius 1. It follows that the area of the three-pointed star is $\sqrt{3} - \frac{\pi}{2}$. Turning to the square, notice that the sectors at the four corners add to a full circle of radius 1. So the area of the four-pointed star inside the square is $4 - \pi$.

2.11. By a formula in Section 2.2, the area of the sector of the circle that θ (in radians) determines is $\frac{1}{2}r^2\theta$. By the solution of Problem 1.9, the center O of the circle lies on the perpendicular bisector of the segment AB. It follows that this perpendicular bisector splits the angle θ into two equal halves. This tells us that $\sin\frac{\theta}{2} = \frac{\frac{1}{2}AB}{r}$ and that $\cos\frac{\theta}{2} = \frac{h}{r}$ where h is the height of the triangle with base AB. Therefore the area of this triangle is $r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. It follows that the shaded section of the circle in Figure 2.40 has area $\frac{1}{2}r^2\theta - r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. By the half-angle formula of Problem 1.26i, $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$.

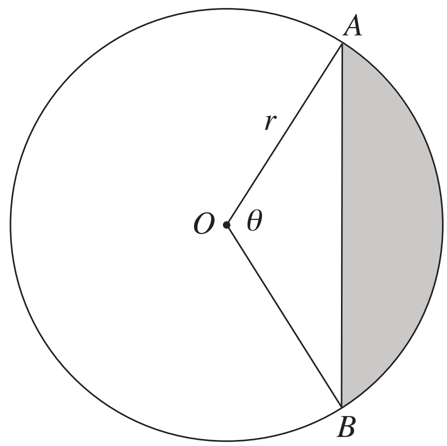


Fig. 2.40

So this area is equal to $\frac{1}{2}r^2(\theta - \sin\theta)$.

2.12. If $r = 7$ and $\theta = 50^\circ$, then (after converting θ to radians) the area is equal to $\frac{1}{2}7^2\big(50^\circ\frac{\pi}{180^\circ} - \sin(50^\circ\frac{\pi}{180^\circ})\big) \approx 2.61$. In the case $r = 5$ and arc $AB = 8$, $\theta = \frac{8}{5}$. In this situation, the area is $\frac{1}{2}5^2\big(\frac{8}{5} - \sin\frac{8}{5}\big) \approx 7.51$.

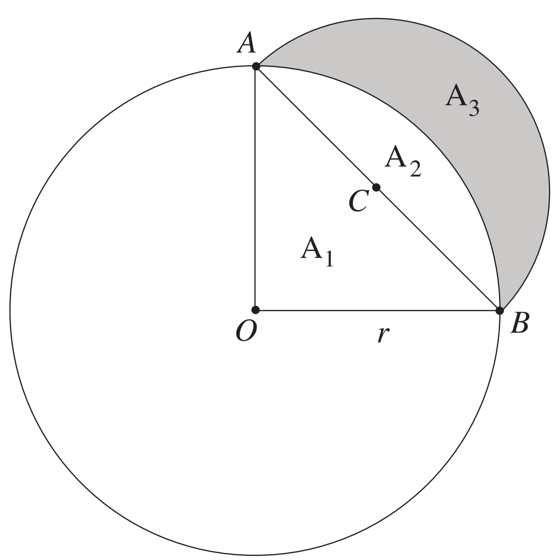


Fig. 2.41

2.13. Let's follow the hint. Figure 2.41 above tells us via the Pythagorean theorem, that $AB = \sqrt{2}r$. So the half-circle with diameter AB has area $\frac{1}{2}\pi\big(\frac{\sqrt{2}r}{2}\big)^2 = \frac{1}{4}\pi r^2$. It follows that if A_1 is the area of ΔAOB, A_3 is the area of the shaded moon shape, and A_2 is the area of the region between them, then $A_1 + A_2 = A_2 + A_3$. So $A_1 = A_3$.

2.14. Continue to refer to Figure 2.41. The area formula of Problem 2.11 with $\theta = \frac{\pi}{2}$ shows that the area A_2 is equal to $\frac{1}{2}r^2(\frac{\pi}{2} - 1)$. Since the area of the shaded lune is equal to the area of the semicircle with diameter AB minus the area A_2, we get $\frac{1}{2}\pi\big(\frac{\sqrt{2}r}{2}\big)^2 - A_2 = \frac{1}{4}\pi r^2 - \frac{1}{2}r^2(\frac{\pi}{2} - 1) = \frac{1}{2}r^2$ for the area of the lune. But this is equal to the area of the right triangle.

