ON THE COBLE QUARTIC

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Abstract. We obtain a short explicit expression for the universal Coble quartic whose partial derivatives give the defining equations for the universal family of Kummer threefolds. The Coble quartic was recently determined completely in [RSSS12], where (Theorem 7.1a) it was computed explicitly, as a polynomial with 372060 monomials of bidegree (28, 4) in theta constants of the second order and theta functions of the second order, respectively. Our expression is in terms of products of theta constants with characteristics corresponding to Göpel systems, and is a polynomial with 134 terms.

Our approach is based on the beautiful geometry studied by Coble [Cob29] and further investigated by Dolgachev and Ortland in [DO88], and highlights the geometry and combinatorics of syzygetic octets of characteristics, the GIT quotient for 7 points in $\mathbb{P}^2$, and the corresponding representations of $\text{Sp}(6, \mathbb{F}_2)$. One new ingredient is the relationship of Göpel systems and Jacobian determinants of theta functions.

In genus 2, we similarly obtain a short explicit equation for the universal Kummer surface, and relate modular forms of level two to binary invariants of six points on $\mathbb{P}^1$.

1. Introduction

In this paper we provide an explicit description for the (universal) Coble quartic. The universal Coble quartic is an equation in $16 = 8 + 8$ variables, such that its partial derivatives with respect to the second variables give the defining equations for the universal family of Kummer varieties. The variety defined by the universal Coble quartic is the moduli space of semistable vector bundles of rank 2 with trivial determinant on genus three curves cf. [NR87], and our work thus provides one of the very few instances where such a moduli space is described completely explicitly. Knowing the Coble quartic explicitly could thus be of further use in investigating the Hitchin integrable system and in particular the Hitchin connection for this case.

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More precisely, the Coble quartic is a bihomogeneous polynomial of bidegree (28,4) in \(16 = 8 + 8\) variables, which we think of as a hypersurface in \(\mathbb{P}^7 \times \mathbb{P}^7\). Here the first \(\mathbb{P}^7\) is the target of the embedding of the moduli space of abelian threefolds (with level structure, \(\mathcal{A}_3(2,4)\)) by theta constants of the second order. The second \(\mathbb{P}^7\) is the projectivization of the space of sections of the bundle \(|2\Theta|\) on the corresponding abelian threefold. We denote theta functions of the second order by

\[
x_\varepsilon := \Theta[\varepsilon](\tau, z) = \theta[\varepsilon_0](2\tau, 2z)
\]

for \(\varepsilon \in \mathbb{F}_2^3\); these are the homogeneous coordinates for the projective space that is the target of the Kummer map. Taking the eight partial derivatives of the Coble quartic with respect to the second set of variables gives the equations defining the Kummer variety of the corresponding abelian threefold.

The existence of the current paper is related to the appearance of the paper [RSSS12], and to the interest of its authors Qingchun Ren, Steven Sam, Gus Schrader, and Bernd Sturmfels in this circle of classical ideas in algebraic geometry. They study the defining ideal of the universal Kummer threefold in \(\mathbb{P}^7 \times \mathbb{P}^7\) (see [RSSS12, Conj. 8.6] for a complete conjectural description of this ideal), and give two explicit generators for the prime ideal of the universal Coble quartic: a form \(S\) of bidegree \((16, 0)\) and a form \(C\) of bidegree \((28, 4)\), i.e polynomials on \(\mathbb{P}^7 \times \mathbb{P}^7\) in theta constants and theta functions of the second order.

Their work inspired us to revisit and reconsider the classical constructions originating with Coble. The investigation of the Coble quartic originates with Coble himself, who in [Cob29, p. 106] gives an implicit equation for the quartic, writing it as

\[
\alpha Q_1 + 2\alpha_1 Q_2 + \ldots + 2\alpha_7 Q_8 + 4\alpha_{423} Q_9 + \cdots + 4\alpha_{456} Q_{15},
\]

where \(Q_i\) are certain quartics in theta functions of the second order, listed explicitly in section 9 of this paper, and about the coefficients \(\alpha\), he states [Cob29, p. 196]:

"The 15 coefficients \(\alpha\) of the quartic spread \(L^4\) in \(S_7\) can be expressed linearly with numerical coefficients in terms of the Göpel invariants of \(\mathbb{P}^7\) and conversely. . . ."

**Remark 1.** As the referee, correctly, pointed out the above cited claim has not been proved by Coble. In particular Coble proved that the 15 coefficients \(\alpha\) satisfy certain cubic equations, but he did not prove that these cubic equations define set-theoretically an irreducible 6-dimensional variety, the so-called Göpel variety [RSSS12]. In the first
version of this paper, we were using Coble’s statement without proof. Following the suggestions of the referee, we tried to obtain a completely geometric proof of this statement of Coble, but were not successful in proving this in a way significantly different from the approach of [RSSS12]. Thus to obtain our results we must use the above claim of Coble, which is an easy consequence of the elaborate computations done in [RSSS12, Thm. 5.1, Thm. 7.1], using the relations among theta constants related to the Prym map.

The aim of this paper is to follow Coble’s original idea and give an equation for the Coble quartic in this way, with explicit formulas for the coefficients $\alpha$. One new tool in our approach compared to [RSSS12] are the gradients at $z = 0$ of theta functions with odd characteristics. It is a classical result that these are related to bitangents of plane quartic curves, see [CS03b] and [CS03a] (and [DO88] for much more of the beautiful classical geometry). This classical relationship gives a geometric viewpoint of our paper, providing a connection (in fact constructing a homomorphism) between the ring of GIT invariants of seven points in $\mathbb{P}^2$ and the ring of modular forms of genus 3 and level 2. This allows us to express the “Göpel invariant” (or Fano Göpel coordinates in [RSSS12]) in terms of theta constants with characteristics. We then use representation theory of $\text{Sp}(6, \mathbb{F}_2)$ to construct the Coble quartic. Indeed, the image of the 15-dimensional vector space spanned by the coefficients $\alpha$ of the Coble quartic is a 15-dimensional space of modular forms of genus 3 and level 2, of weight 14, known to be related to Göpel systems, and for which we thus obtain expressions in terms of Göpel systems. We then perform a straightforward computation in Maple®, yielding the 15 coefficients $\alpha_\ldots$ above explicitly in terms of a basis for the space of modular forms given by Fano configurations — the resulting expression is an explicit polynomial with 134 monomials, presented in theorem 19. Of course the resulting expression agrees with the result of [RSSS12], though it is written in terms of different variables (theta constants with characteristics).

We note that the geometry for the case $g = 2$ in many aspects parallels the situation for $g = 3$, but is of course easier. We develop this story in parallel with the Coble quartic, and in theorem 11 determine explicitly the relation between the ring of modular forms of genus 2
and level 2, and the GIT quotient of 6 points on $\mathbb{P}^1$ (similar to the relationship of modular forms of genus 3 and level 3, and the GIT quotient of 7 points on $\mathbb{P}^2$ discussed above).

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\section{2. Combinatorics of theta characteristics}

In this section we recall some known facts about the action of the symplectic group $\text{Sp}(2g, \mathbb{F}_2)$ on the set of theta characteristics. The main references are the classical books on theta functions, e.g. Wirtinger [Wir95], Krazer [Kra03], Coble [Cob29], with a more modern treatment given by Igusa [Igu72], and many details that we need are investigated in [SM94]. Almost all of these facts are of course discussed in [RSSS12], but we collect here all that we need, as a potential convenient reference for theta constants, and use this to fix notation.

A \textit{theta characteristic} $m$ is an element of $\mathbb{F}_2^{2g}$ (where we think of the elements of $\mathbb{F}_2$ as being 0 and 1), which we will often write as $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$ with $m'$ and $m''$ considered as row vectors in $\mathbb{F}_2^g$. We define the parity as

\begin{equation}
\epsilon(m) := (-1)^{m' \cdot m''}
\end{equation}

and say that $m$ is even or odd according to whether $\epsilon(m)$ is equal to 1 or $-1$, respectively.

For any triple $m_1, m_2, m_3$ of characteristics we set

\begin{equation}
\epsilon(m_1, m_2, m_3) := \epsilon(m_1) \cdot \epsilon(m_2) \cdot \epsilon(m_3) \cdot \epsilon(m_1 + m_2 + m_3),
\end{equation}

\end{document}
and call a triple syzygetic or azygetic depending on whether this number is 1 or \(-1\), respectively.

