COUNTING RANK TWO LOCAL SYSTEMS WITH AT MOST ONE, UNIPOTENT, MONODROMY

YUVAL Z. FLICKER

Abstract. The number of rank two \( \mathbb{Q}_\ell \)-local systems, or \( \mathbb{Q}_\ell \)-smooth sheaves, on \((X - \{u\}) \otimes \mathbb{F}_q\), where \( X \) is a smooth projective absolutely irreducible curve over \( \mathbb{F}_q \), \( u \) is a closed point of \( X \), with principal unipotent monodromy at \( u \), and fixed by \( \text{Gal}(\mathbb{F}/\mathbb{F}_q) \), is computed. It is expressed as the trace of the Frobenius on the virtual \( \mathbb{Q}_\ell \)-smooth sheaf found in [DF13] on the moduli stack of curves with étale divisors of degree \( M \geq 1 \). This completes the work of [DF13] in rank two. This number is the same as that of representations of the fundamental group \( \pi_1((X - \{u\}) \otimes \mathbb{F}_q) \) invariant under the Frobenius \( \text{Fr}_q \) with principal unipotent monodromy at \( u \), or cuspidal representations of \( \text{GL}(2) \) over the function field \( \mathbb{F}_q(X) \) of \( X \) over \( \mathbb{F}_q \) with Steinberg component twisted by an unramified character at \( u \) and unramified elsewhere, trivial at the fixed idèle \( \alpha \) of degree 1. This number is computed in Theorem 4.1 using the trace formula evaluated at \( f_u \prod_{v \neq u} \chi_{K_v} \), with an Iwahori component \( f_u = \chi_{I_u}/|I_u| \), hence also the pseudo-coefficient \( \chi_{I_u}/|I_u| - 2\chi_{K_u} \) of the Steinberg representation twisted by any unramified character, at \( u \).

Theorem 2.1 records the trace formula for \( \text{GL}(2) \) over the function field \( \mathbb{F}_q \). The proof of the trace formula of Theorem 2.1 is given in [F14]. Theorem 3.1 computes, following Drinfeld, the number of \( \mathbb{Q}_\ell \)-local systems, or \( \mathbb{Q}_\ell \)-smooth sheaves, on \( X \otimes \mathbb{F}_q \), fixed by \( \text{Fr}_q \), namely \( \mathbb{Q}_\ell \)-representations of the absolute fundamental group \( \pi_1(X \otimes \mathbb{F}_q) \) invariant under the Frobenius, by counting the nowhere ramified cuspidal representations of \( \text{GL}(2) \) trivial at a fixed idèle \( \alpha \) of degree 1. This number is expressed as the trace of the Frobenius of a virtual \( \mathbb{Q}_\ell \)-smooth sheaf on a moduli stack. This number is obtained on evaluating the trace formula at the characteristic function \( \prod_v \chi_{K_v} \) of the maximal compact subgroup, with volume normalized by \( |K_v| = 1 \).

Section 5, based on a letter of P. Deligne to the author dated August 8, 2012, computes the number of such objects with any unipotent monodromy, principal or trivial, in our rank two case. Surprisingly, this number depends only on \( X \) and \( \deg(S) \), and not on the degrees of the points in \( S_1 \).

1. Introduction

Let \( X \) (denoted by \( X_1 \) in [DF13]) be a smooth projective absolutely irreducible curve over the finite field \( \mathbb{F}_q \) of cardinality \( q \) and characteristic \( p \). Fix a prime \( \ell \neq p \).

Drinfeld [D81] computed the number of isomorphism classes of two dimensional irreducible \( \ell \)-adic representations of the absolute fundamental group \( \pi_1(X \otimes \mathbb{F}_q) \) which are invariant under \( \text{Gal}(\mathbb{F}/\mathbb{F}_q) \) in terms of the zeta function of the curve \( X \).

Date: March 14, 2014.


Key words and phrases. \( \ell \)-adic Galois representations, unipotent monodromy, nowhere ramified automorphic representations, \( \mathbb{Q}_\ell \)-smooth sheaves, \( \text{GL}(2) \), function fields.
To understand this, we computed in [DF13] the number isomorphism classes of \(n\)-dimensional irreducible \(\ell\)-adic representations of the geometric fundamental group \(\pi_1((X - S) \otimes_{\mathbb{F}_q} \mathbb{F})\) which are \(\text{Gal}(\mathbb{F}/\mathbb{F}_q)\)-invariant and with a principal unipotent (rank \(n\)) monodromy at each point of \(S\). Here \(S\) is an \(\text{étale}\) divisor of \(X\), consisting of \(N > 1\) closed points, and \(\mathbb{Z} \ni n > 1\). In [DF13], \(N, S\) are denoted by \(N_1, S_1\).

When \(n = 2\) the result of [DF13] asserts that the number of isomorphism classes of the irreducible rank two \((\mathbb{Q}_l\text{-smooth sheaves})\) on \((X - S) \otimes_{\mathbb{F}_q} \mathbb{F}\), invariant under the Frobenius, with unipotent monodromy at each point of \(S\), where \(N = |S| > 1\), is equal to the trace of the Frobenius \(\text{Fr}_q\) on the virtual \(\mathbb{Q}_l\text{-smooth sheaf} \left(\sum_{i \geq 0} (-1)^i \bigwedge^i \mathcal{H}\right) \otimes \left(\sum_{j \geq 0} (-1)^j q(-j) \sum_{k > 0} \bigwedge^k \mathcal{H}_\ell\right)\).

Here \(\mathcal{H}_\ell\) is the local system of \(H^1_c(X - S), \mathcal{H}\) of \(H^1(X)\), and \(q(-j) = q(-1)^j\), where \(q(-1)\) is the Tate local system, on the moduli stack \(\mathcal{M}_{g,[M_1,M_2]}\) over \(\mathbb{Z}\) of triples \((X; S_1, S_2)\) consisting of a curve \(X\) of genus \(g\) and disjoint sets \(S_1, S_2\) of \(M_1, M_2 > 0\) geometric points on \(X\), \(S = S_1 \cup S_2\). Recall that \(M_i = \text{deg}(S_i)\) is \(\sum_{v \in S_i} \text{deg}(v)\), where \(\text{deg}(v)\) is the residual exponent of \(v\). If \(F = \mathbb{F}_q(X)\) is the function field of \(X\) over \(\mathbb{F}_q\), \(F_v\) is its completion at \(v\), the residue field \(\mathbb{F}_{q,v}\) is \(q^{\text{deg}(v)}\).

The first aim of this work is to extend this result from the covering \(\mathcal{M}_{g,[M_1,M_2]}\) to the moduli stack \(\mathcal{M}_{g,[M]}\) over \(\mathbb{Z}\) of pairs \((X; S)\) consisting of a curve \(X\) of genus \(g\) and an \(\text{étale}\) divisor \(S\) of \(M \geq 1\) geometric points on \(X\). Thus we show that the number \(T\) of the isomorphism classes of the two-dimensional irreducible \(\ell\)-adic representations \(\overline{\rho}\) of \(\pi_1((X - S) \otimes_{\mathbb{F}_q} \mathbb{F})\), the geometric fundamental group of the affine curve \(X - S\), invariant under the Frobenius, that is under \(\text{Gal}(\mathbb{F}/\mathbb{F}_q)\), with a single Jordan block (of rank two, namely \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\)) monodromy at each place of \(S\) [equivalently: the number of isomorphism classes of \(\mathbb{Q}_l\text{-smooth irreducible sheaves}\) of rank two on \((X - S) \otimes_{\mathbb{F}_q} \mathbb{F}\) with principal unipotent local monodromy at each place of \(S\) and fixed by the Frobenius, or equivalently: the number of \(\mathbb{Q}_l\text{-smooth irreducible sheaves}\) of rank two on \(X - S\), with principal unipotent local monodromy at each place of \(S\), up to isomorphism and twisting by a character of \(\text{Gal}(\mathbb{F}/\mathbb{F}_q)\)], is equal to the trace of the Frobenius on the virtual \(\mathbb{Q}_l\text{-smooth sheaf}\) displayed above, on the moduli stack \(\mathcal{M}_{g,[M]}\) over \(\mathbb{Z}\).

The new case, compared with [DF13] where the case \(N > 1\) (denoted by \(N_1\) in [DF13]) is dealt with, is that of \(N = 1\). Here \(N\) is the number of \(\mathbb{F}_q\text{-orbits}\) in \(S\), namely the number of closed points in \(S\). In this case \(S\) consists of a single closed point \(u\) of degree \(d = d_u = \text{deg}(u)\). Our result in this case, recorded as Theorem 4.1 below, is expressed in terms of the \(\zeta\)-function of the curve \(X\). Recall that \(\zeta(X, t) = \prod_{v \in |X|} (1 - t_v)^{-1}, \ t_v = t^{\deg(v)}, \ |X|\) denotes the set of closed points in \(X\), and \(\zeta(X, t) = \frac{h_1(t)}{(1 - q)(1 - qt)}\), where \(h_1(t) = \sum_{0 \leq i \leq 2g} a_i t^i\), \(a_i \in \mathbb{Z}, \ g = \text{genus}(X)\). The result is \(T = h_1 b_u\), where \(h_1 = \sum_{i} a_i = h_1(1)\) and

\[
b_u = h_1(q) \sum_{0 \leq i \leq d - 3} [(d - 1 - j)/2]q^i + [d/2] \sum_{0 \leq i \leq 2g} a_i \left(\sum_{0 \leq j < i} q^j - 1\right) + 1 + (d - 2[d/2]) \left(\sum_{1 \leq i \leq 2g} a_i \sum_{0 \leq j < (i - 1)/2} q^{i-1-2j} - a_0\right) .\]
The proof is based on an explicit evaluation of the trace formula of GL(2) over the function field $F = \mathbb{F}_q(X)$ of $X$ over $\mathbb{F}_q$ at a suitable test function $\otimes_v f_v$. Here $f_u$ is the characteristic function $\chi_{K_v}$ of the maximal compact subgroup $K_v = \text{GL}(2, O_v)$ at each place $v \neq u$ of $F$ (here $O_v$ denotes the ring of integers in the completion $F_v$ of the field $F$ at the place $v$), and a pseudo coefficient $f_u = \chi_{I_u}/|I_u| - 2\chi_{K_v} = (g_u + 1)\chi_{I_u} - 2\chi_{K_v}$ of the Steinberg representation twisted by any unramified character, at $u$. The statement of the trace formula is Theorem 2.1. The proof, which follows work of Drinfeld, is developed in [F14]. Drinfeld computed the trace formula for GL(2) in order to count in [D81] the Frobenius fixed $\ell$-adic local systems, namely smooth sheaves, namely representations of the fundamental group $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$. We construct a virtual $\mathbb{Q}_\ell$-smooth sheaf to express his result as follows.