2.15. Turn to Figure 2.42 and observe:

i. The sum of the areas of the two smaller semicircles is $\frac{1}{2}\pi(\frac{a}{2})^2 + \frac{1}{2}\pi(\frac{b}{2})^2$. Adding the area of the right triangle gives us

$$\tfrac{1}{2}\pi(\tfrac{a}{2})^2 + \tfrac{1}{2}\pi(\tfrac{b}{2})^2 + \tfrac{1}{2}ab = \tfrac{1}{2}\big[\pi(\tfrac{a}{2})^2 + \pi(\tfrac{b}{2})^2 + ab\big]$$

Subtracting from this the area of the semicircle of diameter c and using the fact that $c^2 = a^2 + b^2$, tells us that the sum of the areas of the shaded lunes is

$$\tfrac{1}{2}\big[\pi(\tfrac{a}{2})^2 + \pi(\tfrac{b}{2})^2 + ab\big] - \tfrac{1}{2}\pi(\tfrac{c}{2})^2 = \tfrac{1}{2}ab$$

as we needed to show.

ii. The area of the right triangle with base c is greatest when its height is greatest. A look at Figure 2.42 show that this is so when $a = b$.

iii. Letting b shrink to 0 pushes a to c and $\frac{1}{2}ab$ to $\frac{1}{2}c \cdot 0 = 0$. So the sum of the areas of the lunes can be made as small as desired. So there is no smallest area. (The stipulation that there needs to be a triangle rules out the case $b = 0$.)

2.16. Since AOB is a diameter of a circle and C is on the circle, ΔABC is a right triangle by a result in Section 1.3. Figure 2.43a denotes the lengths AP and PB by a and b respectively. The radius of the circle is $r = \frac{a+b}{2}$. The distance from C to the vertical radius of the circle is $x = r - b = \frac{a}{2} + \frac{b}{2} - b = \frac{a-b}{2}$ and $y = PC$ is the height of the triangle ΔABC.

i. In the special case of a circle, the semimajor and semiminor axes of an ellipse are both equal to the radius of the circle. Since the radius of the circle of the figure is $r = \frac{a+b}{2}$, an application of Example 2.1 tells us that $x^2 + y^2 = (\frac{a+b}{2})^2$. Since $x = \frac{a-b}{2}$, we get $y^2 = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2 = \frac{2ab}{4} + \frac{2ab}{4} = ab$. Therefore, $y = \sqrt{ab}$.

The triangles ΔACP and ΔCBP are both similar to ΔABC. This is so because ΔABC is a right triangle and because each of these two right triangles shares an acute angle with ΔABC. Since are ΔACP and ΔCBP are similar to each other, the ratio $\frac{y}{b}$ is equal to the ratio $\frac{a}{y}$. So $y^2 = ab$ and hence $y = \sqrt{ab}$.

By applying the Pythagorean Theorem to each of the triangles ΔACP, ΔCBP, and ΔABC, we get $AC^2 = y^2 + a^2, CB^2 = y^2 + b^2$, and $(a+b)^2 = AC^2 + CB^2$. So $(a+b)^2 = y^2 + a^2 + y^2 + b^2$. Therefore $2ab = 2y^2$, and $y = \sqrt{ab}$ follows once more.

ii. That the area of the triangle ΔABC is $\frac{1}{2}(a+b)y = \frac{1}{2}\sqrt{ab}(a+b)$ follows from (i).

iii. The shaded area of Figure 2.43b is equal to $\frac{1}{2}\pi(\frac{a+b}{2})^2 - \frac{1}{2}\pi(\frac{a}{2})^2 - \frac{1}{2}\pi(\frac{b}{2})^2 = \frac{1}{2}\pi\left[\frac{(a+b)^2}{4} - \frac{a^2}{4} - \frac{b^2}{4}\right] = \frac{1}{2}\pi\frac{ab}{2} = \pi(\frac{y}{2})^2$. But this is equal to the area of the circle with diameter CP.