The action of \(\text{Sp}(2g, \mathbb{F}_2)\) on the set of characteristics is given by

\[
\gamma \cdot m := \left( \begin{array}{cc} D & -C \\ -B & A \end{array} \right) \left[ \begin{array}{c} m' \\ m'' \end{array} \right] + \left( \begin{array}{c} \text{diag}(C'D) \\ \text{diag}(A'B) \end{array} \right) ,
\]

where as usual we write \(\gamma \in \text{Sp}(2g, \mathbb{F}_2)\) in the block form as \(\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)\). This affine-linear action preserves the parity, azygy/syzygy, and linear relations with an even number of terms. That is to say, the orbits of this action on tuples of characteristics are described as follows: there exists an element of \(\text{Sp}(2g, \mathbb{F}_2)\) mapping a sequence of characteristics \(m_1, \ldots, m_l\) to a sequence of characteristics \(n_1, \ldots, n_l\) if and only if for any \(1 \leq i, j, k \leq l\) we have \(e(m_i) = e(n_i), e(m_i, m_j, m_k) = e(n_i, n_j, n_k)\), and linear relations among \(m_i\)'s with an even number of terms are in bijection with such relations for the \(n_i\)'s, see [Igu72, Prop. 2, p. 212] or [SM94].

A fundamental system of characteristics is a set of \(2g + 2\) characteristics such that any triple is azygetic; we will consider fundamental systems as unordered sets of theta characteristics. By the above, two fundamental systems belong to the same \(\text{Sp}(2g, \mathbb{F}_2)\) orbit if and only if they contain the same number of odd characteristics. In fact, a fundamental system with \(k\) odd theta characteristics exists if and only if \(k\) is congruent to \(g\) modulo \(4\), see [Fay79]. A special fundamental system is a fundamental system containing \(g\) odd characteristics and \(g + 2\) even characteristics — all special fundamental systems form one \(\text{Sp}(2g, \mathbb{F}_2)\) orbit.

In this paper, we are mostly interested in the cases \(g = 2\) and \(g = 3\). For genus 2, fundamental systems consist of 6 characteristics, of which either 2 or 6 may be odd. The unique fundamental system with 6 odd characteristics is simply the set of all 6 odd characteristics:

\[
\begin{bmatrix}
01 \\
01 \\
11 \\
10 \\
10 \\
11
\end{bmatrix}.
\]

In genus 2, we also have the following easy combinatorial lemma.

**Lemma 2.** For any azygetic triple of odd characteristics \(m_1, m_2, m_3\) there exist a unique even characteristic \(n_0\) such that the quadruple \(n_0, m_1, m_2, m_3\) is azygetic, and it is given by \(n_0 = m_1 + m_2 + m_3\).

The rest of this section is devoted to the case \(g = 3\). Then fundamental systems consist of 8 characteristics, of which either 7 or 3 may
be odd. In the first case, we have fundamental systems of the form

\[ n_0, m_1, m_2, \ldots, m_7 \]

with \( n_0 \) even and all \( m_i \) odd. Classically, in this case the set of odd characteristics \( m_1, m_2, \ldots, m_7 \) is called an Aronhold set. We refer to [DO88] for a beautiful modern exposition of the theory and many classical and new results. Thus the group \( \text{Sp}(6, \mathbb{F}_2) \) acts transitively and faithfully on the collection of Aronhold sets, and the number of ordered different Aronhold sets is equal to \(|\text{Sp}(6, \mathbb{F}_2)| = 36 \cdot 8! = 1451520.\)

An example of an Aronhold set is the following:

\[
(5) \quad \begin{bmatrix} 111 \\ 111 \\ 100 \\ 001 \\ 110 \\ 011 \\ 010 \\ 011 \end{bmatrix} 
\]

This Aronhold set is completed to a fundamental system by adding the zero characteristic \( \begin{bmatrix} 000 \\ 000 \end{bmatrix} \).

We now collect some needed results about the combinatorics of Aronhold sets for \( g = 3 \):

**Lemma 3.** For any azygetic triple of odd characteristics \( m_1, m_2, m_3 \) there exist 6 even characteristics \( n_1, \ldots, n_6 \) such that the quadruple \( m_1, m_2, m_3, n_j \) is azygetic. One of these characteristics (label it \( n_6 \)) is equal to the sum \( n_6 = m_1 + m_2 + m_3 \). Then the set

\[ m_1, m_2, m_3, n_1, \ldots, n_5 \]

forms a special fundamental system. In particular, every azygetic triple of odd characteristics is contained in (and defines) a unique, up to permutations, special fundamental system.

**Lemma 4.** If \( m_1, m_2, m_3 \) and \( m_1, m_4, m_5 \) are two azygetic triples of odd characteristics (sharing one characteristic) that are subsets of a fundamental system \( n_0, m_1, m_2, \ldots, m_7 \), then the intersection of the special fundamental systems defined by \( m_1, m_2, m_3 \), and by \( m_1, m_4, m_5 \), respectively, consists of \( m_1 \) and \( n_0 \).

Since the group \( \text{Sp}(6, \mathbb{F}_2) \) acts transitively on azygetic triples of odd characteristics, it is enough to verify these lemmas for just one such odd triple \( m_1, m_2, m_3 \). This can be checked by hand, but more easily follows from the following combinatorial

**Lemma 5.** For any fundamental system \( n_0, m_1, \ldots, m_7 \), the other 21 odd characteristics are obtained as all possible sums \( n_0 + m_i + m_j \) with \( 1 \leq i < j \leq 7 \), while the 36 even characteristics are \( n_0 \) together with the 35 sums \( m_i + m_j + m_k \) with \( 1 \leq i < j < k \leq 7 \).
Remark 6. Explicitly, if an azygetic odd triple $m_1, m_2, m_3$ is part of the fundamental system $n_0, m_1, m_2, \ldots, m_7$, then the corresponding special fundamental system is given by

$$\{m_1, m_2, m_3, n_0, m_4 + m_5 + m_6, m_4 + m_5 + m_7, m_4 + m_6 + m_7, m_5 + m_6 + m_7\}.$$  

We define the symplectic form on $F_6^2$ by

$$e(m, n) := (-1)^{m'n'' - m''n'}.$$  

Then the number of Lagrangian subspaces of $F_6^2$, called Göpel systems, is equal to 135, see [SM85]. We call a Göpel system a Fano configuration if all 8 characteristics in it are even, and call it a Pascal configuration if four characteristics in the Lagrangian space are even, and four are odd. In genus 3 there are 30 Fano configurations and 105 Pascal configurations. In both cases, there exists a unique affine subspace of $F_6^2$ that is the translate of the Göpel systems, which consists only of even characteristics — for a Fano configuration, this is the Göpel system itself.

All Fano configurations can be obtained as follows: fix an Aronhold set $m_1, \ldots, m_7$. Then for any set of 7 triples of indices among $1, \ldots, 7$, such that any two triples have exactly one element in common, the seven corresponding triple sums $m_i + m_j + m_k$ form the set of non-zero elements of a Fano configuration, and all Fano configurations are obtained this way. For example, we could choose the 7 triples as

(6) \[ F := \{(123), (145), (167), (247), (256), (346), (357)\}, \]

so that for the Aronhold set example given in (5) the corresponding Fano configuration formed by zero and the above 7 elements is

(7) \[
\begin{bmatrix}
000 & 000 & 001 & 011 & 001 & 010 & 000 \\
010 & 010 & 000 & 000 & 010 & 010 & 000 \\
\end{bmatrix}
\]

Similarly, starting with a fixed Aronhold set, the non-zero elements of any Pascal configuration can be described by taking sums of elements of the following form (so that there are three triple sums all sharing the same one element, each giving an even characteristic lying in the Pascal configuration, and the other 4 odd elements of a Pascal configuration are obtained as double sums forgetting the common characteristic, and the common characteristic itself)

(8) \[ P := \{(123), (145), (167), (1), (23), (45), (67)\}, \]

so that for the Aronhold set given in (5) we get the Pascal configuration

(9) \[
\begin{bmatrix}
000 & 000 & 001 & 101 & 101 & 011 & 111 \\
010 & 010 & 000 & 111 & 111 & 101 & 111 \\
\end{bmatrix}
\]
3. Modular forms

In this section we review the notation on modular forms and theta constants. We denote by $\mathbb{H}_g$ the *Siegel upper half-space* — the space of complex symmetric $g \times g$ matrices with positive definite imaginary part.