Let $u(X)$ denote the number of the isomorphism classes of the two-dimensional irreducible $\ell$-adic representations $\rho$ of the geometric fundamental group $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$ of the curve $X$, invariant under the Frobenius, that is under $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$, (equivalently: the number of isomorphism classes of $\mathbb{Q}_\ell$-smooth irreducible sheaves of rank two on $X \otimes_{\mathbb{F}_q} \mathbb{F}$ fixed by the Frobenius, or equivalently: the number of $\mathbb{Q}_\ell$-smooth irreducible sheaves of rank two on $X$ up to isomorphism and twisting by a character of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$), is equal to the trace of the Frobenius on the virtual $\mathbb{Q}_\ell$-smooth sheaf $\sum_i (-1)^i \bigwedge^i \mathcal{H}$ tensored with

$$
\sum_{0 \leq i \leq 2g} (-1)^i \bigwedge^i \mathcal{H} \otimes \left[ \sum_{0 \leq j \leq i-3} [(i-j-1)/2] \mathbb{Q}(-j) \right] + (g-i) \sum_{0 \leq j < g-i} \mathbb{Q}(-j) + (1-i) \mathbb{Q} - \mathbb{Q}
$$

where $\mathcal{H}$ is the local system of $H^1(X)$, and $\mathbb{Q}(-j) = \mathbb{Q}(-1)^i$, on the moduli stack $\mathcal{M}_g$ over $\mathbb{Z}$ of the curves $X$ of genus $g$. We deduce Drinfeld’s original result [D81], that $u(X) = bh_1$ with $h_1 = \sum_i a_i$ and

$$
b = \sum_{0 \leq j \leq i \leq 2g} a_i q^j [(i-j-1)/2] + \sum_{0 \leq i \leq 2g} a_i (g-i) \sum_{0 \leq j < g-i} q^j + (1-g)h_1 - 1,
$$

in Theorem 3.1 from the trace formula of Theorem 2.1 upon evaluation at the test function $f = \otimes_v f_v$, $f_v$ being the characteristic function of $\text{GL}(2, O_v)$ for all $v$, as Drinfeld did.

As is clear from [F14], the trace formula is a heavy analytic machine. The point of [DF13] is that considering objects with suitable ramification at least at two points, and we take at [DF13] principal unipotent ramification, one can transfer the question to an anisotropic form of the group, where the compact quotient trace formula is easy to compute. It will of course be of much interest to compute the trace formula for GL(n) and a test function $f_{u} \otimes \otimes_{v \neq u} f_v$, $f_u = \chi_{K_v}$ for all $v \neq u$ and $f_u$ pseudo coefficient of the Steinberg representation twisted by any unramified character, and extend Theorem 4.1 to all $n \geq 2$. More demanding is the case of the test function $\otimes_v f_v$ with $f_v = \chi_{K_v}$ for all $v$, which would permit extending Drinfeld’s nowhere ramified case of Theorem 3.1 to all $n \geq 2$.

Section 5, based on a letter of P. Deligne to the author dated August 8, 2012, computes the number of such objects with any unipotent monodromy, principal or trivial, in our rank two case. It concludes the surprising fact that this number depends only on the geometric $X$ and $\deg(S)$, not on the number of points in the
rational divisor \( S_1 \), nor on the degrees of the points in \( S_1 \) (notations as in [DF13]). Only the total degree (cardinality of \( S \)) is relevant. This does not hold when we consider in [DF13] only principal unipotent local monodromy. That the position of the points does not matter is interesting too, but found already in [DF13].

Our terminology follows that of [DF13], where detailed definitions are given, except that \( X, \mathcal{S}, N, M \) of our (rational) sections 1-4 are denoted by \( X_1, S_1, N_1, \mathcal{N} \) in [DF13] and in section 5 below (where \( X, \mathcal{S}, N \) signify the absolute avatars).

The author is a 2013 Simons Fellow. He is deeply grateful to Vladimir Drinfeld for his permission to include section 3, to Pierre Deligne for his suggestion to include “locally constant”, and to David Kazhdan, Takayuki Oda, Dipendra Prasad for invitations to discuss this work at the Hebrew University, University of Tokyo, TIFR.

2. Statement of the Trace Formula

Let us write the trace formula for \( \text{GL}(2) \) over a function field \( F \) of a smooth projective geometrically connected curve \( X \) over a finite field \( \mathbb{F}_q \), and a test function \( f \) in \( C^\infty_c(\text{GL}(2, \mathbb{A})) \) (subscript \( c \) for “compactly supported”, superscript \( \infty \) for “locally constant”, \( \mathbb{A} \) denotes the ring of adèles of \( F \)). Fix an idèle \( \alpha \) of degree 1. Let \( \tau_0 \) be the representation of \( \text{GL}(2, \mathbb{A}) \) by right translation on the space \( \mathcal{A}_{\alpha, \alpha} \) of cusp forms on \( \alpha \cdot \text{GL}(2, \mathbb{F}) \backslash \text{GL}(2, \mathbb{A}) \), and \( \tau_0(f) = \int f(g) \tau_0(g)dg \,(g \in \text{GL}(2, \mathbb{A})) \) the convolution operator; \( dg = \otimes_v dg_v \) is a Haar measure.

A cusp form is a function \( \phi : \text{GL}(2, \mathbb{F}) \backslash \text{GL}(2, \mathbb{A}) \rightarrow E \) (\( E \) is a fixed algebraically closed subfield of \( \mathbb{C} \)) which is invariant on the right by some open compact subgroup of \( \text{GL}(2, \mathbb{A}) \), and \( \int_{N(\mathbb{F}) \backslash N(\mathbb{A})} \phi(nx)dn = 0 \) for all \( x \) in \( \text{GL}(2, \mathbb{A}) \). Here \( N \) denotes the unipotent upper triangular subgroup of \( \text{GL}(2) \). We also write \( A \) for the diagonal subgroup, and \( A' = A - Z \) where \( Z \) is the center of \( \text{GL}(2) \).

**Theorem 2.1.** For any \( f \) in \( C^\infty_c(\text{GL}(2, \mathbb{A})) \) we have \( \text{tr} \, \tau_0(f) = \sum_{1 \leq i \leq 8} S_i(f) \). Here

\[
S_1(f) = |\alpha^\mathbb{Z} \cdot \text{GL}(2, \mathbb{F}) \backslash \text{GL}(2, \mathbb{A})| \sum_{\gamma \in \alpha^\mathbb{Z} \cdot \mathbb{F}} f(\gamma).
\]

\[
S_2(f) = \sum_{F_2} S_{2,F_2}(f),
\]

\[
S_{2,F_2}(f) = |\text{Aut}_F F_2|^{-1} \sum_{\gamma \in \alpha^\mathbb{Z} \cdot (F_2 - F)} \int_{\text{GL}(2, \mathbb{A})/\alpha^\mathbb{Z} \cdot F_2^\circ} f(x\gamma x^{-1})dx.
\]

Here \( F_2 \) ranges over the set of isomorphism classes of quadratic extensions of the field \( F \). For each \( F_2 \) we fix an embedding \( F_2 \hookrightarrow M(2, \mathbb{F}) \) into the ring of \( 2 \times 2 \) matrices over \( F \).

\[
S_3(f) = \sum_{\gamma \in \alpha^\mathbb{Z} \cdot A'(\mathbb{F})} \int_{A(\mathbb{A}) \backslash \text{GL}(2, \mathbb{A})} f(x^{-1} \gamma x)v(x)dx.
\]

Any \( x \) in \( \text{GL}(2, \mathbb{A}) \) can be written in the form ank, \( a \in A(\mathbb{A}), k \in \text{GL}(2, O_\mathbb{A}), \) \( n = (\begin{smallmatrix}a & b \\ 0 & 1 \end{smallmatrix}) \), \( b \) is determined uniquely by \( x \) up to \( b \mapsto ub + w \), \( u \in O_\mathbb{A}^times, w \in O_\mathbb{A} \). Put \( v(x) = \sum_v \log(max(1, |b_v|_v)). \)

\[
S_4(f) = \sum_{a \in F^\times \cdot \alpha^\mathbb{Z}} \tilde{\theta}_{a,f}(1), \quad \tilde{\theta}_{a,f}(t) = \frac{1}{2}(\theta_{a,f}(t) + \theta_{a,f}(t^{-1})),
\]

\[
\theta_{a,f}(t) = \int_{F^\times \cdot \alpha^\mathbb{Z} \cdot N(\mathbb{F}) \backslash \text{GL}(2, \mathbb{A})} f\left(x^{-1} \begin{pmatrix}a & 0 \\ 0 & a^{-1} \end{pmatrix} x\right) t^{ht^+(x)}dx,
\]
\[ a, b \in h_t \]

Notations are as in \[ M \]

The product is well defined as the local operator maps the function in the source whose argument used in the proof of Proposition 3.9 shows that for any \( S \) represents the trace of the convolution operator associated with \( f \). Then by analytic continuation, as it is a rational function in \( z \), it follows from \([F14]\) Proposition 4.11, for \( R \) and \( R^{-1} \) from \([F14]\) Corollary 4.28.

\[ \sum_{\mu_1, \mu_2} \int_{|z|=1} \text{tr} I(\mu_1 \nu_z, \mu_2 \nu_{z^{-1}}, f) R(\mu_1, \mu_2, z)^{-1} \frac{dz}{dz} R(\mu_1, \mu_2, z) \]

It is a GL(2, \( \mathbb{A} \))-module by right translation, and \( \text{tr} I(\mu_1 \nu_z, \mu_2 \nu_{z^{-1}}, f) \) is the trace of the indicated convolution operator.

\[ S_6(f) = \frac{-1}{4\pi i} \sum_{\mu_1, \mu_2} \int_{|z|=1} \text{tr} \left[ I(\mu_1 \nu_z, \mu_2 \nu_{z^{-1}}, f) \cdot R(\mu_1, \mu_2, z)^{-1} \frac{dz}{dz} R(\mu_1, \mu_2, z) \right] \]

Notations are as in \( S_5(f) \), and \( R(\mu_1, \mu_2, z) : I(\mu_1 \nu_z, \mu_2 \nu_{z^{-1}}) \rightarrow I(\mu_2 \nu_{z^{-1}}, \mu_1 \nu_z) \) is an operator, rational in \( z \), defined as a product \( \otimes \phi R(\mu_1, \mu_2, \nu_z) \), \( \nu_z(x) = z^{\deg(x)} \). Also \( I(\mu_1, \mu_2) \) is the space of right locally constant functions \( \phi \) on \( \text{GL}(2, \mathbb{A}) \) with

\[ \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = |a/b|^{1/2} \mu(a) \mu(b) \phi(x) \quad (x \in \text{GL}(2, \mathbb{A}); \ a, b \in \mathbb{A}^\times; \ c \in \mathbb{A}) \]

It is rational and may have at most 1 pole, for each \( S \). Hence \( \sum S_8 \) is over all automorphic one dimensional representations of \( \mathbb{A}^\times \text{GL}(2, \mathbb{A}) \). The integral there represents the trace of the convolution operator associated with \( f \).

The terms \( S_1(f) \) and \( S_2(f) \) are finite by \([F14]\), Propositions 3.5, 3.6, 3.9. The argument used in the proof of Proposition 3.9 shows that for any \( \gamma \in \mathbb{A}^2(A(F) - Z(F)) \) the function \( x \mapsto f(x^{-1} \gamma x) \) on \( A(\mathbb{A}) \text{GL}(2, \mathbb{A}) \) has compact support, hence the integral in \( S_3(f) \) converges.

By \([F14]\) Proposition 3.11 the function \( \theta(x,t) \) is rational and may have at \( t = 1 \) a pole of order at most 1, for each \( a \in \mathbb{A}^\times \). Hence \( \theta(x,t) \) is regular at \( t = 1 \). From \([F14]\) Proposition 3.5 it follows that the sums in \( S_3(f) \) and \( S_4(f) \) are finite, so these terms are well defined.

For any \( f = \otimes f \) in \( C^\infty_0(\text{GL}(2, \mathbb{A})) \), the operator \( I(\mu_1, \mu_2, f) \) is zero unless \( \mu_i \) are unramified at each \( v \) where \( f_v \) is \( \text{GL}(2, O_v) \) bi-invariant. This implies that the sums in \( S_i(f) \) (5 \( i \leq 8 \)) are finite, for a given \( f \). To see that \( S_5(f) \) and \( S_6(f) \) are well defined, note that the rational functions \( m(\mu, t) \), \( R(\mu_1, \mu_2, t) \), \( R(\mu_1, \mu_2, \nu_z)^{-1} \) are regular on \( \nu_z = 1 \) for all characters \( \mu, \mu_1, \mu_2 \) of \( \mathbb{A}^\times \text{GL}(2, \mathbb{A}) \). For \( m(\mu, t) \) this follows from \([F14]\) Proposition 4.11, for \( R \) and \( R^{-1} \) from \([F14]\) Corollary 4.28.
The distributions [linear forms on $C^\infty_c(\text{GL}(2, \mathbb{A}))$] $f \mapsto \text{tr} r_0(f)$, $S_i(f)$ ($i = 1, 2, 5, 7, 8$) are invariant, namely take the same value at $f$ and $f^h(x) = f(h^{-1}xh)$, $h \in \text{GL}(2, \mathbb{A})$. For $i = 3, 4, 6$ we have $S_i(f^h) = S_i(f)$ if $h \in \text{GL}(2, \mathbb{O}_A)$, but $S_i$ is not invariant.

