Confluent $A$-hypergeometric functions and rapid decay homology cycles

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Abstract

We study confluent $A$-hypergeometric functions introduced by Adolphson [1]. In particular, we give their integral representations by using rapid decay homology cycles of Hien [17] and [18]. The method of toric compactifications introduced in [27] and [31] will be used to prove our main theorem. Moreover we apply it to obtain a formula for the asymptotic expansions at infinity of confluent $A$-hypergeometric functions.

1 Introduction

The theory of $A$-hypergeometric systems introduced by Gelfand-Graev-Kapranov-Zelevinsky [13], [14] is a vast generalization of that of classical hypergeometric differential equations. We call their holomorphic solutions $A$-hypergeometric functions. As in the case of classical hypergeometric functions, $A$-hypergeometric functions admit $\Gamma$-series expansions ([14]) and integral representations ([15]). Moreover this theory has deep connections with many other fields of mathematics, such as toric varieties, commutative algebra, projective duality, period integrals, mirror symmetry, enumerative algebraic geometry and combinatorics. Also from the viewpoint of the $D$-module theory (see [22]), $A$-hypergeometric $D$-modules are very elegantly constructed in [15]. For the recent development of this subject see [40] and [41]. In [5], [6], [15], [20] and [44] the monodromies of their $A$-hypergeometric functions were studied. Recall that in the theory of Gelfand-Graev-Kapranov-Zelevinsky [13], [14] they assumed that $A$ is homogeneous (see Remark 2.1). By removing this homogeneous condition, Adolphson [1] generalized their $A$-hypergeometric systems to the confluent (i.e. irregular) case and proved many important results. However his construction of the confluent $A$-hypergeometric $D$-modules is not given by the standard operations of $D$-modules as in [14], [15]. This leads us to some difficulties in obtaining the integral representations of confluent $A$-hypergeometric

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functions. Indeed, in the confluent case almost nothing is known about their global properties. In this paper, we construct Adolphson’s confluent $A$-hypergeometric $D$-modules as in [15]. Note that recently the same problem was solved more completely in Saito [39] and Schulze-Walther [42], [43] by using commutative algebras. Our approach is based on sheaf-theoretical methods and totally different from theirs. Moreover we also construct an integral representation of the confluent $A$-hypergeometric functions by using the theory of rapid decay homology groups introduced recently in Hien [17] and [18]. Recall that $A = \{a(1), a(2), \ldots, a(N)\} \subset \mathbb{Z}^n$ is a finite subset of a lattice $\mathbb{Z}^n$ and Adolphson’s confluent $A$-hypergeometric system is defined on $\mathbb{C}^A = \mathbb{C}_x^N$. Then our integral representation of confluent $A$-hypergeometric functions

$$u(z) = \int_{\gamma} \exp\left(\sum_{j=1}^{N} z_j x^{a(j)} \right) x_1^{c_1 - 1} \cdots x_n^{c_n - 1} dx_1 \wedge \cdots \wedge dx_n$$  \hspace{1cm} (1.1)$$

coincides with the one in Adolphson [1, Equation (2.6)], where $\gamma = \{\gamma^z\}$ is a family of real $n$-dimensional topological cycles $\gamma^z$ in the algebraic torus $T = (\mathbb{C}^*)^n$ on which the function $\exp(\sum_{j=1}^{N} z_j x^{a(j)} \right) x_1^{c_1 - 1} \cdots x_n^{c_n - 1}$ decays rapidly at infinity. More precisely $\gamma^z$ is an element of Hien’s rapid decay homology group. See Sections 3 and 4 for the details. Adolphson used the formula (1.1) without giving any geometric condition on the cycles $\gamma^z$ nor proving the convergence of the integrals. In our Theorem 4.5 we could give a rigorous justification to Adolphson’s formula [1, Equation (2.6)] by using rapid decay homology cycles. This integral representation can be considered as a natural generalization of those for the classical Bessel and Airy functions etc. Note that in the case of hypergeometric functions associated to hyperplane arrangements the same problem was precisely studied by Kimura-Haraoka-Takano [25]. We hope that our geometric construction would be useful in the explicit study of Adolphson’s confluent $A$-hypergeometric functions. In the proof of Theorem 4.5, we shall use the method of toric compactifications introduced in [27] and [31] for the study of geometric monodromies of polynomial maps. Moreover we introduce Proposition 3.4 which enables us to calculate Hien’s rapid decay homologies by usual relative twisted homologies. By Proposition 3.4 and Lemmas 3.5 and 3.6 we can calculate the rapid decay homologies very explicitly in many cases. Let $\Delta \subset \mathbb{R}^n$ be the convex hull of $A \cup \{0\}$ in $\mathbb{R}^n$ and $h_z : T = (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ the Laurent polynomial on $T$ defined by $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$. Then in Section 5, assuming the condition $0 \in \text{Int}(\Delta)$ and using the twisted Morse theory we construct also a natural basis of the rapid decay homology group indexed by the critical points of $h_z$. Furthermore we apply it to obtain a precise formula for the asymptotic expansions at infinity of Adolphson’s confluent $A$-hypergeometric functions. The formula that we obtain in Theorem 5.6 will be very similar to that of the classical Bessel functions. Finally in Sections 6 and 7, removing the condition $0 \in \text{Int}(\Delta)$ we construct another natural basis of the rapid decay homology group. We thus partially solve the famous problem in Gelfand-Kapranov-Zelevinsky [15] of constructing a basis of the twisted homology group in their integral representation of non-confluent $A$-hypergeometric functions, in the more general case of confluent ones. Moreover, recently in [2] (a slight modification of) this result was effectively used for the study of the monodromies at infinity of confluent $A$-hypergeometric functions. See [2] for the details.