2.17. Turn to Figure 2.44 and let $AC = DB = a$ and $CD = b$. Using the fact that a semicircle of diameter d has area $\frac{1}{2}\pi(\frac{d}{2})^2 = \frac{1}{8}\pi d^2$, we get that the area of the curving

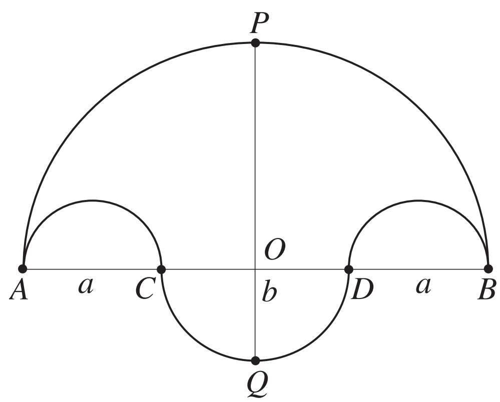


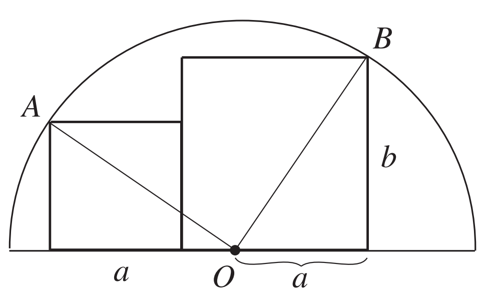
Fig. 2.44

figure bounded by the three semicircles is equal to

$$\frac{1}{8}\pi(2a+b)^2 - 2\cdot\frac{1}{8}\pi a^2 + \frac{1}{8}\pi b^2 = \frac{1}{8}\pi(4a^2+4ab+b^2-2a^2+b^2) = \frac{1}{8}\pi(2a^2+4ab+2b^2) = \pi\frac{(a+b)^2}{4}.$$

Since $PQ = a + \frac{b}{2} + \frac{b}{2} = a + b$, the circle of diameter PQ also has area $\pi(\frac{a+b}{2})^2$.

2.18. With O placed on the diameter by marking off a distance a from the lower right vertex of the square on the right, draw in the two segments OA and OB from O to the upper corners of the two squares as in the figure below. By the Pythagorean theorem, the lengths of OB and OA are $\sqrt{a^2+b^2}$ and $\sqrt{(a+(b-a))^2+a^2} = \sqrt{a^2+b^2}$ and hence equal. Let C be the center of the circle. So CA and CB are both equal to r. That $C = O$



follows easily. If C were to lie to the left of O, then $r = CA < OA = OB < CB = r$, a contradiction. In the same way, C cannot lie to the right of O. Since $r^2 = a^2 + b^2$, it follows that the ratio of the area of the semicircle over the sum of the areas of the two rectangles is $\frac{\frac{1}{2}\pi r^2}{r^2} = \frac{\pi}{2}$.

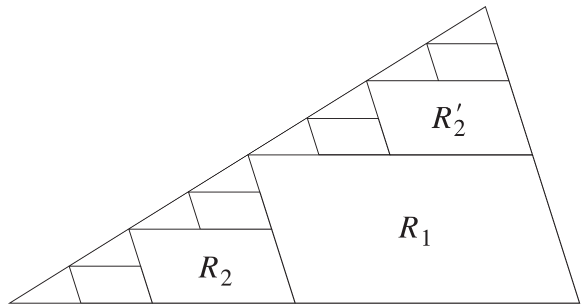
2.19. Substituting successively 1, 2, 3, and 4, for i in the expression $\displaystyle\sum_{i=1}^{4} i$ and adding after each substitution gives us the sum $1+2+3+4$. Doing a similar thing for $\displaystyle\sum_{i=2}^{5} i^2$ gives $2^2+3^2+4^2+5^2$, doing so with $\displaystyle\sum_{i=3}^{6} i^i$ gives us the sum $3^3+4^4+5^5+6^6$, and, finally, doing this with $\displaystyle\sum_{i=4}^{6} i^{2i}$ gives the sum $4^8+5^{10}+6^{12}$.

2.20. The black areas of the four squares (from left to right) are $\frac{1}{2}$, $\frac{1}{2}+\frac{1}{4}$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ and $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ or, in rewritten form, $\frac{1}{2}$, $\frac{1}{2}+\frac{1}{2^2}$, $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}$ and $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$, or again, $\displaystyle\sum_{i=1}^{1}\left(\tfrac{1}{2}\right)^i, \sum_{i=1}^{2}\left(\tfrac{1}{2}\right)^i, \sum_{i=1}^{3}\left(\tfrac{1}{2}\right)^i$, and $\displaystyle\sum_{i=1}^{4}\left(\tfrac{1}{2}\right)^i$. By repeating this for a large number of steps, say $n =$ 1,000,000 or more, we see that the black area $\displaystyle\sum_{i=1}^{n}\left(\tfrac{1}{2}\right)^i$ grows imperceptibly close to the area of the 1×1 square. Letting n go to ∞, we get $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\tfrac{1}{2}\right)^i = 1$.