If $f \in C^\infty_c(\text{GL}(2, \mathbb{A}))$ takes values in $\mathbb{Q}$ then $\text{tr} r_0(f) \in \mathbb{Q}$, since the representation $r_0$ is defined over $\mathbb{Q}$. For $i = 1, 2, 3, 4, 8$ it is clear that $S_i(f) \in \mathbb{Q}$. For $i = 7$ the integrand contains the factor $\mu(ab)|a/b|^{1/2}$ which involves $\sqrt{q}$. However the sum includes with $\mu$ also $\mu \varepsilon$, $\varepsilon(\alpha) = -1$, and so the sum of the terms indexed by $\mu$ and $\mu \varepsilon$ can be written as an integral over the domain where $|a/b|$ is in $q^{2\mathbb{Z}}$.

To see that $S_5(f)$ is rational, we put $a(\mu_1, \mu_2) = \frac{1}{2\pi i} \int_{|t|=1} f(\mu_1, \mu_2, t)dt$ where

$$f(\mu_1, \mu_2, t) = \text{tr} I(\mu_1 \nu_t, \mu_2 \nu_{t-1}, f) \cdot \frac{d}{dt} \ln m(\mu_1/\mu_2, t^2),$$

and claim that for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one has $\sigma(a(\mu_1, \mu_2)) = a(\sigma \mu_1, \sigma \mu_2)$. Note that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the group of characters on $\mathbb{A}^\times / \mathbb{F}^\times \cdot \alpha^Z$ as they are all $\mathbb{Q}$-valued. Now $a(\mu_1, \mu_2)$ is the sum of the residues of $f(\mu_1, \mu_2, t)$ at the points of the unit disc. We have that $\sigma(f(\mu_1, \mu_2, t)) = f(\sigma \mu_1, \sigma \mu_2, \varepsilon(\sigma) \cdot t)$ with $\varepsilon(\sigma) = \sigma(\sqrt{q})/\sqrt{q}$. However, if $f(\mu_1, \mu_2, t)$ has a pole at $t = t_0$ and $|t_0| < 1$, then by [F14] Proposition 4.11, $|\sigma(t_0)| < 1$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence $S_5(f) \in \mathbb{Q}$.

To see that $S_6(f) \in \mathbb{Q}$ one proceeds similarly, using the results of [F14] Corollary 4.28 on the poles of $R(\mu_1, \mu_2, t)$ and $R(\mu_1, \mu_2, t^{-1})$.

3. Unramified representations

Let $Y$ be the set of isomorphism classes of two dimensional irreducible $\ell$-adic representations of $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$. Our aim is to compute the number $u(X)$ of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$-invariant elements in $Y$ in terms of the $\zeta$-function of the curve $X$. Recall that $\zeta(X, t) = \prod_{v \in |X|} (1 - t_v)^{-1}$, $t_v = t^{\text{deg}(v)}$, and $\zeta(X, t) = \frac{h_1(t)}{(1-q)(1/qt)}$ where

$$h_1(t) = \sum_{0 \leq i \leq 2g} a_i t^i, \quad a_i \in \mathbb{Z}, \quad g = \text{genus}(X).$$

**Theorem 3.1.** We have $u(X) = bh_1$ with $h_1 = \sum_i a_i$ and

$$b = \sum_{0 \leq j \leq i \leq 2g} a_i q^i [(i - j - 1)/2] + \sum_{0 \leq j \leq 2g} a_i (g - i) \sum_{0 \leq j < g - i} q^j + (1 - g)h_1 - 1.$$

It is equal to the trace of the Frobenius on the virtual $\overline{\mathbb{Q}}_\ell$-smooth sheaf $\sum_i (-1)^i \wedge^i \mathcal{H}$ tensored with

$$\sum_{0 \leq i \leq 2g} (-1)^i \wedge^i \mathcal{H} \otimes \left[ \sum_{0 \leq j \leq i - 3} [(i - j - 1)/2] \mathbb{Q}(-j) + (g - i) \sum_{0 < j < g - i} \mathbb{Q}(-j) + (1 - i)\mathbb{Q} \right] - \mathbb{Q}$$

where $\mathcal{H}$ is the local system of $H^1(X)$, and $\mathbb{Q}(-j) = \mathbb{Q}(-1)^j$ where $\mathbb{Q}(-1)$ is the Tate local system, on the moduli stack $\mathcal{M}_g$ over $\mathbb{Z}$ of the curves $X$ of genus $g$.

The Weil group $\pi_1^W(X)$ is the pullback of $\mathbb{Z} \subset \hat{\mathbb{Z}} = \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ in

$$1 \rightarrow \pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}) \rightarrow \pi_1(X) \rightarrow \text{Gal}(\mathbb{F}/\mathbb{F}_q) \rightarrow 1.$$

Recall that $\alpha$ is a fixed idèle of degree 1. An $\ell$-adic representation $\rho$ of $\pi_1^W(X)$ is called normalized if $(\det \rho)(\alpha) = 1$. 

Put $h_m = \# \text{Pic}^0(X)(\mathbb{F}_q^m)$. Recall that $\text{Pic}(X)(\mathbb{F}_q) = \pi_1^W(X)^{\text{ab}} = \mathbb{A}^\times/F^\times O^\times_k$.

Let $V$ denote the space of right-GL$(2, O_k)$ invariant automorphic cusp forms on GL$(2, \mathbb{A})/\mathbb{A}^\times \mathbb{Z}^\times$. To prove Theorem 3.1, we first prove Lemmas 3.2-3.9.

**Lemma 3.2.** We have $u(X) = \frac{1}{2} \dim_{\mathbb{Q}_l} V + \frac{1}{2}(h_1 - h_2)$. Here $h_2 = \# \text{Pic}^0(X)(\mathbb{F}_q^2)$.

**Proof.** Denote by $Z$ the set of isomorphism classes of two dimensional irreducible $\ell$-adic representations of $\pi_1^W(X)$ which are normalized. Denote by $Z_1$ the set of elements of $Z$ whose restriction to $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$ is irreducible. Put $Z_2 = Z - Z_1$. By the automorphic-Galois reciprocity law, $\# Z = \dim_{\mathbb{Q}_l} V$. The natural map $Z_1 \to Y$ is onto, its fibers consist of two elements each, obtained from each other by twisting by the nontrivial character of $\pi_1^W(X)/\pi_1^W(X \otimes_{\mathbb{F}_q} \mathbb{F}_q)$.

Hence $u(X) = \frac{1}{2} \# Z_1$. It remains to show that $\# Z_2 = \frac{1}{2}(h_2 - h_1)$. For that we note the following

1) An irreducible two dimensional $\ell$-adic representation $\rho$ of $\pi_1^W(X)$ becomes reducible upon restriction to the subgroup $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$ iff $\rho$ is induced from a character $\chi$ of $\pi_1^W(X \otimes_{\mathbb{F}_q} \mathbb{F}_q)$: we write $\rho = \text{Ind}(\chi)$ in this case, and note that $\chi$ is determined uniquely by $\rho$ up to replacement with $\sigma \chi$, where $\sigma$ is the nontrivial automorphism of $\mathbb{F}_q$ over $\mathbb{F}$.

2) The representation $\text{Ind}(\chi)$ of $\pi_1^W(X)$ is irreducible iff $\chi \neq \sigma \chi$.

3) If $\rho = \text{Ind}(\chi)$ then $\det \rho$ is a character of $\pi_1^W(X)^{\text{ab}}$, with $\det \rho = (\chi \circ t)\omega$. Here $\omega$ is the nontrivial character of $\pi_1^W(X)$ whose restriction to $\pi_1^W(X \otimes_{\mathbb{F}_q} \mathbb{F}_q)$ is trivial, and $t : \pi_1^W(X)^{\text{ab}} \to \pi_1^W(X \otimes_{\mathbb{F}_q} \mathbb{F}_q)^{\text{ab}}$ is the transfer homomorphism, which by class field theory coincides with the natural embedding $\text{Pic}(X(\mathbb{F}_q)) = \pi_1^W(X)^{\text{ab}} \to \pi_1^W(X \otimes_{\mathbb{F}_q} \mathbb{F}_q)^{\text{ab}} = \text{Pic}(X(\mathbb{F}_q^2))$.

It follows from 1)-3) that $\# Z_2$ is $\frac{1}{2}(x - y)$, where $x$ denotes the number of characters $\chi$ of $\text{Pic}^0(X)(\mathbb{F}_q)$, and $y$ is the number of those $\chi$ with $\sigma \chi = \chi$. Then $x = h_2$ and $y = h_1$, as $\sigma \chi = \chi$ implies that $\chi = \eta \circ t$, $\eta$ being a character of $\text{Pic}^0(X)(\mathbb{F}_q)$. \hfill $\Box$

Now $\dim_{\mathbb{Q}_l} V = \text{tr} r_0(f)$ where $r_0$ is the representation of GL$(2, \mathbb{A})$ in $A_{0, \alpha}$ and $f$ denotes the characteristic function of GL$(2, O)$ in $C_c^\infty(\text{GL}(2, \mathbb{A}))$. We work with the measure on GL$(2, \mathbb{A})$ which is normalized by $|\text{GL}(2, O)| = 1$.

By the trace formula $\dim_{\mathbb{Q}_l} V = \sum_{1 \leq i \leq 8} S_i(f)$. We then compute these terms.

**Lemma 3.3.** We have $S_i(f) = 2h_1h_1(q)/(q - 1)(q^2 - 1)$.

**Proof.** Recall that $S_1(f) = |\alpha^Z \cdot \text{GL}(2, F) \setminus \text{GL}(2, \mathbb{A})| \sum_{\gamma \in \alpha^Z \cdot F^\times} f(\gamma)$.

For any $\gamma$ in $\alpha^Z \cdot F^\times$ we have $f(\gamma) = 0$ unless $\gamma \in F_q^\times$, where $f(\gamma) = 1$. Then $\sum_{\gamma} f(\gamma) = q - 1$. Also $|\text{GL}(2, \mathbb{A})/\alpha^Z \cdot \text{GL}(2, F)|$ is $|\text{SL}(2, \mathbb{A})/\text{SL}(2, F)| \cdot |\mathbb{A}^\times/F^\times \cdot \alpha^Z| = 2|\text{SL}(2, \mathbb{A})/\text{SL}(2, F)| \cdot |\mathbb{A}^\times/F^\times \cdot \alpha^Z|$. We normalize the Haar measures on SL$(2, \mathbb{A})$ and $\mathbb{A}^\times$ by $|\text{SL}(2, O)| = 1$, $|O^\times| = 1$.

The exact sequence

$$1 \to \mathbb{F}_q^\times \to O^\times \to \mathbb{A}^\times/\alpha^Z \cdot F^\times \to (\text{Pic} X)/J(= \text{Pic}^0 X) \to 1$$

shows that $|\mathbb{A}^\times/F^\times \cdot \alpha^Z| = h_1 \cdot |O^\times|/(q - 1)$. It remains to show that $|\text{SL}(2, \mathbb{A})/\text{SL}(2, F)| = \zeta(X, q)$.