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2 Adolphson’s results

First of all, we recall the definition of the confluent $A$-hypergeometric systems introduced by Adolphson [1] and their important properties. In this paper, we essentially follow the terminology of [22]. Let $A = \{a(1), a(2), \ldots, a(N)\} \subset \mathbb{Z}^n$ be a finite subset of the lattice $\mathbb{Z}^n$. Assume that $A$ generates $\mathbb{Z}^n$ as in [14] and [15]. Following [1] we denote by $\Delta$ the convex hull $\text{conv}(A \cup \{0\})$ of $A \cup \{0\}$ in $\mathbb{R}^n$. By definition $\Delta$ is an $n$-dimensional polytope.

Let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ be a parameter vector. Moreover consider the $n \times N$ integer matrix $A := \begin{pmatrix} t_{a(1)} & t_{a(2)} & \cdots & t_{a(N)} \end{pmatrix} = (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq N} \in M(n, N, \mathbb{Z})$ (2.1) whose $j$-th column is $t_a(j)$. Then Adolphson’s confluent $A$-hypergeometric system on $X = \mathbb{C}^A = \mathbb{C}^N$ associated with the parameter vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ is

$$
\left( \sum_{j=1}^{N} a_{i,j} z_j \frac{\partial}{\partial z_j} + c_i \right) u(z) = 0 \quad (1 \leq i \leq n),
$$

$$
\left\{ \prod_{\mu_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} u(z) = 0 \quad (\mu \in \text{Ker}A \cap \mathbb{Z}^N). \quad (2.2)
$$

Remark 2.1. The above $A$-hypergeometric system was introduced first by Gelfand-Graev-Kapranov-Zelevinsky [13], [14] under the homogeneous condition on $A$ i.e. when there exists a linear functional $l : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $l(\mathbb{Z}^n) = \mathbb{Z}$ and $A \subset l^{-1}(1)$. In this case, Hotta [21] proved that it is regular holonomic i.e. non-confluent.

Remark 2.2. In [1] Adolphson does not assume that $A$ generates $\mathbb{Z}^n$. However we need this condition to obtain a geometric construction of his confluent $A$-hypergeometric systems. Even when $A$ does not generate $\mathbb{Z}^n$, by a suitable linear coordinate change of $\mathbb{R}^n$ we can get an equivalent system for $A' \subset \mathbb{Z}^n$ and $c' \in \mathbb{C}^n$ such that $A'$ generates $\mathbb{Z}^n$. Namely our condition is not restrictive at all.

Let $D(X)$ be the Weyl algebra over $X$ and consider the differential operators

$$
Z_{i,c} := \sum_{j=1}^{N} a_{ij} z_j \frac{\partial}{\partial z_j} + c_i \quad (1 \leq i \leq n),
$$

$$
\Box_\mu := \prod_{\mu_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-\mu_j} \quad (\mu \in \text{Ker}A \cap \mathbb{Z}^N) \quad (2.5)
$$

in it. Then the above system is naturally identified with the left $D(X)$-module

$$
M_{A,c} = D(X) / \left( \sum_{1 \leq i \leq n} D(X) Z_{i,c} + \sum_{\mu \in \text{Ker}A \cap \mathbb{Z}^N} D(X) \Box_\mu \right). \quad (2.6)
$$
Let $\mathcal{D}_X$ be the sheaf of differential operators over the algebraic variety $X$ and define a coherent $\mathcal{D}_X$-module by

$$M_{A,c} = \mathcal{D}_X / \left( \sum_{1 \leq i \leq n} \mathcal{D}_X Z_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathbb{Z}^N} \mathcal{D}_X \Box_{\mu} \right).$$  \hspace{1cm} (2.7)

Then we have an isomorphism $M_{A,c} \simeq \Gamma(X; \mathcal{M}_{A,c})$ (see [22, Proposition 1.4.4]). Adolphson [1] proved that $M_{A,c}$ is holonomic. In fact, he proved the following more precise result.