2.21. Is it the case that if the pattern of parallelograms of Figure 2.47 is continued, then each point of the triangle will eventually lie in one of these parallelograms? Does the infinite array of parallelograms fill out the entire triangle? Put another way, is the sum of the areas of all these parallelograms equal to the area of the triangle? The parallelogram R_1 with base $\frac{1}{2}b$ has height $\frac{1}{2}h$ since the triangle above it is similar to the given triangle. So R_1 has area $\frac{1}{4}bh$. The two parallelograms of the next tier both have base $\frac{1}{2}\cdot\frac{1}{2}b = \frac{1}{4}b$ and height $\frac{1}{2}\cdot\frac{1}{2}h = \frac{1}{4}h$ (again, because the triangles above the two parallelograms are similar to the original triangle). So the area of each these parallelograms is $\frac{1}{16}bh$. Since there are two of them, the area that is added by the second tier is $\frac{1}{8}bh$. The area that the four parallelograms of the next tier add is $4 \times (\frac{1}{8}b)(\frac{1}{8}h) = \frac{1}{16}bh$. In the same way, the area that each tier of parallelograms adds is $\frac{1}{2}$ the area of those of the previous tier. The unfolding pattern tells us that the area of the triangle is equal to

$$\tfrac{1}{4}bh + \tfrac{1}{8}bh + \tfrac{1}{16}bh + \tfrac{1}{32}bh + \ldots \;=\; \tfrac{1}{2}bh(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots).$$

In view of the conclusion of Problem 2.20 this is equal to $\frac{1}{2}bh$. So the sum of the areas of all the parallelograms of the infinite array is equal to the area of the triangle. In the case of an acute triangle do the same thing by working with the figure shown below.



2.22. Notice first that the shaded squares together yield $\frac{1}{3}$ of the 1×1 square. On the other hand the areas of these squares add to

$$(\tfrac{1}{2})^2 + (\tfrac{1}{4})^2 + (\tfrac{1}{8})^2 + (\tfrac{1}{16})^2 + \cdots = \tfrac{1}{2^2} + \tfrac{1}{2^4} + \tfrac{1}{2^6} + \tfrac{1}{2^8} + \cdots = \tfrac{1}{4} + \tfrac{1}{4^2} + \tfrac{1}{4^3} + \tfrac{1}{4^4} + \ldots$$

This verifies the result.

2.23. The identity $S_n = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$ holds for any positive integer n and any real number $x \neq 1$. That $\lim\limits_{n\to\infty} S_n = \frac{1}{1-x}$ when $|x| < 1$ is established in the narrative of the problem. So the sum $1 + x + x^2 + \cdots + x^{n-1} + \cdots$ of infinitely many terms adds up to the finite number $\frac{1}{1-x}$. For $x = 1$, $S_n = n$, and for $x = -1$, $S_n = 1$ when n is odd and $S_n = 0$ when n is even. In neither case does S_n add to a fixed finite number when n is pushed to infinity. The same is true when $|x| > 1$, because in this case, the absolute value $|x^n|$ of x^n keeps getting larger and larger when when n is pushed to infinity.

2.24. Since $a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots = a(1+r+r^2+r^3+\cdots+r^{n-1}+\cdots)$ and $|r| < 1$, it follows from Problem 2.23, that $a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots = a\big(\frac{1}{1-r}\big) = \frac{a}{1-r}$.

2.25. The sum $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is of the form $1 + x + x^2 + x^3 + \cdots$ with $x = \frac{1}{2}$. So by Problem 2.23, the sum $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ adds to the finite number $\frac{1}{1-x} = \frac{1}{1-\frac{1}{2}} = 2$. It follows that $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ adds to 1. For $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$, take $x = \frac{1}{4}$ to see that $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{1-x} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$.

2.26. The information provided tells us that the triangle inscribed in the parabolic section has base 7 and height 4. So the area of the triangle is $\frac{1}{2} \cdot 7 \cdot 4 = 14$. Therefore, by Archimedes's theorem, the area of the parabolic section is $\frac{4}{3} \cdot 14 = \frac{64}{3}$.