The definition of the Tamagawa measure on an algebraic group $G$ of dimension $n$ over $F$ requires a nonzero invariant $n$-form $\omega$ on $G$. For any closed point $v \in |X|$
fix the Haar measure $dx_v$ on $F_v$ so that $\int_{K_v/F_v} dx_v = 1$. The form $\omega$ and the measure $dx_v$ determine canonically a Haar measure $\mu_v$ on $G(F_v)$ for each $v \in |X|$. The group $G$ is the generic fiber of a group scheme $G'$ over $X - \text{Fin}$, where Fin is a finite set in $X$. Hence for almost all $v \in |X|$, the groups $G'_v = G'(O_v) \subset G(F_v)$ are defined. Of course replacing $G'$ by a different group scheme with generic fiber $G$ leads to changing at most finitely many of the $G'_v$. Assume that the product $\prod_v \mu_v$ converges. Then the measure $\mu_T = \prod_v \mu_v$ on $G(\mathbb{A})$ is well defined. It is named the Tamagawa measure. It depends neither on the choice of the measures $dx_v$ nor on the choice of the form $\omega$, by the product formula.

The Tamagawa volume of $\text{SL}(2, \mathbb{A})/\text{SL}(2, F)$ is well known to be $1$. So it remains to show that the Tamagawa volume $\text{vol}_T(\text{SL}(2, O))$ of $\text{SL}(2, O)$ is $\zeta(X, q)^{-1}$, as

$$\text{vol}_T(\text{SL}(2, \mathbb{A})/\text{SL}(2, F))/\text{vol}_T(\text{SL}(2, O)) = \left| \text{SL}(2, \mathbb{A})/\text{SL}(2, F) \right|/\left| \text{SL}(2, O) \right|.$$

With the Haar measure $dx_v$ on $F_v$ as above, put $a_v = |O_v|$. Then $\prod_v a_v = |O|/|k/F| = q^{1-g}$ (see the end of proof of [F14] Proposition 3.11). The form $\omega$ on $\text{SL}(2)$ can be chosen to be defined over the subfield $\mathbb{F}_q \subset F$. Then $|\text{SL}(2, O_v)| = (a_v/q_v)^3|\text{SL}(2, \mathbb{F}_q)| = a_v^3(1 - q_v^{-2})$. Hence $\text{vol}_T(\text{SL}(2, O)) = \prod_v a_v^3(1 - q_v^{-2}) = q^{3-3g}(X, q^{-2})^{-1}$. Using the functional equation $\zeta(X, t) = t^{g-1}q^{g-2}\zeta(X, 1/tq)$ one deduces that $\text{vol}_T(\text{SL}(2, O)) = \zeta(X, q)^{-1}$, as required. 

**Lemma 3.4.** We have $S_2(f) = qh_2/2(q + 1)$.

**Proof.** Recall that $S_2(f) = \sum_{F_2} S_{2, F_2}(f)$, where $F_2$ range over a set of representatives for the isomorphism classes of quadratic extensions of $F$, embedded in $M(2, F)$, and

$$S_{2, F_2}(f) = \left| \text{Aut}_F F_2 \right|^{-1} \sum_{\gamma \in \alpha^{2}(F_2 - F)} \int_{\text{GL}(2, \mathbb{A})/\alpha^{2}, F_2} f(x\gamma x^{-1}) dx.$$

If $\gamma \in \alpha^{2}(F_2 - F)$ and $f(x\gamma x^{-1}) \neq 0$, then $\text{tr} \gamma \in O, \det \gamma \in O^\times$. If $\gamma = \alpha^m \gamma', m \in \mathbb{Z}, \gamma' \in F_2 - F$, then $\alpha^{2m} \det \gamma' \in O^\times$, hence $m = 0$, so $\gamma \in F_2 - F$, $\text{tr} \gamma \in O \cap F = F_q, \det \gamma \in O^\times \cap F^\times = F_q^\times$. Hence $\gamma \in F_{q^2} - F_q, F_2 = F \otimes_{\mathbb{F}_q} F_q^\times$. If $|\text{Aut}_F F_2| = 2$. For such $\gamma$ we have $x\gamma x^{-1} \in \text{GL}(2, O)$ if $x \in \text{GL}(2, O) \cdot (A \otimes_{\mathbb{F}_q} F_q^\times)$. Hence $S_2(f) = \frac{1}{2}||F_q^2 - F_q|| \text{vol}(Z) = \frac{1}{2}(q^2 - q) \text{vol}(Z)$, where $Z = AB/C, A = \text{GL}(2, O), B = (A \otimes_{\mathbb{F}_q} F_q^\times) \supset C = (F \otimes_{\mathbb{F}_q} F_q^\times) \alpha^{2}.\n
As |A| = 1 and $A \cap B = (O \otimes_{\mathbb{F}_q} F_q^\times)$, using $A/A \cap C \times B/C \cdot A \cap B \cong AB/C$ we get

$$\text{vol} Z = \left| B/A \cap C \right| = \frac{|(A \otimes_{\mathbb{F}_q} F_q^\times) \times (O \otimes_{\mathbb{F}_q} F_q^\times) \times (F \otimes_{\mathbb{F}_q} F_q^\times) \alpha^{2}|}{|F \otimes_{\mathbb{F}_q} F_q^\times| \cdot \alpha^{2} \cap (O \otimes_{\mathbb{F}_q} F_q^\times)|} = \frac{\# \text{Pic}^0(X)(\mathbb{F}_q^\times)}{\# F_q^2} = \frac{h_2}{q^2 - 1}.\n
$$

**Lemma 3.5.** We have $S_3(f) = 0$.

**Proof.** Recall that $S_3(f)$ is the sum of $\int_{A(\mathbb{A}) \setminus \text{GL}(2, \mathbb{A})} f(x^{-1}\gamma x)v(x)dx$ over $\gamma$ in $\alpha^{2} \cdot A'((F), A)$ where $A$ is the diagonal subgroup of $\text{GL}(2)$ and $A'$ is the set of diag(a, b) with $a \neq b$. The function $v : \text{GL}(2, \mathbb{A}) \to \mathbb{Z}$ takes the value $0$ on $A(\mathbb{A}) \text{GL}(2, O)$. Now $f(x^{-1}\gamma x) \neq 0$ for $\gamma$ in $\alpha^{2} \cdot A'((F)$ and $x \in \text{GL}(2, \mathbb{A})$ implies, as in the proof of
the previous Lemma, that $\gamma \in A'(F_q)$. Writing $x = nk$, $n = (\frac{1}{k} b)$, $k \in \text{GL}(2, O)$, and multiplying $\gamma^{-1} n^2 \gamma$, we conclude from $f(x^{-1} \gamma x) \neq 0$ that $x \in \text{GL}(2, O)$, hence $v(x) = 0$. □

Lemma 3.6. We have $S_4(f) = h_1 w$, $w = \frac{1}{2} \lim_{t \to 1} [t^{2g-2} \zeta(X, 1/t) + t^{2g-2} \zeta(X, t)]$

\[= \lim_{t \to 1} \frac{1}{2} \left[ \frac{h_1(t)^{t^{2g-2}}}{(1-t)(1-q/t)} + \frac{h_1(t^{-1})^{t^{2g-2}}}{(1-t^{-1})(1-q/t)} \right] = h_1^2(1) - h_1 \left[ \frac{2g - \frac{1}{2}}{q-1} + \frac{1}{(q-1)^2} \right]. \]

Proof. Recall that $S_4(f)$ is the sum over $a \in F^\times \cdot \alpha^\times$ of the value at $t = 1$ of $\frac{1}{2}(\theta_{a,f}(t) + \theta_{a,f}(t^{-1}))$, where

\[\theta_{a,f}(t) = \int_{\alpha^\times \cdot F^\times \cdot N(F) \setminus \text{GL}(2, O)} f(x^{-1} (a \cdot x)) dht(x) dt.\]

We have $\theta_{a,f}(t) = 0$ if $a \notin F_q^\times$, and $\theta_{a,f}(t) = \theta_{1,f}(t)$ if $a \in F_q^\times$. By [F14] Proposition 3.11 we have $\theta_{1,f}(t) = h_1 q^{g-1} \zeta(X, t/q)/(q-1)$. Then the functional equation $q^{g-1} \zeta(X, t/q) = t^{2g-2} \zeta(X, 1/t)$ gives the first expression for $w$ in the lemma. Using $\zeta(X, t) = h_1(t)/(1-t)(1-q/t)$, where $h_1(t)$ is a polynomial of degree $2g$ with integral coefficients, we get the 2nd, while the 3rd and last follows from l'Hôpital's rule. □

Lemma 3.7. We have $S_5(f) = -h_1^2(2g - 2) - h_1$.

Proof. Recall that $S_5(f)$ is the sum over the characters $\mu_1, \mu_2$ of $\alpha^\times \cdot F^\times \setminus A^\times$, of

\[\frac{-1}{4\pi i} \oint_{|t|=1} \text{tr} I(\mu_1 \mu_2, t) \frac{dt}{dt} \ln m(\mu_1/\mu_2, t^2) dt,\]

where $m(\mu, t) = L(\mu, t)/L(\mu, t/q)$. The cardinality of the set $U$ of characters of $\mathbb{A}^\times / \mathbb{A}^\times \cdot \alpha^\times \cdot O^\times$ is $h_1$, and $\text{tr} I(\mu_1 \mu_2, \mu_2 \mu_1^{-1}, f)$ is 1 if $\mu_1, \mu_2$ are in $U$, and 0 if not. Hence $S_5(f)$ is

\[\frac{-1}{4\pi i} \sum_{\mu_1, \mu_2 \in U} \oint_{|t|=1} \frac{dt}{dt} \ln m(\mu_1/\mu_2, t^2) dt = \frac{-h_1}{4\pi i} \sum_{\mu \in U} \oint_{|t|=1} \frac{dt}{dt} \ln m(\mu, t^2) dt\]

\[= - \frac{h_1}{2\pi i} \sum_{\mu \in U} \oint_{|t|=1} \frac{dt}{dt} \ln m(\mu, t) dt = -h_1 \sum_{\mu \in U} (Z'_{\mu} - P'_{\mu}),\]

where $Z'_{\mu}$ is the number of zeroes of $m(\mu, t)$ in $|t| < 1$, and $P'_{\mu}$ the number of poles.

Now (see [F14] Proposition 4.11), $L(\mu, t/q)$ has neither zeroes nor poles in $|t| < 1$. Hence $Z'_{\mu} - P'_{\mu} = P_{\mu} - Z_{\mu}$, where $Z_{\mu}$ is the number of zeroes of $L(\mu, t)$ on $|t| \geq 1$, and $P_{\mu}$ the number of poles. From the functional equation for $L(\mu, t)$, the order of pole of $L(\mu, t)$ at $t = \infty$ is $2g - 2$. If $\mu \neq 1$, the function $L(\mu, t)$ has no poles in $1 \leq |t| < \infty$. If $\mu = 1$ then in this domain $L(\mu, t)$ has only one pole, at $t = 1$, of order one. The function $L(\mu, t)$ has no zeroes in $1 \leq |t| < \infty$. Hence $P_{\mu} - Z_{\mu}$ equals $2g - 2$ if $\mu \neq 1$, and $2g - 1$ if $\mu = 1$. All-in-all we get $-h_1((h_1 - 1)(2g - 2) + (2g - 1))$. □

Lemma 3.8. We have $S_6(f) = 0$.