An element $\tau \in \mathbb{H}_g$ is called a *period matrix*, and defines the complex abelian variety $X_\tau := C^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$. The group $\Gamma_g := \text{Sp}(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by automorphisms: for 

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$$

the action is $\gamma \circ \tau := (A\tau + B)(C\tau + D)^{-1}$. The quotient of $\mathbb{H}_g$ by the action of the symplectic group is the moduli space of principally polarized abelian varieties (ppav): $A_g := \mathbb{H}_g / \text{Sp}(2g, \mathbb{Z})$. We define the *level subgroups* of the symplectic group to be 

$$\Gamma_g(n) := \{ \gamma \in \text{Sp}(2g, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \}$$

The corresponding *level $n$ moduli space of ppav* is denoted $A_g(n) := \mathbb{H}_g / \Gamma_g(n)$.

A function $F : \mathbb{H}_g \to \mathbb{C}$ is called a *modular form of weight $k$ and multiplier $\chi$ with respect to a subgroup $\Gamma \subset \Gamma_g$* if

$$F(\gamma \circ \tau) = \chi(\gamma) \det(C\tau + D)^k F(\tau), \quad \forall \gamma \in \Gamma, \forall \tau \in \mathbb{H}_g.$$ 

We shall write $[\Gamma, k, \chi]$ for this space. We omit the multiplier if it is trivial. We shall define the ring of modular forms as

$$A(\Gamma) = \bigoplus_{k=0}^{\infty} [\Gamma, k].$$

This is a finitely generated graded ring. For any theta characteristic $m \in \mathbb{F}_2^g$, we define the *theta function with characteristic $m$* to be the map $\theta_m : \mathbb{H}_g \times C^g \to \mathbb{C}$ given by

$$\theta_m(\tau, z) := \sum_{p \in \mathbb{Z}^g} \exp \pi i \left[ \left( p + \frac{m'}{2} \right)^t \left( \tau (p + \frac{m'}{2}) + 2(z + \frac{m''}{2}) \right) \right].$$

Sometimes we will write $\theta \left[ \begin{array}{c} m' \\ m'' \end{array} \right] (\tau, z)$ for $\theta_m(\tau, z)$.

For $\varepsilon \in \mathbb{F}_2^g$ we also define the *second order theta function with characteristic $\varepsilon$* to be

$$\Theta[\varepsilon](\tau, z) := \theta \left[ \begin{array}{c c} \varepsilon \\ 0 \end{array} \right] (2\tau, 2z).$$
The transformation law for theta functions under the action of the symplectic group is given in [Igu72]:

$$\theta \left[ \gamma \begin{pmatrix} m' \\ m'' \end{pmatrix} \right] (\gamma \circ \tau, (C \tau + D)^{-1}z) =$$

$$\phi(m, \gamma) \det(C \tau + D)^{1/2} e^{\pi i (C \tau + D)^{-1}Cz} \theta \left[ \begin{pmatrix} m' \\ m'' \end{pmatrix} \right](\tau, z),$$

where $\phi$ is some complicated explicit function (which we will discuss in more detail before theorem 19) not depending on $z$ and $\tau$, and the action of $\gamma$ on characteristics is taken modulo 2 (and the exponent $-t$ means the inverse of the transpose). It is further known (see [Igu72], [SM83]) that $\phi|_{z=0}$ does not depend on $\tau$. Thus the values of theta functions at $z = 0$, called theta constants, are modular forms of weight one half with multiplier, with respect to $\Gamma_g(2)$. Similarly it is known that the theta constants of second order are modular forms of weight one half with respect to a certain normal subgroup $\Gamma_g(2, 4) \subset \text{Sp}(2g, \mathbb{Z})$ (containing $\Gamma_g(4)$ and contained in $\Gamma_g(2)$).

All odd theta constants with characteristics vanish identically, as the corresponding theta functions are odd functions of $z$, and thus there are $2^{g-1}(2^g + 1)$ non-trivial theta constants with characteristics, corresponding to even theta characteristics $m$. All theta functions of the second order are even with respect to $z$, and all $2^g$ theta constants of the second order are not identically zero.

Given a set of $g$ odd characteristics $m_1, \ldots, m_g$, one constructs the corresponding Jacobian determinant by taking the determinant of the $g \times g$ matrix formed by the $g$ gradients of the functions $\theta_{m_1}$ with respect to $z$, evaluated at $z = 0$ (i.e. the matrix with $i, j$ entry being $\partial_{z_i} \theta_{m_1}(z)|_{z=0}$). We write it as

$$D(m_1, \ldots, m_g) = \prod_{i=1}^{g} \theta_{m_i}(\tau, z) \wedge \ldots \wedge \theta_{m_g}(\tau, z)|_{z=0},$$

and it turns out to be a modular form of weight $\frac{2g+2}{2}$ with respect to $\Gamma_g(2)$, with a suitable multiplier.

In genus 1, the famous Jacobi derivative formula

$$D \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = -\pi \theta \begin{pmatrix} 0 \\ 0 \end{pmatrix} \theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

expresses the only Jacobian determinant as a product of theta constants. This formula has been generalized to genus 2 and 3 by Igusa [Igu80], and the results are as follows. In genus 2, for any special fundamental system $m_1, m_2, n_1, n_2, n_3, n_4$ we have

$$D(m_1, m_2) = -\pi^2 \theta_{n_1} \theta_{n_2} \theta_{n_3} \theta_{n_4}. $$
In genus 3, for any azygetic triple of odd characteristics \(m_1, m_2, m_3\) forming a special fundamental system with \(n_1, \ldots, n_5\) (see lemma 3), we have

\[
D(m_1, m_2, m_3) = -\pi^3 \theta_{n_1} \theta_{n_2} \theta_{n_3} \theta_{n_4} \theta_{n_5}.
\]

In general, for higher genus it is known that the Jacobian determinant is not a polynomial in theta constants, and various results were obtained by Igusa [Igu80] and Fay [Fay79]. Moreover, a Jacobian determinant associated to a non-azygetic set of characteristics is not a polynomial in the theta constants whenever \(g \geq 3\).

In the remaining sections of the text, we will use the generalized Jacobi derivative formulas for genera 2 and 3 to relate modular forms, configuration spaces of points, and the Coble quartic. We first deal with the easier case of genus 2 in the following two sections.

4. Abelian surfaces with level 2 structure, and the GIT quotient of \((\mathbb{P}^1)^6\)

The moduli space of abelian surfaces with a level two structure, \(A_2(2) = \mathbb{H}/\Gamma_2(2)\) admits various compactifications. Indeed, the space of modular forms \([\Gamma_2(2), 2]\) for \(\Gamma_2(2)\) of weight 2 is 5-dimensional, spanned by the fourth powers of theta constants with characteristics \(\theta_n^4\). Choosing an appropriate set of characteristics \(n_1, \ldots, n_5\) such that \(X_0 := \theta_{n_1}^4(\tau), \ldots, X_4 := \theta_{n_5}^4\) are linearly independent defines an embedding \(A_2(2) \hookrightarrow \mathbb{P}^1\). The closure of the image — the Satake compactification \(A_2(2)^{Sat}\) — is given by one equation, the Igusa quartic, see [Igu64]:

\[
I(X_0, \ldots, X_4) := (X_0X_1 + X_0X_2 + X_1X_2 - X_3X_4)^2 - 4X_0X_1X_2(X_0 + X_1 + X_2 + X_3 + X_4).
\]

This result implies that the ring \(A(\Gamma_2(2))\) of modular forms of genus 2, level 2, and even weight is generated by the fourth powers of 5 theta constants with characteristics, with the only relation being the Igusa quartic:

\[
A(\Gamma_2(2)) = \bigoplus_{k=0}^{\infty} [\Gamma_2[2], 2k] = \mathbb{C}[X_0, \ldots, X_4]/I(X_0, \ldots, X_4).
\]

Alternatively, note that any indecomposable abelian surface is a hyperelliptic Jacobian, and thus is determined by the 6 branch points of the hyperelliptic cover of \(\mathbb{P}^1\) (up to automorphisms of \(\mathbb{P}^1\)); the level two structure corresponds to an ordering of these 6 points, and thus an
alternative birational model of $A_2(2)$ is obtained as the GIT quotient of $(\mathbb{P}^1)^6$ under the action of $\text{PSL}(2, \mathbb{C})$. We now recall this construction.