2.27. The distance from the vertex to the cut is the height h of the inscribed triangle. So the area of the inscribed triangle is $\frac{1}{2}5h$. By Archimedes's theorem, $16 = \frac{4}{3} \cdot \frac{5}{2}h = \frac{10}{3}h$. So $h = \frac{48}{10} = \frac{24}{5}$.

2.28. The height h of the triangle in Figure 2.49 satisfies, $\sin\alpha = \frac{h}{c}$. So the area of the triangle is $\frac{1}{2}bc\sin\alpha$. Therefore by Archimedes's theorem, the area of the parabolic section ABC is $\frac{4}{3} \cdot \frac{1}{2}bc\sin\alpha = \frac{2}{3}bc\sin\alpha$.

2.29. Refer to Figure 2.50. Since the points S and S' on the parabola are 7 units from the directrix, they are also both 7 units from the focal point F. Consider the triangle $\Delta SFS'$ and notice that its height is $7-3=4$. Since $\Delta SFS'$ is isosceles, the focal axis divides the triangle $\Delta SFS'$ into two congruent right triangles. By applying the Pythagorean theorem, we get $(\frac{1}{2}SS')^2 = 7^2 - 4^2$ and $\frac{1}{2}SS' = \sqrt{33}$. The length of the cut SS' is $2\sqrt{33}$. Let V be the vertex of the parabolic section. Since $FV = \frac{3}{2}$, it follows that the area of $\Delta SVS'$ is equal to $\sqrt{33}(4+\frac{3}{2}) = \frac{11}{2}\sqrt{33}$. Hence the area of the parabolic section is $\frac{4}{3}\cdot\frac{11}{2}\sqrt{33} = \frac{22}{3}\sqrt{33}$.

2.30. Consider Figure 2.51. By applying Proposition E1 to the circle (so that both focal points are at the center O), we get that the radius OV is perpendicular to the tangent at V. By the conclusions of Problem 1.9, O lies on the perpendicular bisector of SS'. Since $\angle OSS'$ and $\angle OS'S$ are equal, it follows that $\angle SOV$ and $\angle S'OV$ are also equal. So the perpendicular bisector of SS' coincides with the radius OV. Therefore, the tangent at V and the segment SS' are both perpendicular to OV.

i. The area A of the circular section SVS' is the difference between the area of the sector $OSVS'$ and the triangle $\Delta SS'O$. So $\mathrm{A} = \frac{1}{2}\theta r^2 - r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. By the half-angle formula of Problem 1.26i, $\mathrm{A} = \frac{1}{2}r^2(\theta - \sin\theta)$. Consider the triangle $\Delta SVS'$. Its height relative to the base SS' is $r - r\cos\frac{\theta}{2}$. It follows that its area B is $(r\sin\frac{\theta}{2})(r - r\cos\frac{\theta}{2}) = r^2(\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}) = \frac{1}{2}r^2(2\sin\frac{\theta}{2} - \sin\theta)$.

ii. In view of i, $\frac{\mathrm{A}}{\mathrm{B}} = \frac{\theta - \sin\theta}{2\sin\frac{\theta}{2} - \sin\theta}$. With $\theta = \pi$, $\sin\theta = 0$, $\sin\frac{\theta}{2} = 1$, so that $\frac{\mathrm{A}}{\mathrm{B}} = \frac{\pi}{2}$. With $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, $\sin\frac{\theta}{2} = \frac{\sqrt{2}}{2}$, so that $\frac{\mathrm{A}}{\mathrm{B}} = \frac{\frac{\pi}{2}-1}{\sqrt{2}-1}$.

2.31. The distance d_2 has to satisfy $5\cdot 3 = 7\cdot d_2$. So $d_2 = \frac{15}{7}$.

2.32. On the one hand, $80\cdot d_1 = 15\cdot d_2$ and on the other, $d_1 + d_2 = 9$. So $80\cdot d_1 = 15\cdot(9-d_1)$. Therefore, $95d_1 = 135$. Hence $d_1 = \frac{135}{95} = \frac{27}{19} = 1\frac{8}{19}$ and $d_2 = 9 - \frac{27}{19} = \frac{171-27}{19} = \frac{144}{19} = 7\frac{11}{19}$.