Proof. Recall: $S_6(f)$ is $-1$ times the sum over the characters $\mu_1, \mu_2$ of $\mathbb{A}^\times / \mathbb{A}^\times \cdot \alpha^\times$, of

\[\oint_{|t|=1} \text{tr} I(\mu_1 \mu_2 \mu_2, t) \cdot R(\mu_1, \mu_2, t)^{-1} \cdot \frac{dt}{dt} R(\mu_1, \mu_2, t) dt.\]

Note that the restriction of the operator $R(\mu_1, \mu_2, t)$ to the space of $\text{GL}(2, O)$-fixed vectors in $I(\mu_1 \mu_2, \mu_2 \mu_1)$ does not depend on $t$, in the sense that if $\varphi \in$
Lemma 3.9. We have $S_7(f) = h_1/2$ and $S_8(f) = -2h_1$.

This is clear.

Proof of Theorem 3.1. By Lemmas 3.2-3.9, $u(X)$ is equal to

$$
\frac{h_1 - h_2}{4} + \frac{1}{2}h_1 \zeta(X, q) + \frac{qh_2}{2(q+1)} + h_1 w - h_1^2(2g-2) - h_1 + \frac{h_1}{2} - 2h_1
$$

with $h_1 = \# \text{Pic}^0(X)(\mathbb{F}_q)$, $h_2 = \# \text{Pic}^0(X)(\mathbb{F}_{q^2})$, $w$ as in $S_4(f)$ above.

Note that $h_2 = h_1(1)h_1(-1)$, since $h_m(t) = \prod_i (1-tX)^m$. Thus $u(X) = bh_1$ and

$$
b = \frac{h_1(1)}{(q-1)(q^2-1)} - \frac{h_1(-1)}{4(q+1)} + \frac{h_1'(1)}{2(q-1)} - h_1(1) \left( g - 1 + \frac{g - \frac{1}{2}}{q-1} + \frac{1}{2(q-1)^2} \right) - 1,
$$

but it is not clear from this expression that $u(X)$ is integral.

Recall that $h_1(t) = \sum_{0 \leq i \leq 2g} a_i t^i$. From $h_1(1/qt) = q^{-gt^{-2}h_1(t)}$ we have $a_{2g-i} = q^{g-i}a_i$ and $h_1 = \sum_i a_i$. To express $b$ in terms of the $a_i$ note that the coefficient of $h_1(q)$ in $b$ is $1/2(q-1)^2 - 1/4(q-1) + 1/4(q+1) = 1/(q-1)(q^2-1)$. Thus $b$ can be expressed as the sum of

$$
b_1 = \frac{h_1(q) - h_1(1) - h_1'(1)(q-1)}{2(q-1)^2} - \frac{h_1(q) - h_1(1)}{4(q-1)} + \frac{h_1(q) - h_1(-1)}{4(q+1)}
$$

and

$$
b_2 = \frac{h_1'(1) - gh_1(1)}{q-1} + (1-g)h_1(1) - 1.
$$

The first expression in $b_1$ is the sum over $i$ ($0 \leq i \leq 2g$) of $1/2a_i$ times

$$
\frac{q^i - 1 - i(q-1)}{(q-1)^2} = \frac{q^{i-1} + q^{i-2} + \cdots + q + (1-i)}{q-1} = q^{i-2} + 2q^{i-3} + 3q^{i-4} + \cdots + kq^{i-k-1} + \cdots + (1-i).
$$

The next two, multiplied by 2, are

$$
-\frac{h_1(q) - h_1(1)}{2(q-1)} + \frac{h_1(q) - h_1(-1)}{2(q+1)} = -\sum_{0 \leq i \leq 2g} a_i (q^{i-2} + q^{i-4} + q^{i-6} + \ldots).
$$

Then

$$
b_1 = \sum_{0 \leq i \leq 2g} a_i (q^{i-3} + q^{i-4} + 2q^{i-5} + 2q^{i-6} + \ldots) = \sum_{0 \leq j < i \leq 2g} a_i q^i [(i-j-1)/2].
$$

From the relation $a_{2g-i} = q^{g-i}a_i$ it follows that

$$
\frac{h_1'(1) - gh_1(1)}{q-1} = \sum_{0 \leq i \leq 2g} \frac{i-q}{q-1} a_i = \sum_{0 \leq i \leq g} \left( \frac{i-q}{q-1} a_i + \frac{q-i}{q-1} a_{2g-i} \right)
$$

$$
= \sum_{0 \leq i \leq 2g} (g-i) q^{q-i-1} a_i.
$$

Since $b = b_1 + b_2$, Theorem 3.1 follows.
4. Single Steinberg component

Let $u$ be a closed point of $X$, thus a fixed place of $F$, of degree $d = d_u = \deg(u)$. Recall that $h_1(t) = \sum_{0 \leq i \leq 2g} a_i t^i$, $a_i \in \mathbb{Z}$.

**Theorem 4.1.** The number of the isomorphism classes of the two-dimensional irreducible $\ell$-adic representations $\rho$ of $\pi_1(X^u \otimes_{\mathbb{F}_q} \mathbb{F})$, the geometric fundamental group of the affine curve $X^u = X - \{u\}$, invariant under the Frobenius, that is under $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$, with a single Jordan block of rank two, namely $(0 \ 1 \ 0 \ 1)$ monodromy at $u$ [equivalently: the number of isomorphism classes of $\mathbb{Q}_l$-smooth irreducible sheaves of rank two on $X^u \otimes_{\mathbb{F}_q} \mathbb{F}$ with principal unipotent local monodromy at $u$ and fixed by the Frobenius, or equivalently: the number of $\mathbb{Q}_l$-smooth irreducible sheaves of rank two on $X^u$, with principal unipotent local monodromy at $u$, up to isomorphism and twisting by a character of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$], is $T = h_1 b_u$, where

$$
\quad b_u = h_1(q) \sum_{0 \leq j \leq d-3} \left( \frac{(d - 1 - j)}{2} q^j + \frac{d}{2} \sum_{0 \leq i \leq 2g} a_i \left( \sum_{0 \leq j < i} q^j - 1 \right) + 1 
\right)
+ (d - 2 \frac{d}{2}) \left( \sum_{1 \leq j \leq 2g} a_i \sum_{0 \leq j < (i-1)/2} q^{i-2j} - a_0 \right).
$$

It is equal to the trace of the Frobenius on the virtual $\mathbb{Q}_l$-smooth sheaf

$$
\left( \sum_{i} (-1)^i \bigwedge^i \mathcal{H} \right) \otimes \left( \sum_{j > 0} (-1)^j \mathbb{Q}(-j) \sum_{k > 0} \bigwedge^k \mathcal{H}_c \right),
$$

where $\mathcal{H}_c$ is the local system of $H^1_c(X - \{u\}), \mathcal{H}$ of $H^1(X)$, and $\mathbb{Q}(-j) = \mathbb{Q}(-1)^j$, where $\mathbb{Q}(-1)$ is the Tate local system, on the moduli stack $\mathcal{M}_{g,d}$ over $\mathbb{Z}$ of pairs $(X,S)$ consisting of a curve $X$ of genus $g$ and an étale divisor $S$ consisting of $d$ points on $X$.

**Proof.** A representation $\overline{\rho}$ of $\pi_1 = \pi_1(X^u \otimes_{\mathbb{F}_q} \mathbb{F}) = \text{Gal}(\mathbb{F}^u / \mathbb{F} \otimes_{\mathbb{F}_q} \mathbb{F})$, the geometric fundamental group, is invariant under the Frobenius precisely when it extends to a representation, say $\rho$, of the arithmetic fundamental group $\pi_1(X^u)$. See [DF13], Lemma 1.9. Here $\mathbb{F}^u$ denotes the maximal extension of $\mathbb{F}$ in a fixed algebraic closure $\overline{\mathbb{F}}$ which is unramified outside $u$. For any $\alpha$ in $\pi_1^0(X^u)_{ab} = W(\mathbb{F}^u / \mathbb{F})_{ab} \simeq \mathbb{A}^\times / \mathbb{F}^\times O_{\mathbb{A}}$, which corresponds by CFT to a fixed idèle of degree 1 in $\mathbb{A}^\times / \mathbb{F}^\times O_{\mathbb{A}}$, $\rho$ can be chosen to have det $\rho(\alpha) = 1$. If $\rho$ is irreducible, so is $\overline{\rho}$. Conversely, if $\overline{\rho}$ is irreducible, $\rho$ is too, by the condition that we put at $u$. Any other representation which restricts to $\overline{\rho}$ is of the form $\rho \otimes \chi$, $\chi$ a character of $\text{Gal}(\mathbb{F} / \mathbb{F}_q)$, and $\rho \otimes \chi \simeq \rho$ iff $\chi = 1$.

The number of $\overline{\rho}$ stated in the theorem is the number of equivalence classes of irreducible representations $\rho$ of $\pi_1(X^u)$, with det $\rho(\alpha) = 1$ and of dimension 2, with principal unipotent monodromy at $u$, up to twisting by a character of $\text{Gal}(\mathbb{F} / \mathbb{F}_q) = \pi_1(X^u / \overline{\mathbb{F}})$. If det $\rho(\alpha) = 1$, and $1 = \det(\rho \otimes \chi)(\alpha) = \chi(\alpha)^2 \det \rho(\alpha) = \chi(\alpha)^2$, there are two possibilities for the unramified $\chi$. The number of $\rho$ not up to twisting is twice that of the theorem.

By the automorphic-Galois reciprocity law we translate the question of the theorem, to compute $T$, to the question of counting cuspidal automorphic representations of $\text{GL}(2, \mathbb{A})$ which are unramified at all places except at $u$, whose component
12 Yuval Z. Flicker

Thus we compute the number presented by the sum of the numbers in the components at each value is measure normalized by $v$. We have $\dim V_u = \text{tr} r_0(f_u)$ where $f_u = \otimes f_v$, $f_v$ being the characteristic function $\chi_{K_v}$ of $K_v = \text{GL}(2, O_v)$ in $\text{GL}(2, F_v)$ if $v \neq u$, and $f_u$ is the quotient $\chi_{I_u}/|I_u|$ of the characteristic function $\chi_{I_u}$ of $I_u$ by the volume $|I_u|$ of $I_u$. Note that $K_u = I_u \cup \bigcup_{v \in O_v/\pi_v} (1 \cdot 0 \cdot w)I_u$, $w = (q^{1/2} \cdot q^{-1})$, thus $[K_u : I_u] = 1 + q_u$, so $|I_u| = 1/(1 + q_u)$ and $f_u = (q_u + 1)|\chi_{I_u}|$. By the trace formula, $\dim V_u = \sum_{1 \leq i \leq 8} S_i(f_u)$.

We shall compute $S_i(f_u)$ in Lemmas 4.2-4.9 below.

To compute the number of cuspidal representations of $\text{GL}(2, F)$ whose components at each $v \neq u$ are unramified, while their component at $u$ is an unramified twist of the Steinberg representation, with central character trivial at $\alpha$, we compute the trace formula at the global function $f_{\alpha} = \otimes f_v$, where $f_v = \chi_{K_v}$ for all $v \neq u$, and $f_u = \chi_{I_u}/|I_u| - 2\chi_{K_u} = (q_u + 1)|\chi_{I_u}| - 2\chi_{K_u}$. We work with the Haar measure normalized by $|K_v| = 1$ for all $v$. Indeed, the function $f_u$ has the property that for every irreducible representation $\pi_u$ of $\text{GL}(2, F_u)$ the trace $\text{tr} \pi_u(f_u)$ is 0, except if $\pi_u$ is an unramified twist of the Steinberg representation of $\text{GL}(2, F_u)$, where the value is 1, or $\pi_u$ is an unramified one dimensional representation, where the value is $-1$. But no cuspidal representation of $\text{GL}(2, F)$ has a one dimensional component. Thus we compute the number presented by the sum of the numbers in Lemmas 4.2-4.9 below, minus twice that of Lemmas 3.3-3.9. Write $S_{u,i}$ for $S_i(f_{\alpha})$.