Denote $\mathcal{X}(6) \subset (\mathbb{P}^1)^6$ the subset where all the points are distinct; the action of $\text{PSL}(2, \mathbb{C})$ on $\mathbb{P}^1$ extends to its action (diagonally) on $\mathcal{X}(6)$. The configuration space is then defined as

$$X(6)^o = \mathcal{X}(6)/\text{PSL}(2, \mathbb{C}).$$

In non-homogeneous coordinates $x_1, \ldots, x_6$ on $\mathbb{C}^6 \subset (\mathbb{P}^1)^6$, degree $k$ forms on $X(6)^o$, which we call binary invariants of degree $k$, are given by polynomials $P \in \mathbb{C}[x_1, \ldots, x_6]$ such that

$$P(\gamma \cdot x) = \prod_{i=1}^{6} (cx_i + d)^{-k} \cdot P(x)$$

where we denote

$$(\gamma \cdot x)_i = (ax_i + b)(cx_i + d)^{-1}$$

for every

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}).$$

We denote by $S(6)_k$ the space generated by such forms, and let $S(6) := \bigoplus_{k=0}^{\infty} S(6)_k$ be the graded ring of binary invariants.

Recall that a tableau is a way of filling a matrix $\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$ by the numbers from 1 to 6 in such a way that we have

$$i_1 < i_2 < i_3, \quad i_1 < j_1, \quad i_2 < j_2, \quad i_3 < j_3.$$ 

A tableau is called standard if moreover it satisfies $j_1 < j_2 < j_3$.

There are 5 standard tableaux enumerated as follows:

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

To any tableau $N$ we associate the invariant

$$B(N) := (x_{i_1} - x_{j_1})(x_{i_2} - x_{j_2})(x_{i_3} - x_{j_3}) \in S(6)_1.$$ 

The space spanned by all these polynomials has dimension 5, and a basis is given by polynomials associated to the standard tableaux, which we denote $T_0(x), \ldots, T_4(x)$ in the above ordering. Kempe [Kem] proved that $T_i(x)$ generate the ring of invariants $S(6)$; the invariants $T_i$ thus define a smooth embedding

$$T : X(6)^o \to \mathbb{P}^4,$$

and we denote by $X(6)$ the closure of the image $T(X(6)^o)$. This is called the GIT quotient $X(6) = (\mathbb{P}^1)^6//\text{PSL}(2, \mathbb{C})$, and gives an alternative
compactification of the moduli space of indecomposable abelian surfaces with a level two structure. It turns out that the ideal of relations among binary invariants is generated by the Segre cubic polynomial

\[ S(T_0, \ldots, T_4) := T_1T_2T_4 - T_3(T_0T_4 + T_1T_2 - T_0T_1 - T_0T_2 + T_0^2) \]

(see [DO88]), and hence

\[ X(6) = \text{Proj} \left( \mathbb{C}[T_0, \ldots, T_4]/S(T_0, \ldots, T_4) \right). \]

**Remark 7.** It is in fact known that the Igusa quartic and Segre cubic are dual hypersurfaces up to projectivities, see [DO88, p. 183]. Moreover, if one embeds a Kummer surface as a quartic surface in \( \mathbb{P}^3 \) by using theta functions of the second order, its equation is of the form

\[ K(\tau, z) = \alpha_0(x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4) + 2\alpha_1(x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2) \]

\[ + 2\alpha_2(x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2) + 2\alpha_3(x_{00}^2x_{11}^2 + x_{10}^2x_{01}^2) + 4\alpha_4(x_{00}x_{01}x_{10}x_{11}), \]

where \( x_\varepsilon = \Theta[\varepsilon](\tau, z) \). It then turns out that the coefficients \( \alpha_i \) satisfy the Segre cubic equation and certain inequalities, see [DO88].

The above equation for the universal Kummer surface can be obtained by writing down the Fourier-Jacobi expansion of each theta constant of the second order, and then substituting them into the unique degree 16 polynomial (the Schottky-Igusa form) in theta constants of the second order for genus 3 that vanishes identically; see [RSSS12] or [DPSM13]. In the following section we compute the coefficients \( \alpha_i \) explicitly using a different approach — which then generalizes to the case of genus 3.

5. **Modular forms of genus 2 and level 2, and binary invariants of six points on \( \mathbb{P}^1 \)**

We now investigate the relation between the two birational models of \( \mathcal{A}_2(2) \) constructed in the previous section. That is to say, we will investigate the relation between modular forms of genus 2 and level 2, and binary invariants of 6 points on \( \mathbb{P}^1 \), constructing a pair of dual homomorphisms from one graded algebra to the other. One of these — the homomorphism associating a binary invariant to a modular form was classically studied by Thomae [Tho70] in the nineteenth century, with further work done by Igusa [Igu67] and the second author [SM00] (there is also recent work of Howard, Millson, Snowden, and Vakil (see [HMSV09] and references therein) on invariants of arbitrary numbers of points on a line, which yields a beautiful picture in general, which we do not require here).

Indeed, working in arbitrary genus, Thomae defined a homomorphism from the ring \( A(\Gamma_g(2)) \) of modular forms of genus \( g \) and level 2
to the ring $S(2g + 2)$ of binary invariants of points on $\mathbb{P}^1$. This homomorphism is obtained essentially by restricting the modular forms to the locus of hyperelliptic Jacobians. The corresponding expression for theta constants of hyperelliptic Jacobians is known as Thomae’s formula, and the result in genus 2 is as follows:

**Theorem 8** (Thomae [Tho70], see also Igusa [Igu67]). For $g = 2$ for any even theta characteristic $n$ write

$$n = m_{i_1} + m_{i_2} + m_{i_3} = m_{i_4} + m_{i_5} + m_{i_6}$$

where $i_1, \ldots, i_6$ is a suitable ordering of the odd characteristics. Let

$$\psi^* (\theta^4_n) := \pm (x_{i_1} - x_{i_2})(x_{i_2} - x_{i_3})(x_{i_3} - x_{i_4})(x_{i_4} - x_{i_5})(x_{i_5} - x_{i_6})(x_{i_6} - x_{i_1})$$

and extend $\psi^*$ to a homomorphism of the ring of modular forms of genus 2 and level 2. Then the map $\psi^*$ defines an injective degree preserving homomorphism

$$\psi^* : A(\Gamma_2(2)) \to S(6).$$

For degree reasons the associated rational map

$$\psi : X(6) \dashrightarrow A_2(2)^{Sat}$$

interpreted as a map from the Segre cubic to the Igusa quartic is a map given via quadrics.

We will now construct a homomorphism of graded algebras

$$\phi^* : S(6) \to A(\Gamma_2(2))$$

such that the associated rational map

$$\phi : A_2(2)^{Sat} \dashrightarrow X(6)$$

will be the inverse of the rational map $\psi$.

To construct the homomorphism $\phi^*$, we will use the Jacobian determinants discussed above. Indeed, for any binary invariant written as a polynomial $P = P(\ldots, x_i - x_j, \ldots)$ in the variables $x_i - x_j$ for $1 \leq i < j \leq 6$, we set

$$\phi^*(P) := P(D(m_i, m_j)),$$

where the resulting expression is then written as a scalar modular form by utilizing the generalized Jacobi derivative formula (11).

**Proposition 9.** With the above notation we have for any even theta characteristic $n$

$$\phi^*(\psi^*(\theta^4_n)) = \chi_5^2 \theta^4_n$$
with
\[ \chi_5 = \prod_{n \text{ even}} \theta_n. \]

Vice versa,
\[ \psi^* (\phi^* (x_{i_1} - x_{j_1}) (x_{i_2} - x_{j_2}) (x_{i_3} - x_{j_3})) = \Delta^{1/2} \cdot (x_{i_1} - x_{j_1}) (x_{i_2} - x_{j_2}) (x_{i_3} - x_{j_3}) \]
with
\[ \Delta^{1/2} = \prod_{1 \leq i < j \leq 6} (x_i - x_j). \]

Proof. Using equation (16) defining the map \( \psi^* \), the computations with Jacobian determinants done by Fiorentino [Fio11], and lemma 2, we get
\[
D(m_{i_1}, m_{i_2}) D(m_{i_1}, m_{i_3}) D(m_{i_2}, m_{i_4})
= \pm D(m_{i_4}, m_{i_5}) D(m_{i_4}, m_{i_6}) D(m_{i_5}, m_{i_6})
= \pm \pi^6 \chi_5 \theta_m^2.
\]
Thus choosing a suitable ordering of the odd characteristics and multiplying the first two lines of the formula we obtain \( \pi^{12} \chi_5^2 \theta_1^4 \) proving the formula for \( \phi^* \circ \psi^* \).