**Definition 2.3.** (Adolphson [1, page 274], see also [36]) For $z \in X = \mathbb{C}^A$ we say that the Laurent polynomial $h_z(x) = \sum_{j=1}^N z_j x^{a(j)}$ is non-degenerate if for any face $\Gamma$ of $\Delta$ not containing the origin $0 \in \mathbb{R}^n$ we have

$$\left\{ x \in T = (\mathbb{C}^*)^n \mid \partial h_{\Gamma}^1(x) = \cdots = \partial h_{\Gamma}^n(x) = 0 \right\} = \emptyset,$$

where we set $h_{\Gamma}^j(x) = \sum_{j:a(j) \in \Gamma} z_j x^{a(j)}$. \hspace{1cm} (2.8)

**Remark 2.4.** Since in the definition above we consider only faces $\Gamma \prec \Delta$ such that $\dim \Gamma \leq n - 1$ and $h_z$ are quasi-homogeneous, our condition (2.8) is equivalent to the weaker one in [1, page 274]. See Kouchnirenko [26, Definition 1.19].

Let $\Omega \subset X$ be the Zariski open subset of $X$ consisting of $z \in X = \mathbb{C}^A$ such that the Laurent polynomial $h_z(x) = \sum_{j=1}^N z_j x^{a(j)}$ is non-degenerate. Then Adolphson’s result in [1, Lemma 3.3] asserts that the holonomic $\mathcal{D}_X$-module $\mathcal{M}_{A,c}$ is an integrable connection on $\Omega$ (i.e. the characteristic variety of $\mathcal{M}_{A,c}$ is contained in the zero section of the cotangent bundle $T^*\Omega$). Now let $X^\text{an}$ (resp. $\Omega^\text{an}$) be the underlying complex analytic manifold of $X$ (resp. $\Omega$) and consider the holomorphic solution complex $\text{Sol}_X(\mathcal{M}_{A,c}) \in D^b(X^\text{an})$ of $\mathcal{M}_{A,c}$ defined by

$$\text{Sol}_X(\mathcal{M}_{A,c}) = R\text{Hom}_{\mathcal{D}_{X^\text{an}}}((\mathcal{M}_{A,c})^\text{an}, \mathcal{O}_{X^\text{an}}) \hspace{1cm} (2.9)$$

(see [22] for the details). Then by the above Adolphson’s result, $\text{Sol}_X(\mathcal{M}_{A,c})$ is a local system on $\Omega^\text{an}$. Moreover he proved the following remarkable result. Let $\text{Vol}_{\mathbb{Z}}(\Delta) \in \mathbb{Z}$ be the normalized (or simplicial) $n$-dimensional volume of $\Delta$ i.e. $n!$ times the Euclidean volume of $\Delta \subset \mathbb{R}^n$ with respect to the canonical embeddings $\mathbb{Z}^n \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$.

**Theorem 2.5.** (Adolphson [1, Corollary 5.20]) Assume that the parameter vector $c \in \mathbb{C}^n$ is semi-nonresonant (for the definition see [1, page 284]). Then the rank of the local system $H^0\text{Sol}_X(\mathcal{M}_{A,c})|_{\Omega^\text{an}}$ on $\Omega^\text{an}$ is equal to $\text{Vol}_{\mathbb{Z}}(\Delta)$. \hspace{1cm} (2.10)

This is a generalization of the famous result of Gelfand-Kapranov-Zelevinsky in [14] to the confluent case. Later Matushevich-Miller-Walther [33] generalized it further by showing that the holonomic rank of $\mathcal{M}_{A,c}$ does not jump outside the union of finitely many subspaces of $\mathbb{C}^n$ of codimension at least two. We call the sections of the local system $H^0\text{Sol}_X(\mathcal{M}_{A,c})|_{\Omega^\text{an}}$ confluent $A$-hypergeometric functions (associated to the parameter $c \in \mathbb{C}^n$).
3 Hien’s rapid decay homologies

In this section, we review Hien’s theory of rapid decay homologies invented in [17] and [18]. For the theory of twisted homology groups we refer to Aomoto-Kita [3] and Pajitnov [37]. Let $U$ be a smooth quasi-projective variety of dimension $n$ and $(\mathcal{E}, \nabla) (\nabla : \mathcal{E} \rightarrow \Omega_U^1 \otimes_{\mathcal{O}_U} \mathcal{E})$ an integrable connection on it. We consider $(\mathcal{E}, \nabla)$ as a left $\mathcal{D}_U$-module and set

$$\text{DR}_U(\mathcal{E}) = \Omega^\bullet_{U, \text{an}} \otimes_{\mathcal{O}_{U, \text{an}}} \mathcal{E}^\text{an} \simeq \Omega^\bullet_{U, \text{an}} \otimes_{\mathcal{O}_{U, \text{an}}} \mathcal{E}^\text{an}[n].$$