Then $S_{u,i} = 0$ for $i = 3, 5, 7$; $S_{u,8} = 2h_1$, $S_{u,6} = -d_u h_2^1$ and $S_{u,4} = -d_u h_2^2/q - 1$ since $\lim_{s \to 1} (2q^3 - 3q^2 + 1) q^s - 3q^s + 1$ is $2h_1$. Now $S_{u,1} = 2h_2 h_1(q - 1)/(q - 1)(q - 1)$, and $S_{u,2}$ is 0 if 2 divides $\deg(u)$, and it is $-q h_2 h_1 q h_2^1 q^1 h_2^1$ if $2, \deg(u) = 1$. Then $T_u = \sum_{i} S_i(f_{\alpha})$ is

$$\frac{2h_1 h_2(q - 1)}{(q - 1)(q - 1)} - \left(d_u - 2q(h_u/2)\right) q h_2 h_1 q^1 \frac{h_1 q^1}{h_2(q - 1)} - d_u h_2^2 q - d_u h_2^2 q^1 + 2h_1.$$

We claim that $T_u$ has the form $2h_1 b_u$, where $b_u \in \mathbb{Z}$. Recall that $h_1(1) = h_1$ and $h_1(t) = \sum a_i t^i$. When $d = d_u = \deg(u) = 2r$ is even, $T_u$ is then $2h_1$ times

$$h_1(q)(q^{2(r - 1)} + \cdots + q^2 + 1) - r h_1 (q - 1) - r h_1 + 1$$

$$= h_1(q) \sum_{0 \leq i < r} q^{2i} - \frac{r h_1(q - 1)}{q - 1} - h_1 + 1$$

$$= h_1(q) \sum_{0 \leq i < r} q^i + r \sum_{0 \leq i \leq 2g} a_i \sum_{0 \leq j < i} q^j - r h_1 + 1.$$

Note that $\sum_{0 \leq i < r - 1} \sum_{0 \leq j < 2q^i} c_j q^j$ is equal to $t - [t] = [t - 1]/2$ as $i < [t]$. When $d = 2r + 1$ is odd, $T_u/h_1$ is

$$\frac{2h_1(q)(q^{d - 1} + \cdots + q + 1)}{(q - 1)(q + 1)} - \frac{q b_2(q - 1)}{q - 1} - \frac{d h_1}{q - 1} - d h_1 + 2.$$
This is $2A + B$, where
\[
A = h_1(q) \left( \sum_{0 \leq i < r} \frac{q^{g_i^2} - 1}{q^{g_i} - 1} + q \sum_{0 \leq i < r} \frac{q^{g_i} - 1}{q^{g_i} - 1} \right) = h_1(q) \sum_{0 \leq j \leq d-3} \left[ \frac{d-1-j}{2} \right] q^j
\]
and
\[
B = \frac{2h_1(q)(r + 1 + qr)}{q^2 - 1} - \frac{qh_1(-1)}{q + 1} - \frac{dh_1}{q - 1} - dh_1 + 2.
\]
Then $B$ is the sum of $2r(h_1(q) - h_1)/(q - 1)$, $2 - 2rh_1$, and
\[
\frac{2h_1(q)}{q^2 - 1} - \frac{qh_1(-1)}{q + 1} - \frac{h_1}{q - 1} = h_1(q) - h_1(q) + \frac{h_1 + h_1(-1)}{q + 1} - h_1
\]
\[
= \sum_{0 \leq i \leq 2g} a_i \frac{q^i - 1}{q - 1} - \sum_{0 \leq i \leq 2g} a_i \frac{q + (-1)^i q}{q + 1} - \sum_{0 \leq i \leq 2g} a_i
\]
\[
= \sum_{1 \leq i \leq 2g} a_i \left[ \sum_{0 \leq j \leq i - 1} q^j + q(-1)^i \sum_{0 \leq j \leq i - 2} (-q)^j - 1 \right] - 2a_0
\]
\[
= 2 \sum_{1 \leq i \leq 2g} a_i \sum_{0 \leq j \leq \frac{i-1}{2}} q^{i-1-2j} - 2a_0,
\]
and the claim that $T_q = 2h_1 b_n$ follows, subject to Lemmas 4.2-4.9 below.

To verify the last claim of the theorem we note that it is proven in [DF13] when the $d$ geometric points $u$ make at least two Frobenius $Fr_q$ orbits. It remains to deal with the case that $u$ makes a single $Fr_q$ orbit. As in [DF13], denote the multiset (set with multiplicities, unordered sequence) of eigenvalues of the Frobenius on $H^i(X - \{u\})$ by $\{\beta\}$. It consists of the eigenvalues $\lambda_i$ ($1 \leq i \leq 2g$) of the Frobenius on $H^i(X)$ and the $d - 1$ $d$th roots of 1 other than 1, $\eta \neq 1$. Thus we claim that $T_q/2h_1 (h_1$ is clearly the trace of the Frobenius on $\sum_i (-1)^i \Lambda^i H$), namely $\sum_{j>0} (-q)^j \sum_{k>0} \Omega_{j+k}$, where

\[
\Omega_m = \prod_{1 \leq i_1 < \ldots < i_m \leq 2g + d - 1} \beta_{i_1} \ldots \beta_{i_m}, \quad \beta_i \in (\lambda_1, \ldots, \lambda_{2g}, \eta, \eta^2, \ldots, \eta^{d-1}),
\]
is equal to

\[
\frac{h_1(q)(1 - q^d)}{(q^2 - 1)(1 - q)} - \prod_{\eta \neq 1} (1 + \eta) \cdot \frac{q(q - 1)}{2(q^2 - 1)} h_1(-1) - \frac{dh_1 q(q + 1)}{2(q^2 - 1)} + 1.
\]

The last expression is

\[
\frac{1}{q^2 - 1} \left[ \prod_{\lambda} (1 - \lambda q) \cdot \prod_{\eta \neq 1} (1 - \eta q) - \prod_{\lambda} (1 + \lambda) \cdot \prod_{\eta \neq 1} (1 + \eta) \cdot \frac{q(q - 1)}{2}
\]
\[
- \prod_{\lambda} (1 - \lambda) \cdot \prod_{\eta \neq 1} (1 - \eta) \cdot \frac{q(q + 1)}{2} \right] + 1.
\]

This is

\[
\frac{1}{q^2 - 1} \left[ \prod_{\beta} (1 - \beta q) - \prod_{\beta} (1 + \beta) \cdot \frac{q(q - 1)}{2} - \prod_{\beta} (1 - \beta) \cdot \frac{q(q + 1)}{2} \right] + 1.
\]
Now \[
\prod_{\beta}(1 + \beta) = \sum_{0 \leq m < 2g + d} \Omega_m, \quad \prod_{\beta}(1 - \beta) = \sum_{0 \leq m < 2g + d} (-1)^m \Omega_m.
\]
Note that \[
\sum_{0 \leq m < 2g + d} \left[ \frac{q^2 - q}{2} + (-1)^m \frac{q^2 + q}{2} \right] \Omega_m = \sum_{0 \leq m < 2g + d} (-q)^m \Omega_m.
\]
As \[
\prod_{\beta}(1 - \beta q) = \sum_{0 \leq m < 2g + d} (-q)^m \Omega_m
\]
our expression is \[
\sum_{0 \leq m < 2g + d} \frac{q^2 - q^2}{q^2 - 1} \Omega_m - \sum_{0 \leq m < 2g + d, m = 2\ell + 1} \frac{q^{2\ell + 1} - q}{q^2 - 1} \Omega_m + 1
\]
\[
= \sum_{\ell \geq 1} \sum_{0 \leq i \leq -2} (-q)^{2(\ell + 1)} \Omega_{2\ell} + \sum_{\ell \geq 1} \sum_{0 \leq i \leq -1} (-q)^{2i + 1} \Omega_{2\ell + 1}
\]
\[
= \sum_{j \geq 1} (-q)^j \sum_{k \geq 1} \Omega_{j + 2k},
\]
where we note that here \(j\) is even or \(2i + 1\) and \(\ell = i + k\), thus \(2\ell = 2i + 2k\) or \(2\ell + 1 = 2i + 1 + 2k\) is even. In any case, this is the required trace of the Frobenius, and the last claim of the theorem follows.

It remains to compute the trace formula, thus the following Lemmas 4.2-4.9.

**Lemma 4.2.** We have \(S_1(f_u) = 2h_1(q_u + 1)\xi(X, q)\).

**Proof.** This is the same as Lemma 3.3 except that \(f_u(\gamma) = q_u + 1\) for \(\gamma \in O_u^\times\), and not 1 as there.

**Lemma 4.3.** We have \(S_2(f_u) = 0\) if \(\text{deg}(u)\) is odd, \(S_2(f_u) = qh_2/(q + 1)\) if \(\text{deg}(u)\) is even.

**Proof.** Since \(\text{supp}(f_u) \subset K_u\), we argue as in Lemma 3.4 to see that the sum in each \(S_2, F_2, f_u(f_u)\) ranges over \(\gamma \in \mathbb{F}_{q^2} - \mathbb{F}_q\), and \(F_2 = F(\gamma)\) is unique. When \(\text{deg}(u)\) is odd, \(F_{2,u} = F_u(\gamma) = F_u \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}\) is a quadratic extension of \(F_u\), hence no conjugate of \(\gamma\) lies in the support of \(f_u\). If \(\text{deg}(u)\) is even, \(F_{2,u} = F_u \times F_u\), and we may conjugate \(\gamma \in G_u = \text{GL}(2, F_u)\) to assume \(\gamma = \text{diag}(a, b)\). For such \(\gamma\), we have to find the \(x \in G_u\) with \(x^\gamma x^{-1} \in I_u\). In particular \(x^\gamma x^{-1} \in I_u\). Thus \(x \in K_u A_u / A_u\). But \(K_u = I_u \cup s \in O_u / \pi_u, I_u w (\frac{1}{s} 0 \frac{a}{0} 1)\) so \(w (\frac{1}{s} 0 \frac{a}{0} 1) (\frac{1}{s} 0 -s \frac{b}{0} 1) w^{-1} = (b \frac{b}{s a-b} 0)\) lies in \(I_u\) iff \(s = 0\) in \(O_u / \pi_u\). Thus \(x^\gamma x^{-1} \in I_u\) iff \(x \in I_u\) or \(I_u w\), hence \(\int_{G_u / A_u} f_u(x^\gamma x^{-1}) dx = 2\) (as \(|I_u / I_u \cap A(O_u)| = |I_u|\) since \(|O_u^\times| = 1\). We get the result of Lemma 3.3 multiplied by 2.

**Lemma 4.4.** We have \(S_3(f_u) = 0\).

**Proof.** Since \(\text{supp}(f_u) \subset K_u\), the proof of Lemma 3.5 applies.

**Lemma 4.5.** We have \(S_4(f_u) = h_1 w_u\), where \(t_u = t^{\text{deg}(u)}\) and
\[
w_u = \frac{1}{2} \lim_{t \to 1} [t^g - 2 \zeta(X, 1/t)(t_u + 1) + t^g - 2 \zeta(X, t)(1 + 1/t_u)].
\]
Proof. We argue as in Lemma 3.6. We are led to [F14] Proposition 3.11, which computes $\theta_1, f_u(t_v)$ as the product of the local integrals $\theta_1, f_v(t_v) = \sum_{n \in \mathbb{Z}} \tau_n(f_v)(t_v/q_v)^n$, where
\[
\tau_n(f_v) = \int_{\text{GL}(2, \mathbb{O}_v)} f_v(y \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} y^{-1}) \, dy.
\]
When $f_v = \chi_{K_v}$, $\theta_1, f_v(t_v) = (1 - t_v/q_v)^{-1}$. At $v = u$, with our $f_u = (q_u + 1)\chi_{I_u}$, we use $y \in K_u = I_u \cup \sum_{s \in \mathcal{O}_u/\text{largest}}RI_uw(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix})$ to see that when $n = 0$, $y \in I_u$ if $f_u \neq 0$, while if $n \geq 1$ then $w(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) w^{-1} = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ lies in $I_u$, so $\theta_1, f_u(t_u) = \sum_{n \in \mathbb{Z}} \tau_n(f_u)(t_u/q_u)^n = 1 + (q_u + 1) \sum_{n \geq 1} (t_u/q_u)^n = (1 + t_u)/(1 - t_u/q_u)$. This is why $\zeta(X, 1/t)$ of Lemma 1.6 gets multiplied by $1 + t_u$. □

Lemma 4.6. We have $S_0(f_u) = -2(b_2^2(2g - 2) + h_1)$.