Vice versa, we compute
\[
\phi^* (x_{i_1} - x_{j_1}) (x_{i_2} - x_{j_2}) (x_{i_3} - x_{j_3}) =
\theta_{m_{i_1} + m_{j_1} + m_{i_2}} \theta_{m_{i_1} + m_{j_2} + m_{i_3}} \theta_{m_{i_1} + m_{j_2} + m_{j_3}} \cdot
\theta_{m_{i_2} + m_{j_1} + m_{i_3}} \theta_{m_{i_2} + m_{j_2} + m_{j_3}} \theta_{m_{i_2} + m_{j_2} + m_{j_2}}.
\]

Applying \( \psi^* \) and using the computations in [Fio11], we finally get the formula for \( \psi^* \circ \phi^* \). \( \blacksquare \)

Remark 10. We observe (see [Igu67]) that
\[ \psi^* (\chi_5^2) = c \Delta, \]
and the zero set of either side is the complement of the locus of indecomposable abelian surfaces (Jacobians of smooth hyperelliptic curves) with level 2 structure, in both varieties. We note also that the map \( \psi^* \) can be extended so as to be applied to \( \chi_5 \) itself, then producing the alternating form \( \Delta^{1/2} \), up to a constant factor.

As a consequence of the above discussion we have

Theorem 11. The maps \( \phi \) and \( \psi \) restricted to the locus of indecomposable ppav within \( \mathcal{A}_2(2) \), i.e. to the locus of Jacobians of smooth hyperelliptic genus two curves, are the inverses of each other. Hence choosing suitable bases in the space of binary invariants of degree one and in the space of modular forms of weight two induces the duality maps between the Igusa quartic and the Segre cubic.
As an application, we compute the coefficients $\alpha_i$ of the universal Kummer surface (15), proving along the way that they are modular forms of weight six for $\Gamma_2(2)$. To this end, we apply the homomorphism $\phi^*$ to these coefficients of the universal Kummer surface. A very similar strategy applies in the more difficult case of genus 3, yielding the equation of the Coble quartic. Here we only sketch the argument for genus 2, referring to the genus 3 case studied below for a more detailed discussion.

It turns out that the five coefficients $\alpha_0, \ldots, \alpha_4$ of the universal Kummer surfaces, and the 5 polynomials given in (15) quartic polynomials in $x_{ij}$ each span a five-dimensional representation of $Sp(4, \mathbb{F}_2)$. Let us denote these representations by $W$ and $W'$, respectively; it turns out that these representations are isomorphic, and that the tensor product $W \otimes W'$ contains a unique copy of the trivial representation of $Sp(4, \mathbb{F}_2)$. We define a suitable subgroup $\Gamma_{2,0}(2)$ of $Sp(4, \mathbb{Z})$ of index 15 containing $\Gamma_2(2)$ (see (22) below for the analogous definition in genus 3) such that each $W$ and $W'$ contain a unique $\Gamma_{2,0}(2)$-invariant vector, which we denote $v$ and $v'$, and then the symmetrization of $v \otimes v'$ under $Sp(4, \mathbb{Z})$ must give the element of $W \otimes W'$ that corresponds to the trivial representation of $Sp(4, \mathbb{F}_2)$, which is then equal to the universal Kummer surface. This situation is completely analogous to the case of the Coble quartic, discussed in much more detail below, and the resulting expression for the universal Kummer surface is

$$K(\tau, z) = s_1(x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4) - 2(s_1 + 2s_2)(x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2)$$

$$- 2(s_1 + 2s_3)(x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2) - 2(s_1 + 2s_4)(x_{00}^2x_{11}^2 + x_{10}^2x_{01}^2)$$

$$+ 8(s_1 + s_2 + s_3 + s_4 + 2s_5)(x_{00}x_{01}x_{10}x_{11}),$$

where

$$s_1 = (\theta [00] \theta [00] \theta [00] \theta [00])^{-2} \chi_5^2,$$

$$s_2 = (\theta [00] \theta [00] \theta [01] \theta [01])^{-2} \chi_5^2,$$

$$s_3 = (\theta [00] \theta [00] \theta [01] \theta [00])^{-2} \chi_5^2,$$

$$s_4 = (\theta [00] \theta [01] \theta [01] \theta [01])^{-2} \chi_5^2,$$

$$s_5 = (\theta [00] \theta [01] \theta [00] \theta [01])^{-2} \chi_5^2,$$

and we recall that $x_\varepsilon = \Theta[\varepsilon](\tau, z)$.

This completes our results for the case of genus 2. We note that in the expression above the coefficients of the equation defining the universal Kummer surface are polynomials in the $\theta_{m}^2$. Using Riemann relations one can actually rewrite them as polynomials in the $\theta_{m}^2$, but the formula above looks nicer, and has more similarities with the genus 3 case that we treat below. The rest of the paper will be devoted to a
detailed exposition of the case of genus 3, culminating in formulas for the Coble quartic.

6. The Configuration Space of Seven Points on \( \mathbb{P}^2 \)

Similar to the above relation between modular forms of genus 2 and invariants of points on \( \mathbb{P}^1 \), the G"opel systems discussed above are related to configurations of points on \( \mathbb{P}^2 \), and we now recall this construction, from [Cob29], [DO88], [Kon11].

Let \( \mathbb{P}_7^2 \) be the GIT quotient of \(( \mathbb{P}^2)^7 \) under the diagonal action of \( \text{PGL}(3, \mathbb{C}) \), and let \( R_7^2 \) be its ring of invariants. It turns out that \( R_7^2 \) is finitely generated and its structure is implicitly described in [RSSS12]. We now recall and explicitly state this description. To any G"opel system one can associate (see [DO88] and [Cob29]) a 15-dimensional subspace \( V \subset R_7^2 \) of invariants of degree 3 as follows.

Let \( p_i, p_j, p_k \in \mathbb{P}^2 \), and \( v_i, v_j, v_k \in \mathbb{C}^3 \setminus \{0\} \) be such that \( v_i \mapsto p_i \) under the projection. Denote then \((ijk) := v_i \wedge v_j \wedge v_k\), and associate to the Fano configuration given by (6) the function

\[
G_F := (123)(145)(167)(256)(346)(357),
\]

and associate to the Pascal configuration given by (8) the function

\[
\]

It turns out that the 135 functions \( G_F \) and \( G_P \) span a 15-dimensional space of degree 3 invariants \( V \), a basis of which is given by \( G_{F1}, \ldots, G_{F15} \) corresponding to suitable 15 Fano configurations. The linear relations among the \( G_F \) and the \( G_P \) are related to two-dimensional isotropic subspaces of \( \mathbb{F}_2^6 \), see [Cob29], [DO88], or [Kon11].

This 15-dimensional space \( V \) is an irreducible representation of \( \text{Sp}(6, \mathbb{F}_2) \), see [DO88, Prop. 9, p. 191]. Moreover, choosing a basis in it given by suitable 15 \( G_F \)'s defines a \( \text{Sp}(6, \mathbb{F}_2) \)-equivariant rational map from the GIT quotient \( \phi_v : \mathbb{P}_3^2 \to \mathbb{P}^{14} \), birational onto the image (a similar map is defined also from a six-dimensional ball quotient, see [Kon11] for details). The image of this map, \( \mathcal{G} \) the so-called G"opel variety, is described completely in [RSSS12].

**Theorem 12.** The six-dimensional G"opel variety \( \mathcal{G} \) has degree 175 in \( \mathbb{P}^{14} \). The homogeneous coordinate ring of \( \mathcal{G} \) is Gorenstein, and its defining prime ideal is minimally generated by 35 cubics and 35 quartics.

As immediate consequence of this fact we get that

**Corollary 13.** The above described ring is the ring of invariants \( R_7^2 \).
Proof. We know from [RSSS12] that the ideal is prime, moreover the
ring is Cohen-Macaulay, since it is Gorenstein, and regular in codimen-
sion 1, hence it is normal.

Moreover, according to the proof of the previous theorem, the Göpel
variety is defined set theoretically by the 35 cubic equations that are
an irreducible representation of the group Sp(6, \mathbb{F}_2).

Now, according to Coble [Cob29, p. 109], these cubic equations are
also satisfied by the 15-dimensional representation space spanned by
the coefficients of the Coble quartic.

In [RSSS12] the authors determine a 15-dimensional space \( W \) of
modular forms of weight 14 (spanned by some polynomials of degree
28 in theta constants with characteristics) isomorphic to \( V \). We will
do the same, by using and interpreting the constructions and results
originating with Coble, in a more geometric way.