2.33. Go through the proof and note that the location of XZ is irrelevant.

2.34. Let h be the distance (in feet) from the water level to the bottom of the block. So the volume of the submersed part of the block is $(0.6)(0.6)h = 0.36h$. It follows that $0.36h = 0.16$ and hence that $h = \frac{0.16}{0.36} \approx 0.44$ feet. Because $\frac{0.44}{0.6} \approx 0.73$, about 73% of the block is submersed. Note the misprint in the statement of the problem: it should be “along the 0.6-foot height of the block” not “along the 1-foot height of the block”.

2.35. The weight of the water that volume of 5,000 cubic feet displaces is $5000 \times 62.5 =$ 312,500 pounds. This is the maximum upward force on the boat. Since he hull weighs 100,000 pounds, this leaves a maximum of 212,000 pounds for the cargo.

2.36. In the situation where $w_1 = \frac{1}{3}w$ and hence $w_2 = \frac{2}{3}w$, the gold in the crown will lose $\frac{1}{3}f_1$ pounds and the silver will lose $\frac{2}{3}f_2$ pounds. Therefore $f = \frac{1}{3}f_1 + \frac{2}{3}f_2$. Since

$$\frac{f_2-f}{f-f_1} = \frac{f_2-(\frac{1}{3}f_1+\frac{2}{3}f_2)}{(\frac{1}{3}f_1+\frac{2}{3}f_2)-f_1} = \frac{\frac{1}{3}f_2-\frac{1}{3}f_1}{\frac{2}{3}f_2-\frac{2}{3}f_1} = \frac{1}{2} = \frac{w_1}{w_2}.$$

If $w_1 = cw$, then $w_2 = w - w_1 = (1-c)w$. Now the gold in the crown will lose cf_1 pounds and the silver $(1-c)f_2$ pounds. So $f = cf_1 + (1-c)f_2$, and hence

$$\frac{f_2-f}{f-f_1} = \frac{f_2-(cf_1+(1-c)f_2)}{(cf_1+(1-c)f_2)-f_1} = \frac{cf_2-cf_1}{(1-c)f_2-(1-c)f_1} = \frac{c}{1-c} = \frac{w_1}{w_2}.$$

2.37. Continue to let the pound be the unit of weight and let the cubic foot be the unit of volume. With the crown displacing v pounds per cubic foot, the upward force of the water on the immersed crown is $62.5v$ pounds. In the same way the upward forces on the immersed lumps of gold and silver are $62.5v_1$ and $62.5v_2$ pounds, respectively. Inserting this information into the discussion of Problem 2.36, we get $f = 62.5v, f_1 = 62.5v_1$, and $f_2 = 62.5v_2$ pounds, respectively. If the crown is made of w_1 pounds of gold and w_2 pounds of silver, then by a substitution, $\frac{w_1}{w_2} = \frac{v_2-v}{v-v_1}$.

2.38. Let w_1 and w_2 be the the amounts of gold and silver in the crown in pounds. So $w_1 + w_2 = 3$. Since 0.698 pounds of gold displaces 1 cubic inch, 1 pound of gold displaces $\frac{1}{0.698}$ cubic inches and 3 pounds displace $\frac{3}{0.698} \approx 4.30$ cubic inches. Similarly, 3 pounds of silver displace $\frac{3}{0.379} \approx 7.92$ cubic inches. Inserting $v = 5, v_1 = 4.30$, and $v_2 = 7.92$ into the formula that Problem 2.37 provides, we get that $\frac{w_1}{w_2} \approx \frac{2.92}{0.70} \approx 4.17$. So $4.17w_2 + w_2 \approx 3$, hence $w_2 \approx \frac{3}{5.17} \approx 0.58$ pounds and $w_1 \approx 2.42$ pounds. Since $\frac{2.42}{3} \approx 0.81$, about 81% of the crown is gold.

2.39. $85 = \pi\varepsilon$, $842 = \omega\mu\beta$, $34547 = \gamma\mathrm{M}͵\delta\phi\mu\zeta$, $2{,}875{,}739 = \sigma\pi\zeta\mathrm{M}͵\varepsilon\psi\lambda\theta$, and $99{,}999{,}999 =$

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2.40. For $n = 4$, $n^{n^2} = 4^{16} = 4{,}294{,}967{,}296$, with $n = 5$, $n^{n^2} = 5^{25} = 298{,}023{,}223{,}876{,}953{,}125$, and for $n = 6, n^{n^2} = 6^{36} = 10{,}314{,}424{,}798{,}490{,}535{,}546{,}171{,}949{,}056$. So even with the small numbers $n = 5$ and $n = 6$ Archimedes scheme is already huge.