Proof. The only difference from Lemma 3.7 is that $\text{tr} I(\mu_1u\nu_{t_u}, \mu_2u\nu_{t_u}^{-1}, f_u) = 2$ here, not 1 as there. □

Lemma 4.7. We have $S_0(f_u) = -d_u h_2^2$, where $d_u = \deg(u)$.

Proof. $S_0(f_u)$ is $-2^{-1}$ times the sum over the characters $\mu_1, \mu_2$ of $\mathbb{A}^\times/F^\times \cdot \alpha^Z$, of
\[
\int_{|t| = 1} \text{tr} I(\mu_1u_1, \mu_2u_1^{-1}, f_u) \cdot R(\mu_1, \mu_2, t)^{-1} \cdot \frac{d}{dt} R(\mu_1, \mu_2, t)dt.
\]

We write $R(\mu_1, \mu_2, t)$ as a product $\otimes v R(\mu_{1v}, \mu_{2v}, t_v)$. Applying $\frac{d}{dt}$ we obtain a sum over $v$ of $R(\mu_{1v}, \mu_{2v}, t_v)^{-1} \cdot \frac{d}{dt} R(\mu_{1v}, \mu_{2v}, t_v)$ in the integral. Since $f_u$ is the characteristic function of $K_u$, the operator $I(\mu_1u_1, \mu_2u_1^{-1}, f_u)$ acts as the projection to the space of $\text{GL}(2, \mathbb{O}_u)$-fixed vectors in $I(\mu_1u_1, \mu_2u_1^{-1}, f_u)$, a one dimensional space independent of $t$. Hence $R(\mu_1u_1, \mu_2u_1^{-1}, f_u)$ is independent of $t$, as explained in the proof of Lemma 3.8, and its derivative is 0. There remains in the integrand then only the summand indexed by $u$, and the sum ranges over the characters $\mu_1, \mu_2$ of $\mathbb{A}^\times/F^\times \cdot \alpha^Z$. Since $f_u$ is $\chi_{K_u}/|I_u|$, the operator $I(\mu_1u_1, \mu_2u_1^{-1}, f_u)$ acts as the projection to the space of $I_u$-invariant vectors in $J = I(\mu_1u_1, \mu_2u_1^{-1}, f_u)$.

Recall that $\nu_{t_u}(a) = t_u^{\nu(a)}$, $t_u = t^{\deg(u)}$. The vectors $\phi$ in $J$ satisfy
\[
\phi(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}) = \mu_1u(a)\mu_2u(b) \begin{pmatrix} a & \nu(a) \\ b & \nu(b) \end{pmatrix}^{1/2} t_u^{\nu(a) - \nu(b)} \phi(k),
\]
so from $K_u = I_u \cup \sum_{s \in \mathcal{O}_u/\text{largest}} RI_uw(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ we see that the space $J^\times$ of right $I_u$-invariant $\phi$ is spanned by the basis $\phi_1, \phi_2$, where $\phi_1$ is supported on $B_uI_u$ normalized by $\phi_1(1) = 1$, and $\phi_2$ is supported on $B_uI_u$, normalized by $\phi_2(w) = 1$.

We now study the action of $R = R_u = R(\mu_1u_1, \mu_2u_1, t_u)$ on the basis $\phi_1, \phi_2$. Here
\[
R(\mu_1u_1, \mu_2u_1, t_u) = \frac{L(t_u^2/q_u, \mu_1u_1/\mu_2u_1)}{L(t_u^2, \mu_1u_1/\mu_2u_1)} M(\mu_1u_1, \mu_2u_1, t_u)
\]
where $L(t, \mu) = 1/(1 - \mu_0(\pi_u) t)$ and $M = M_u = M(\mu_1u_1, \mu_2u_1, t_u)$ is the operator from $I(\mu_1u_1, \mu_2u_1^{-1}, \mu_1u_1, \mu_2u_1^{-1})$ to $I(\mu_2u_1^{-1}, \mu_1u_1, \mu_2u_1^{-1})$ defined by
\[
(M\phi)(x) = \int_N \phi(w \begin{pmatrix} 1 & \nu(a) \\ 0 & 1 \end{pmatrix} x) \, dy.
\]
To compute $M$ on $J_{t_u}$ we compute the $a, b, c, d$ in $M \phi_1 = a \phi_1 + c \phi_w$, $M \phi_w = b \phi_1 + d \phi_w$. Put $\mu_u = \mu_{1u}/\mu_{2u}$ and $\mu = \mu_u(\pi_u)$. Then

\[
(M \phi_1)(1) = \int_1 \phi_1 (m_1 y) \, dy = \int_1 \mu_{2u}(y) y^{-2} \text{val}_u(y) \, dy \bigg|_{|y| > 1} \mu_{1u}(y) y^{-2} \text{val}_u(y) \, dy \bigg|_{|y| < 1} = (1 - q_u^{-1}) \sum_{\alpha \geq 1} (\mu t_u^\alpha)^n = (1 - q_u^{-1}) \frac{1 - \mu t_u^{-1}}{1 - \mu t_u^{-2}};
\]

\[
(M \phi_1)(w) = \int_1 \phi_1 (w (1 y) y) \, dy = \int_1 \phi_1 (y) \, dy = q_u^{-1};
\]

\[
(M \phi_w)(1) = \int_1 \phi_w (w (1 y) y) \, dy = \int_1 \phi_w (w (1 y) y) \, dy = 1;
\]

\[
(M \phi_w)(w) = \int_1 \phi_w (w (1 y) y) dy = \int_1 \phi_w (w (1 y) y) dy = (1 - q_u^{-1}) \sum_{\alpha \geq 1} (\mu t_u^\alpha)^n = \frac{1 - q_u^{-1}}{1 - \mu t_u^{-1}}.
\]

We conclude that

\[
R = \frac{L(t_u^2/q_u, \mu_u)}{L(t_u^2, \mu_u)} \left( \frac{(1 - q_u^{-1}) \mu t_u^2}{1 - \mu t_u^2} \right) = \frac{1 - q_u^{-1}}{1 - \mu t_u^{-1}} - \frac{(1 - q_u^{-1}) \mu t_u^2}{1 - \mu t_u^2}.
\]

Then $det R = -(1 - q_u \mu t_u^2)/(1 - \mu t_u^2/q_u)$ and, recalling that $R = R(t_u)$,

\[
R^{-1} = \frac{-q_u}{1 - q_u \mu t_u^2} \left( -q_u^{-1} (1 - \mu t_u^2) \right)
\]

and

\[
dt = d_u t_u^{-1} \left( \frac{1 - q_u^{-1}}{1 - \mu t_u^2/q_u} \right) \left( \frac{1 - q_u^{-1}}{1 - \mu t_u^2/q_u} \right).
\]

So

\[
R^{-1} R' = \frac{-2 t_u d_u t_u^{-1} q_u \mu (1 - q_u^{-1})}{(1 - q_u \mu t_u^2)(1 - \mu t_u^2/q_u)} \left( \frac{1 - q_u^{-1}}{1 - \mu t_u^2/q_u} \right).
\]

and $tr J(f_u) R^{-1} R' = \frac{-2 t_u d_u t_u^{-1} q_u \mu (1 - q_u^{-2})}{(1 - q_u \mu t_u^2)(1 - \mu t_u^2/q_u)}$. We need to integrate this against $dt$ over $|t| = 1$, multiplied by $1/2 \pi i$. We change variables: $y = t^2$. Then $dy = d_u t_u^{-1} dt$. As $t$ goes over the circle $|t| = 1$ once, $y$ goes $d_u$ times around it. Using $\frac{2 y}{y^2 - \rho^2} = \frac{1}{y - \rho} + \frac{1}{y + \rho}$ with $\rho^2 = 1/q_u$, we get

\[
d_u \int_{|y|=1} \frac{-q_u (1 - q_u^{-2}) 2y \, dy}{\mu ((y - 1/q_u) \mu)} = \frac{-2 d_u q_u (1 - q_u^{-2})}{\mu ((1/q_u) - \mu)} = 2 d_u,
\]

so $S_0(f_u) = -d_u h_1^2$. \qed

**Lemma 4.8.** We have $S_7(f_u) = h_1$.

**Proof.** Recall that $S_7(f)$ is the sum of $t_2 \text{tr} I(\mu, \mu, f)$ over the characters $\mu$ of $K^\times/F^\times \cdot \alpha^{2Z}$. For our $f = f_u$, the character $\mu$ contributes 0 unless it is over $K^\times/F^\times \cdot O^\times \cdot \alpha^{2Z}$. The cardinality of the set of these $\mu$ is $2 h_1$, and $\text{tr} I(\mu, \mu, f_u) = \text{tr} I(\mu_u, \mu_u, f_u) = 2$ since $\text{tr} I(\mu_v, \mu_v, f_v) = 1$ for our $f_v (v \neq u)$. \qed

**Lemma 4.9.** We have $S_8(f_u) = -2 h_1$.
Proof. Again we have a sum over $\mu$ on $\mathbb{A}^n / \mathbb{A}^n \cdot \alpha^{\mathbb{Z}}$, which reduces to a sum over $\mathbb{A}^n / \mathbb{A}^n \cdot \mathcal{O}_X \cdot \alpha^{\mathbb{Z}}$ in view of our choice of $f(u)$, and for each such $\mu$, by our choice of $f(u)$ the integral $\int_{\text{GL}(2, \mathbb{A})} f(u)(x)\mu(\det x)dx$ is equal to 1. \qed

5. COUNTING IN RANK TWO AND ANY UNIPOTENT MONODROMY

This section consists of a slight rewording of a letter from P. Deligne to the author, dated August 8, 2012.

Let $X$ be a connected smooth projective curve over $\mathbb{C}$. Let $S \subset X$ be a finite set of points. If $M$ is the moduli space of irreducible rank $n$ complex local systems on $X - S$, with principal unipotent local monodromy at each $s \in S$, it is natural to complete $M$ into the moduli space $M_0$ of irreducible rank $n$ complex local systems, with unipotent local monodromy at each $s \in S$. It would be even more natural to complete $M$ into $M_2$ where “irreducible” is dropped, replaced by “up to semi-simplification”, but we shall stay with $M_0$. The space $M_0$ is singular, with singularities looking like those of (product of nilpotent cones) $\times$ affine space.

Let $X_1$ be a geometrically connected smooth projective curve over $\mathbb{F}_q$, with a reduced divisor $S_1$. Notation is as in [DF13]. These are denoted by $X$ and $S$ in sections 1-4 of this paper, to simplify the notation. Dropping the index 1 indicates here – as in [DF13] – extension of scalars to an algebraic closure $F$ of $\mathbb{F}_q$. The complex case suggests to try to count the rank $n$ smooth $\mathbb{Q}_l$-sheaves on $X_1 - S_1$, with unipotent local monodromy at each $s_1 \in S_1$ (trivial local monodromy is in particular allowed; for $n = 2$, it is the only new possibility at each $s_1 \in S_1$).