7. Del Pezzo surfaces of degree 2 and plane quartics

We recall from [Cob29] or [DO88] that the double cover of \( \mathbb{P}^2 \) branched
along a smooth plane quartic is a degree 2 del Pezzo surface. Con-
versely, the anticanonical model of any degree 2 del Pezzo surface \( S \) is
a double cover of \( \mathbb{P}^2 \) branched along a smooth quartic.

A del Pezzo surface of degree 2 is obtained by blowing up seven points
\( p_1, \ldots, p_7 \in \mathbb{P}^2 \) in general position. The Picard lattice of \( S \) is isomor-
phic to \( (1) \oplus (-1)^{\otimes 7} \), and it is generated by \( h_0 \) (the total transform
of the hyperplane section) and the exceptional curves \( h_1, \ldots, h_7 \). The
orthogonal complement of the anti-canonical class \( 3h_0 - h_1 - \ldots - h_7 \) in
\( \text{Pic}_{\mathbb{Z}}(S) \) then turns out to be isomorphic to the lattice \( E_7 \). The surface
\( S \) contains 56 exceptional \(-1\) curves, which split into 28 pairs, such
that the curves in each pair are interchanged by the deck transforma-
tion of the cover \( S \to \mathbb{P}^2 \). For any set of seven disjoint \(-1\) curves,
contracting them defines a morphism \( \pi : S \to \mathbb{P}^2 \), called a geometric
marking of the degree 2 del Pezzo surface. The number of contractions
is equal to the order of the Weyl group \( W(E_7) \). Moreover, there is an
exact sequence

\[
1 \to (\pm 1) \to W(E_7) \to \text{Sp}(6, \mathbb{F}_2) \to 1,
\]

where the kernel of the map to \( \text{Sp}(6, \mathbb{F}_2) \) is the deck transformation of
the cover. Hence, we recall from [DO88, Cor., p. 168] a geometric mark-
ing of a degree 2 del Pezzo surface corresponds to a level 2 structure on
the Jacobian of the plane quartic \( C \). Under the anticanonical map the
chosen seven pairs of disjoint \(-1\) curves map to seven bitangents of \( C \)
that form an Aronhold set of bitangents, that is to a \( 7 \)-tuple \( \{ \ell_1, \ldots, \ell_7 \} \).
of bitangents such that for each triple \( \ell_i, \ell_j, \ell_k \) the corresponding six points of tangency with \( C \) do not lie on a conic.

Recall that bitangents of a plane quartic are in bijection with odd theta characteristics in genus 3 (the two points of tangency define a line bundle that is a square root of the canonical bundle), and Aronhold sets of characteristics thus correspond to Aronhold sets of bitangents. We thus obtain a birational map between \( \mathbb{P}_2^7 \) and \( \mathcal{A}_3(2) \) (the moduli space of genus 3 curves with a level 2 structure), defined away from the hyperelliptic locus in \( \mathcal{M}_3 \), see [CS03b],[Leh05].

8. Relations among modular forms in genus 3

We denote by \( \mathcal{A}_3 \) and \( \mathcal{A}_3(2) \) the Satake compactifications of \( \mathcal{A}_3 \) and \( \mathcal{A}_3(2) \), respectively. Let

\[
\chi_{18}(\tau) := \prod_{m \text{ even}} \theta_m(\tau, 0)
\]

be the product of all even theta constants with characteristics — it is a modular form of weight 18 for the entire group \( \text{Sp}(6, \mathbb{Z}) \), as each of the 36 theta constants is a modular form of weight 1/2. Recall also that hyperelliptic genus 3 curves are characterized by having one vanishing theta constant, and thus the equation \( \chi_{18} = 0 \) defines in \( \mathcal{A}_3 \) the closure of the hyperelliptic locus. Recall also that any non-hyperelliptic genus 3 curve is a plane quartic, and thus the complement in \( \mathcal{A}_3 \) of the zero locus \( \{\chi_{18} = 0\} \) is the moduli space of plane quartics. We denote this space by \( \mathcal{A}_0 \), and similarly denote by \( \mathcal{A}_3(2)^0 \) the level two cover of the moduli of non-hyperelliptic genus 3 curves.

From the discussion in the previous section, it follows that \( \mathcal{A}_3(2)^0 \) is isomorphic to each of the following: the moduli space of plane quartics together with an Aronhold set of bitangents; the moduli space of marked degree 2 del Pezzo surfaces; the open subset of the GIT quotient \( \mathbb{P}_2^7 \) where the points are in general position, i.e. no three lie on a line, and no six lie on a conic.

Using these identifications we associate to the functions \( G_F \) defined by (18) suitable modular forms (with trivial character) for the group \( \Gamma_3(2) \). Indeed, for a fixed Aronhold set \( m_1, \ldots, m_7 \) and a Fano configuration \( F \) given as triples of characteristics \( M_i = (i_1, i_2, i_3) \) with \( 1 \leq i_1 \leq i_2 \leq i_3 \leq 7 \), we set

\[
H(F)(\tau) := \frac{D(M_1) \ldots D(M_7)}{\theta_{m_0}^{i_7}}(\tau),
\]

where

\[
D(M) := \prod_{m \text{ even}} \theta_m(\tau, 0)^{i_1 + i_2 + i_3 - m},
\]

is a modular form of weight \( 18 - m \).
where we denote
\[ D(M_i)(\tau) := D(m_{i1}, m_{i2}, m_{i3})(\tau). \]

By Jacobi’s derivative formula (12), we can express each \( D(M_i) \) as a product of 5 theta constants (thus a modular form of weight \( 5/2 \)), and by the discussion in lemma 4 it follows that each of these products contains \( \theta_{n_0} \), so that each \( D_{M_i} \) is divisible by \( \theta_{n_0} \), and thus \( H(F)(\tau) \) is indeed a (holomorphic) modular form of weight \( 14 = 7 \cdot (5/2 - 1/2) \). Similarly we can define the modular form \( H(P)(\tau) \) corresponding to any Pascal configuration \( P \) as in (19). Summarizing, as a consequence of the previous identifications we have

**Theorem 14.** For any \( \tau \in \mathfrak{A}_3(2)^0 \) let \( X_\tau := (p_1, \ldots, p_7) \in P_7^2 \) be the corresponding 7-tuple of points. Fix an Aronhold set \( m_1, \ldots, m_7 \); then for every Fano configuration \( F \), we have

\[ G_F(X_\tau) = \theta_{n_0}^7(\tau) H(F)(\tau). \]

Moreover, we have

\[ H(F)(\tau) = \frac{\chi_{18}(\tau)}{\prod_{n \in F} \theta_n(\tau)}, \]

and thus all \( H(F)(\tau) \) are modular forms with respect to \( \Gamma_3(2) \) (with trivial character).

**Proof.** Indeed, on \( \mathbb{H}_3 \) by the discussion above we can express \( H(F) \) as a monomial in theta constants by using Jacobi’s derivative formula, yielding the expression above. The fact that the modular form \( H(F)(\tau) \) has trivial multiplier is a consequence of a result of Igusa [Igu66]. □

**Remark 15.** The construction above associating modular forms to Fano configurations depends on the choice of the Aronhold set. For a different Aronhold set, the correspondence between the \( G_F \) and the monomials \( H(F) \) would be different, but the denominator would still be of the form \( \theta_{n_0}^7 \), and thus on the set of plane quartics, which is the complement of the locus \( \{ \chi_{18}(\tau) = 0 \} \), the projective map given by the set of all \( H(F)(\tau) \) would be the same, since no theta constant vanishes.