More precisely, we want to count up to $\mathbb{F}_q$-twists those which remain irreducible over $X$. On the automorphic side, this means counting the cuspidal automorphic representations which remain cuspidal after base change given by an extension of the field $\mathbb{F}_q$ of constants, are unramified outside of $S_1$, and whose Langlands parameter at each $s_1 \in S_1$ is an unramified representation of the Weil group, together with any allowed nilpotent. In purely representation theoretic language: the component at each $s_1 \in S_1$ has a nonzero Iwahori fixed vector, namely it is tame. The count is of course modulo twists by a character of $\mathbb{Z}$.

For $n = 2$, we can make the computation thanks to what is in [DF13], plus the computation for $S_1$ reduced to one closed point (Theorem 4.1 in this paper, mentioned also at [DF13] after 6.29), plus Drinfeld [D81] (Theorem 3.1 in this paper) for empty $S_1$.

Surprise: For $n = 2$, the count depends only on $X_1$ and $\text{deg}(S_1)$. It does not depend on the degrees of the points in $S_1$, nor on their cardinality. Only the total degree (cardinality of $S$) is relevant. This does not hold when we consider only principal unipotent local monodromy in [DF13]. That the position of the points does not matter is interesting too, but something we already had in [DF13].

We use the following notation: $N := \text{deg}(S_1) = |S|; \lfloor x \rfloor := \text{greatest integer} \leq x; [x]^+ := \max(0, [x])$: it is 0 for $x < 0,$ 0 for $x < 1$.

It is easier to first explain the case of $\mathbb{P}^1$, because for $\mathbb{P}^1, H^1 = 0$ and the count is 0 for $N \leq 3$ (so that we do not need Drinfeld [D81], or Theorem 3.1 of this paper).

Theorem 5.1. If $X_1 = \mathbb{P}^1$, the number of rank 2 irreducible smooth $\mathbb{Q}_l$-sheaves on $X_1 - S_1$, which at each $s_1 \in S_1$ are unramified or have principal unipotent local
monodromy, taken up to \( \mathbb{F}_q \)-twists, is
\[
(5.1.1) \quad q^{N-3} + q^{N-4} + 2q^{N-5} + 2q^{N-6} + \ldots + [(N-2)/2]q^2 = \sum_{k \leq N-2} [k/2]q^{N-1-k}.
\]

If one counts only the sheaves as above which at each \( s_1 \in S_1 \) have principal unipotent local monodromy, the answer is given by the formula (6.26.2) of [DF13]. It has the form
\[
(5.1.2)_S \quad \sum_{k=1}^{N-3} c_k(S)q^k
\]

with \( c_k(S) \) given by the action of Frobenius on \( S \): if \( \varepsilon(S) \) is the signature of the Frobenius permutation,
\[
(5.1.3) \quad c_k(S) = (-1)^{N+k+1} \varepsilon(S) \sum_{j \geq 1} \text{Tr}(\text{Frob}, \Lambda^{k+2j}(\mathbb{Q}^S/\mathbb{Q})).
\]

As (5.1.3) vanishes for \( k > N-3 \), we may in (5.1.2)_S sum over all \( k \geq 1 \).

The sign \( \varepsilon(S) \) is the action of Frob on \( \det(\mathbb{Q}^S/\mathbb{Q}) \).

As the representation \( \mathbb{Q}^S/\mathbb{Q} \) of the symmetric group \( S_S \) of \( S \) is self dual,
\[
\det(\mathbb{Q}^S/\mathbb{Q}) \otimes \Lambda^{k+2j}(\mathbb{Q}^S/\mathbb{Q}) \cong \Lambda^{N-1-k-2j}(\mathbb{Q}^S/\mathbb{Q}),
\]

and (5.1.3) can be rewritten as
\[
(5.1.4) \quad c_k(S) = (-1)^{N+k+1} \sum_{j \geq 1} \text{Tr}(\text{Frob}, \Lambda^{N-1-k-2j}(\mathbb{Q}^S/\mathbb{Q})).
\]

Lemma 5.2. As a virtual representation of \( S_S \),
\[
\Lambda^{a}(\mathbb{Q}^S/\mathbb{Q}) = \sum_{i \geq 0} (-1)^i \Lambda^{a-i}(\mathbb{Q}^S).
\]

Proof. From \( \mathbb{Q}^S \cong (\mathbb{Q}^S/\mathbb{Q}) \oplus \mathbb{Q} \), one gets
\[
\Lambda^{a}(\mathbb{Q}^S) \cong \Lambda^{a}(\mathbb{Q}^S/\mathbb{Q}) \oplus a \Lambda^{a-1}(\mathbb{Q}^S/\mathbb{Q}) \text{ hence } \Lambda^{a}(\mathbb{Q}^S/\mathbb{Q}) = \Lambda^{a}(\mathbb{Q}^S) - a \Lambda^{a-1}(\mathbb{Q}^S/\mathbb{Q}),
\]

from which the lemma follows by induction on \( a \). We use the convention that \( \Lambda^0 = 0 \) when \( a < 0 \).

By Lemma 5.2, \( c_k(S) \) is \((-1)^{N+k+1}\) times the sum of the traces of Frobenius on the \( \Lambda^{a}(\mathbb{Q}^S), \ a \geq 2 \), with multiplicity \( \# \{ a = i + 2j; \ j \geq 1, \ i \geq 0 \} = \max(0, j+1/2) = \lceil a/2 \rceil \). Put \( b = N-1-k-a \). Then
\[
(5.2.1) \quad c_k(S) = \sum_{b} (-1)^b \lceil (N-1-k-b)/2 \rceil^+ \text{Tr}(\text{Frob}, \Lambda^{b}(\mathbb{Q}^S)).
\]

Our aim is to compute the sum over \( T \subset S \) stable by Frobenius of the (5.1.2)_{S-T},
that is
\[
(5.2.2) \quad \sum_{k \geq 1} q^k \sum_{T \text{ stable}} c_k(S - T).
\]
In (5.2.2), the coefficient of $q^b$ is

\[(5.2.3)_k \sum_{T \text{ stable}} \sum_{b} (-1)^b[(|S - T| - k - 1 - b)/2]^+ \text{Tr}(\text{Frob}, \Lambda(Q^{S-T})).\]

In (5.2.3)_k, let us consider the subsum corresponding to a given value $M$ of $|T|$ and a given $b$. It is $(-1)^b[(N - M - k - 1 - b)/2]^+$ times

\[(5.2.4) \quad \text{Tr}(\text{Frob}, \bigoplus_{T \text{ stable}} \Lambda(Q^{S-T}))_{|T|=M}.

The sum over all $T \subset S$ such that $|T| = M$ of the $\Lambda(Q^{S-T})$ is a representation of $S_S$ denoted $R(M, b)$. It is the induced representation, from $S_M \times S_{N-M}$, of the representation

$$\text{trivial} \otimes \Lambda(Q^{N-M}).$$

The trace $\text{Tr}(\text{Frob}, R(M, b))$ is equal to (5.2.4). Indeed, Frobenius permutes the summands and maps each new direct summand to a different new direct summand.

This gives a new expression for the coefficient of $q^b$ in (5.2.2):

\[(5.2.5)_k \sum_{M, b} (-1)^b[(N - M - k - 1 - b)/2]^+ \text{Tr}(\text{Frob}, R(M, b)).\]

$$= \sum_a [(N - k - 1 - a)/2]^+ \text{Tr}(\text{Frob}, \sum_{\{M, k, M+b=a\}} (-1)^b R(M, b)).$$

**Lemma 5.3.** If $a > 0$, the virtual representation

$$\sum_{M+b=a} (-1)^b R(M, b)$$

of $S_S$ is zero.

**Proof.** Consider the algebraic de Rham complex of $\mathbb{Q}[(X_s)_{s \in S}]$. It is a resolution of $\mathbb{Q}$. The part of degree $a$ and of degree $\leq 1$ in each variable is a direct summand. It is the direct sum, over $A \subset S$ with $a$ elements, of the part of degree 1 in each $X_s$, for $s \in A$, and of degree 0 in the others. For $a > 0$, it is acyclic. In cohomological degree $b$, and for $M = a - b$, it is spanned by the $X_{s(1)} \ldots X_{s(M)}dX_{s(M+1)} \ldots dX_{s(a)}$ with the $s(i)$ all distinct: as a representation of $S_S$, it is $R(a-b, b)$. The lemma follows. \qed

**Proof of Theorem 5.1.** In (5.2.5)_k, we may therefore cancel all the terms except those with $M = b = 0$, for which $R(M, b)$ is the trivial representation. We get

\[(5.3.1)_k \quad (5.2.5)_k = [(N - k - 1)/2]^+,

proving Theorem 5.1. \qed

**5.4. When $X_1$ is of genus $g$.** Here is how to modify the arguments when $X_1$ is of genus $g$. For $S_1 \neq \emptyset$, (6.26.2) of [DF13] tells that the count, when principal unipotent monodromy is imposed at each $s_1 \in S_1$, is the product of

\[(5.4.1) \quad |(\text{Pic}^0(X_1))(\mathbb{F}_q)| = \sum (-1)^a \text{Tr}(\text{Frob}, \Lambda^a H^1(X))

\]
with
\[
\sum_{k \geq 1} c_k(X_1, S_1) q^k
\]
where
\[
(5.4.3)_k \quad c_k(X_1, S_1) = (-1)^{N+k+1} \varepsilon(S) \sum_{j \geq 1} \text{Tr}(\text{Frob}, H^1(X) \otimes (Q^S_l/Q_l))
\]
\[
= (-1)^{N+k+1} \varepsilon(S) \sum_{j \geq 1} \sum_{a+b=k+2j} \text{Tr}(\text{Frob}, H^1(X) \otimes (Q^S_l/Q_l))
\]
\[
= \sum_a (-1)^a \text{Tr}(\text{Frob}, H^1(X)) \cdot (-1)^{N+a+k+1} \varepsilon(S) \sum_{j \geq 1} \text{Tr}(\text{Frob}, H^1(X) \otimes (Q^S_l/Q_l)).
\]

As before, the coefficient of \((-1)^a \text{Tr}(\text{Frob}, H^1(X))\) is
\[
(5.4.4)_{k,a} \quad (-1)^{N+a+k+1} \sum_{j \geq 1} \text{Tr}(\text{Frob}, H^1(X) \otimes (Q^S_l/Q_l))
\]
\[
= \sum_{b} (-1)^b \left[ (N + a - k - 1 - b)/2 \right]^+ \text{Tr}(\text{Frob}, H^1(X) \otimes (Q^S_l/Q_l)).
\]

All terms indexed by \(b \neq 0\) are zero as in the proof of Theorem 5.1.

If \(S_1\) is empty, thus \(N = 0\), Drinfeld’s formula is similar, but there is a term added to the sum
\[
\sum_a q^k (-1)^a \text{Tr}(\text{Frob}, H^1(X)) \cdot [(a - k - 1)/2]^+.
\]
This term is
\[
(5.4.5) \quad \sum_{a=0}^{g-1} (g-a)(-1)^a \text{Tr}(\text{Frob}, H^1(X)) \cdot \frac{q^{g-a} - 1}{q-1} + |\text{Pic}^0(F_q)| (1-g) - 1.
\]

We will leave alone this additional term (5.4.5). Except for it, we have the same cancellation as before and we get only the term \(b = 0\) in (5.4.4)_{k,a}:

**Theorem 5.4.** Our number is the product of \(\sum (-1)^a \text{Tr}(\text{Frob}, H^1(X))\) by
\[
\sum_{k \geq 1} \sum_{a} (-1)^a q^k \text{Tr}(\text{Frob}, H^1(X)) \cdot [(N + a - k - 1)/2]^+ + (5.4.5).
\]

As claimed, it depends only on \(X_1\) and \(N = \deg(S_1)\).

**References**


The Ohio State University, Columbus, Ohio, 43210, USA
University of Ariel, Ariel, Israel
E-mail address: yzflicker@gmail.com