We know that the 15-dimensional space \( V \) is spanned by the functions \( G_F \). Let us then denote by \( W \) the 15-dimensional space spanned by the functions \( H(F) \). The linear relations among the \( G_F \) and the \( G_P \) induce linear relations among the \( H(F) \) and \( H(P) \). The consistency of these relations follows from Riemann’s quartic addition theorem for theta constants. Indeed, recall (see for example [RF74] or [vGvdG86])
that in genus three Riemann’s quartic addition theorem for theta constants with characteristics has the form
\begin{equation}
r_1 = r_2 + r_3,
\end{equation}
where each \( r_i \) is a product of four theta constants with characteristics forming an even coset of a two-dimensional isotropic space (it turns out that in such a relation the signs are always as above, for the appropriate — and unique — choice of which term is \( r_1 \)) Hence the characteristics appearing in the product \( r_ir_j, i \neq j \) are even cosets of Lagrangian spaces (i.e. affine subspaces consisting only of even characteristics, modeled on Göpel configurations as vector subspaces), and we thus get
\begin{equation}
\frac{\chi_{18}}{r_1r_2} + \frac{\chi_{18}}{r_1r_3} - \frac{\chi_{18}}{r_3r_2} = \frac{\chi_{18}}{r_1r_2r_3} (r_3 + r_2 - r_1) = 0.
\end{equation}
The group
\begin{equation}
\Gamma_{3,0}(2) := \{ \gamma \in \text{Sp}(6, \mathbb{Z}) \mid C \equiv 0 \mod 2 \}
\end{equation}
has index 135 in \( \text{Sp}(6, \mathbb{Z}) \), using Maple\textsuperscript{®} (or it can also be seen by using Igusa’s going down process), we found that the subspace of the 15-dimensional space \( W \) invariant under the action of \( \Gamma_{3,0}(2) \) is one-dimensional, and is thus spanned by the manifestly \( \Gamma_{3,0}(2) \)-invariant modular form
\begin{equation}
H(F_1)(\tau) := \frac{\chi_{18}}{\prod_{m''} \theta \left[ \begin{array}{c} 0 \\ m'' \end{array} \right] (\tau)}
\end{equation}
corresponding to the Fano configuration \( F_1 := \left\{ \left[ \begin{array}{c} 0 \\ m'' \end{array} \right] \right\} \) consisting of all characteristics with top vector equal to zero .

We conclude this section by remarking that a basis of \( W \) induces the rational map
\[ \phi_W : \mathcal{A}_3(2) \to \mathbf{P}^{14} = \mathbf{P}(W). \]
Thus one can easily see [RSSS12, Thm. 7.1 (b)]:

**Proposition 16.** The base locus of \( \phi_W \) consists of reducible points in \( \mathcal{A}_3(2) \).

**Proof.** We can choose a basis of \( W \) consisting of monomials in the theta constants. Since for a Jacobian of a smooth hyperelliptic genus 3 curve, precisely one theta constant vanishes, the map \( \phi_W \) is still well-defined on the locus of hyperelliptic Jacobians. However, for a reducible point at least 6 theta constants with azygetic characteristics vanish (this condition characterizes the reducible locus, see [FSM11]).
The complementary set of characteristics in each monomial defining the map $\phi_W$ is a Göpel system, i.e. a syzygetic octet of characteristics. A Göpel system cannot contain an azygetic triple of characteristics, and thus all monomials defining the map $\phi_W$ vanish identically on the reducible locus, so that the map $\phi_W$ is undefined there.

We observe that the map is defined exactly along $\mathcal{M}_3(2)$. In particular, the map $\phi_W$ is well-defined on the hyperelliptic locus.

9. The Coble Quartic

We combine the results and constructions summarized above to obtain a formula for the Coble quartic as a suitable symmetrization.

We now turn to describing the equation of the Coble quartic. We recall from the introduction the theta functions of the second order. These are coordinates for the projective space that is the target of the Kummer map. We denote by $\Gamma_{3,0}(2)$ the subgroup of $\text{Sp}(6, \mathbb{Z})$ consisting of those matrices for which $C \equiv 0 \mod 2$. Let then $W'$ be the 15-dimensional vector space spanned by quartics $Q_1, \ldots, Q_{15}$ in $x$ invariant under translations, enumerated explicitly in the next section, following Coble’s notation in [Cob29] (and also given in [RSSS12]). Here we only note that

$$Q_1 := x_{000}^4 + x_{001}^4 + x_{010}^4 + x_{100}^4 + x_{101}^4 + x_{011}^4 + x_{110}^4 + x_{111}^4$$

spans the unique $\Gamma_{3,0}(2)/\Gamma_3(2)$ invariant line in $W'$. This 15-dimensional space $W'$ forms an irreducible representation of the group $\text{Sp}(6, \mathbb{F}_2)$. It can be verified that the representation $W'$ is in fact isomorphic to $W$, but we keep the notation distinct. Finally, we set $R(\tau, z) := H(F_1)(\tau)Q_1(\tau, z)$ to be the product of the two such invariants, and obtain a formula for the Coble quartic.

**Theorem 17.** Up to a constant factor, the Coble quartic is equal to

$$\sum_{\gamma \in \text{Sp}(6, \mathbb{Z})/\Gamma_{3,0}(2)} \gamma(R(\tau, z)) = \sum_{\gamma \in \text{Sp}(6, \mathbb{Z})/\Gamma_{3,0}(2)} \det(C\tau + D)^{-16} e^{8\pi iz^1(C\tau + D)^{-1}} R(\gamma\tau, (C\tau + D)^{-1}z).$$

**Proof.** Using Magma®, Eberhard Freitag determined the decomposition of $W \otimes W'$ into irreducible representations under the action of $\text{Sp}(6, \mathbb{F}_2)$. It turns out that the dimensions of the irreducible summands are equal to 1, 35, 84, 105, each occurring with multiplicity one, and since the Coble quartic is invariant under $\text{Sp}(6, \mathbb{F}_2)$, it must generate the 1-dimensional trivial subrepresentation of $W \otimes W'$. (From
Schur’s lemma it follows a priori that there is at most a single copy of
the trivial representation in this tensor product, but the computation
above actually guarantees the existence of this invariant, which thus
proves the existence of the Coble quartic!

Recall that we have \( \Gamma_3(2) \subset \Gamma_{3,0}(2) \subset \text{Sp}(6,\mathbb{Z}) \), so we have

\[
\Gamma_{3,0}(2)/\Gamma_3(2) \subset \text{Sp}(6,\mathbb{F}_2) = \text{Sp}(6,\mathbb{Z})/\Gamma_3(2).
\]

Since the modular form \( H(F_1) \) defined in (23) spans the \( \Gamma_{3,0}(2)/\Gamma_3(2) \)
invariant line in \( W \), and similarly \( Q_1 \) spans the \( \Gamma_{3,0}(2)/\Gamma_3(2) \) invariant line in \( W' \), the expression

\[
v_0 = \sum_{\gamma \in \text{Sp}(6,\mathbb{F}_2)/\Gamma_{3,0}(2)} \gamma(v \otimes v') \in W \otimes W'
\]
is invariant under \( \text{Sp}(6,\mathbb{F}_2) \), and thus, unless it is zero, generates the
one-dimensional irreducible summand of \( W \otimes W' \). Indeed, to prove
that \( v_0 \) is not identically zero, we recall that in fact \( W \) and \( W' \) are
isomorphic representations, this isomorphism must send \( H(F_1) \) to \( Q_1 \),
and it is enough to check that the sum above is non-zero under such
an identification. To this end, we compute using the transformation
formula for theta constants given in [SM89]

\[
\sum_{\gamma \in \text{Sp}(6,\mathbb{F}_2)} \gamma(H(F_1)^2) = |\Gamma_{3,0}(2)/\Gamma_3(2)| \sum_{i=1}^{135} H(G_i)^2,
\]

where \( G_1, \ldots, G_{135} \) is some enumeration of all the Göpel systems. This
form is not identically zero, in fact the first Fourier coefficient (for some
lexicographic ordering) of each \( H(G_i)^2 \) is positive, hence \( v_0 \in W \otimes W' \)
cannot be 0, and must be proportional to the Coble quartic. \( \square \)

**Remark 18.** The referee pointed out that we have three 15-dimensional
irreducible representations of \( \text{Sp}(6,\mathbb{F}_2) \), \( V,W,W' \), which are all iso-
morphic and coincide with the representation \( T \) in [DO88, Remark
7, p. 194]. There, the representation \( T \) is defined on the real vector
space spanned by pairwise products of 7 perpendicular reflection hyper-
planes, which are permuted (up to sign) by \( \text{Sp}(6,\mathbb{F}_2) \). Thus the sum
of the squares of these products provides a quadratic invariant, i.e. a
1-dimensional trivial subrepresentation of \( T \otimes T = W \otimes W' \).

**10. An explicit formula for the Coble quartic**

Using a computer, we compute explicitly the expression for the Coble
quartic given by Theorem 17, by determining the orbit of \( H(F_1)Q_1 \)
under the action of \( \text{Sp}(6,\mathbb{F}_2) \). We note that \( \Gamma_0(2)/\Gamma(2) \) is the stabilizer
of the first element, where we recall that the symplectic group acts on
the theta functions with characteristics and of the second order by the "slash" action according to the following rules

\[ \gamma(\theta_{[\alpha \beta]}(\tau, z)) := \det(C\tau + D)^{-1/2}e^{\pi iz'(C\tau + D)^{-1}z}\theta_{[\alpha \beta]}(\gamma\tau, (C\tau + D)^{-\ell}z). \]

\[ \gamma(\Theta_{[\epsilon]}(\tau, z)) := \det(C\tau + D)^{-1/2}e^{2\pi iz'(C\tau + D)^{-1}z}\Theta_{[\epsilon]}(\gamma\tau, (C\tau + D)^{-\ell}z), \]

and this action restricts to an action on the corresponding theta constants. In working with this action, a recurrent difficulty is the multiplier \( \kappa \) that appears in the transformation formula for theta functions. However, this multiplier depends only on \( \gamma \), and not on the characteristics, and since it is an eighth root of unity, it disappears when working with polynomials in theta constants of degree divisible by 8 — which is always our situation.

Since the conventions in the literature vary, and there are various typos in signs, for easy reference here are the relevant formulas from [Igu72],[Run93],[SM94] (note especially that in the first formula the theta function on the right-hand-side is written with characteristic in \( \mathbb{F}_2^6 \), which accounts for the extra sign \( (-1)^{\frac{\beta + S\alpha - \text{diag} S}{2}} \)). Omitting the multiplier, we then have the following transformation formulas

\[ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \theta_{[\alpha \beta]} = \left( \frac{1 + i}{\sqrt{2}} \right)^{\alpha'(-S\alpha - 2\text{diag} S + 4(\frac{\beta + S\alpha - \text{diag} S}{2}))} \theta_{[\alpha \beta]}(\beta + S\alpha + \text{diag} S), \]

\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \theta_{[\alpha \beta]} = \left( \frac{1 + i}{\sqrt{2}} \right)^{-2\alpha'\beta} \theta_{[\alpha \beta]}, \]

\[ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \Theta_{[\epsilon]} = i^{\epsilon S\tau} \Theta_{[\epsilon]}, \]

\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Theta_{[\epsilon]} = \left( \frac{1}{\sqrt{2}} \right)^g \sum_{\alpha} (-1)^{\alpha\epsilon} \Theta_{[\alpha]} \]

To compute the Coble quartic, we then enumerate the 30 Fano configurations and the 105 Pascal configurations. For each Pascal configuration \( P \) there exists a unique pair of Fano configurations \( F', F'' \) such that there exist three syzygetic quartets of characteristics \( S_1, S_2, S_3 \) satisfying

\[ P = S_2 \sqcup S_3, \quad F' = S_1 \sqcup S_2, \quad F'' = S_1 \sqcup S_3. \]

As described above, Riemann’s quartic addition theorem then yields for any Pascal configuration \( P \) the expression \( H(P) = \pm H(F') \pm H(F'') \), where the signs are given explicitly by Igusa [Igu81]. Once this is done, in the same manner one investigates Riemann’s quartic addition theorem given by 3 syzygetic quartets where each \( S_1 \sqcup S_j \) is a Pascal
configuration — substituting the expressions for $H(P)$ then yields various relations among the modular forms $H(F)$. Solving these allows one to find an explicit basis for the space $W$. Explicitly, such a basis can be chosen to be given by the following 15 Fano configurations:

$$F_1 = \{[000], [001], [010], [011], [100], [101], [110], [111]\},$$

$$F_2 = \{[000], [001], [010], [011], [100], [101], [110], [111]\},$$

$$F_3 = \{[000], [001], [100], [101], [110], [111]\},$$

$$F_4 = \{[000], [001], [110], [111], [100], [101], [010], [011]\},$$

$$F_5 = \{[000], [001], [110], [111], [100], [101], [100], [011]\},$$

$$F_6 = \{[000], [010], [000], [110], [001], [010], [100], [111]\},$$

$$F_7 = \{[000], [010], [011], [100], [001], [101], [101], [011]\},$$

$$F_8 = \{[000], [010], [011], [001], [100], [010], [101], [011]\},$$

$$F_9 = \{[000], [010], [011], [001], [100], [010], [101], [011]\},$$

$$F_{10} = \{[000], [010], [011], [001], [111], [011], [111], [111]\},$$

$$F_{11} = \{[000], [010], [011], [011], [100], [101], [111], [111]\},$$

$$F_{12} = \{[000], [010], [011], [011], [100], [101], [111], [111]\},$$

$$F_{13} = \{[000], [010], [011], [011], [100], [101], [111], [111]\},$$

$$F_{14} = \{[000], [110], [000], [110], [110], [110], [111], [111]\},$$

$$F_{15} = \{[000], [110], [000], [110], [110], [110], [111], [111]\}. $$

We note that a different labeling of the Fano configurations is chosen in [RSSS12], where they are explicitly labeled by permutations of the 7 elements of an Aronhold set. The choice of the above 15 Fano configurations giving a basis of $W$ is of course not canonical, but just one possible choice. Note also that our Fano configurations are unordered and we don’t have signs associated to them — the signs in our computations come from the theta transformation formulas and Riemann’s quartic addition formula.

We now list the 15 invariant quartics in theta functions of the second order as given by Coble [Cob29], and also in [RSSS12]: they are

$$Q_{000} := Q_1 = \sum_{\varepsilon \in F_2^3} x_\varepsilon^4, \quad Q_\alpha := \frac{1}{2} \sum_{\varepsilon \in F_2^3} x_\varepsilon^2 x_{\varepsilon+\alpha}, \quad Q'_\alpha := \frac{1}{4} \sum_{\varepsilon \in F_2^3, \mu \in \alpha^\perp} x_{\varepsilon+\mu},$$

for any $\alpha \in F_2^3 \setminus \{0\}$, where we recall (1) that $x_\varepsilon$ denotes $\Theta(\varepsilon)(\tau, z)$, and by $\alpha^\perp$ we mean the two dimensional subspace of $F_2^3$ orthogonal to $\alpha$ with respect to the usual scalar product, so that for example $011^\perp = \{000, 011, 100, 111\}$. The coefficients $1/2$ and $1/4$ are designed
to make each monomial appear with coefficient one. Though the follow-
ing formula would be simpler without them, we keep the notation
above for easy comparison with [Cob29] and [RSSS12].

The result of performing the computations described above, express-
ing all $H(F)$ in terms of the basis given, and using the theta transfor-
mation formula is then the following

**Theorem 19.** The Coble quartic can be written as a polynomial with
134 monomials as

$$C(\tau, z) = s_1 Q_{000} - 2(s_1 + 2s_6)Q_{001} - 2(s_1 + 2s_3)Q_{010} - 2(s_1 + 2s_9)Q_{011}$$

$$- 2(s_1 + 2s_2)Q_{100} - 2(s_1 + 2s_7)Q_{101} - 2(s_1 + 2s_4)Q_{110}$$

$$- 2(s_1 - 2s_{10})Q_{111} + 8(s_1 + s_2 + s_3 + s_4 + 2s_5)Q'_{001}$$

$$+ 8(s_1 + s_2 + s_6 + s_7 + 2s_8)Q'_{010} + 8(s_1 + s_2 + s_9 - s_{10} + 2s_{11})Q'_{011}$$

$$+ 8(s_1 + s_3 + s_6 + s_9 + 2s_{12})Q'_{100} + 8(s_1 + s_3 + s_7 - s_{10} + 2s_{13})Q'_{101}$$

$$+ 8(s_1 + s_4 + s_6 - s_{10} + 2s_{14})Q'_{110} + 8(s_1 + s_4 + s_7 + s_9 + 2s_{15})Q'_{111},$$

where we denote $s_i := H(F_i)$ for the basis of $W$ given above.

**Remark 20.** Note that our expression for the Coble quartic is in terms
of Fano configurations of characteristics, and as such it has few terms.

As explained in [RSSS12], each $s_i$ is in fact a degree 28 polynomial in
theta constants of the second order, with a huge number of monomials.

A completely explicit formula for the Coble quartic, as a polynomial in
theta constants of the second order and $Q_i$‘s, with 372060 monomials
altogether, is obtained in [RSSS12, Sec. 7]. In their notation we have
$r = H(F_1)$, $s_\sigma = -2H(F_1) \pm 4H(F_\sigma)$, and a similar expression can be
obtained for the $t_\sigma$. Moreover, note that in the above formula if we
take $-s_{10}$ instead of $s_{10}$, then all the signs becomes pluses.

Note that formula (17) for the universal Kummer surface and the
formula above for the Coble quartic, in terms of Göpel systems, are
remarkably similar, and it is of course natural to wonder whether there
may be a similar interesting hypersurface in higher genus. We hope
that our formulas may lead to a better further understanding of the
universal Kummer surface and of the Coble quartic.

All of the computations described above were performed using Maple®,
and take a few minutes for a straightforward unoptimized code on a
regular PC.

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