(3.1)

Assume that $i : U \hookrightarrow Z$ is a smooth projective compactification of $U$ such that $D = Z \setminus U$ is a normal crossing divisor and the extension $i_* \mathcal{E}$ of $\mathcal{E}$ to $Z$ admits a good lattice in the sense of Mochizuki [34]. Such a good compactification of $U$ for $(\mathcal{E}, \nabla)$ always exists by the fundamental theorem recently established by Mochizuki [34]. Now let $\pi : \tilde{Z} \rightarrow Z^\text{an}$ be the real oriented blow-up of $Z^\text{an}$ in [17], [18] and set $\tilde{D} = \pi^{-1}(D^\text{an})$. Recall that $\pi$ induces an isomorphism $\tilde{Z} \setminus \tilde{D} \simeq Z^\text{an} \setminus D^\text{an}$. More precisely, for each point $q \in D^\text{an}$ by taking a local coordinate $(x_1, \ldots, x_n)$ on a neighborhood of $q$ such that $q = (0, \ldots, 0)$ and $D^\text{an} = \{x_1 \cdots x_k = 0\}$ the morphism $\pi$ is explicitly given by

$$(0, \epsilon) \times S^1 \times B(0; \epsilon)^n \rightarrow B(0; \epsilon)^k \times B(0; \epsilon)^n,$$

(3.2)

$$(\{x, e^{-\epsilon} \cdot \theta_i\})_{k \geq 1} \times x_{k+1}, \ldots, x_n) \mapsto \{x, e^{-\epsilon} \cdot \theta_i\} \times (x_{k+1}, \ldots, x_n),$$

(3.3)

where we set $B(0; \epsilon) = \{x \in \mathbb{C} \mid |x| < \epsilon\}$ for $\epsilon > 0$. For $p \geq 0$ and a subset $B \subset \tilde{Z}$ denote by $S_p(B)$ the $\mathbb{C}$-vector space generated by the piecewise smooth maps $c : \Delta^p \rightarrow B$ from the $p$-dimensional simplex $\Delta^p$. We denote by $C^p_{\tilde{Z}, \tilde{D}}$ the sheaf on $\tilde{Z}$ associated to the presheaf

$$V \mapsto S_p(\tilde{Z} \setminus V) = S_p(\tilde{Z})/S_p(\tilde{Z} \setminus V \cup \tilde{D}).$$

(3.4)

Namely $C^p_{\tilde{Z}, \tilde{D}}$ is the sheaf of the relative $p$-chains on the pair $(\tilde{Z}, \tilde{D})$. Now let $\mathcal{L} := H^{-n}\text{DR}_U(\mathcal{E}) = \text{Ker}\{\nabla^\text{an} : \mathcal{E}^\text{an} \rightarrow \Omega^1_{U, \text{an}} \otimes_{\mathcal{O}_{U, \text{an}}} \mathcal{E}^\text{an}\}$ be the sheaf of horizontal sections of the analytic connection $(\mathcal{E}^\text{an}, \nabla^\text{an})$ and $i : U^\text{an} \hookrightarrow \tilde{Z}$ the inclusion. Then $i_* \mathcal{L}$ is a local system on $\tilde{Z}$. We define the sheaf $C^p_{\tilde{Z}, \tilde{D}}(i_* \mathcal{L})$ of the relative twisted $p$-chains on the pair $(\tilde{Z}, \tilde{D})$ with coefficients in $i_* \mathcal{L}$ by $C^p_{\tilde{Z}, \tilde{D}}(i_* \mathcal{L}) = C^p_{\tilde{Z}, \tilde{D}} \otimes_{C^0_{\tilde{Z}, \tilde{D}}} i_* \mathcal{L}$.

**Definition 3.1.** (Hien [17] and [18]) A section $c \otimes s \in \Gamma(V; C^p_{\tilde{Z}, \tilde{D}}(i_* \mathcal{L}))$ is called a rapid decay chain if for any point $q \in c(\Delta^p) \cap \tilde{D} \cap V$ the following condition holds:

In a local coordinate $(x_1, \ldots, x_n)$ on a neighborhood of $q$ in $Z$ such that $q = (0, \ldots, 0)$ and $D^\text{an} = \{x_1 \cdots x_k = 0\}$ by taking a local trivialization $(i_* \mathcal{E})^\text{an} \simeq \oplus_{i=1}^n \mathcal{O}_{Z^\text{an}}(*D^\text{an})e_i$ with respect to a basis $e_1, \ldots, e_r$ and setting $s = \sum_{i=1}^r f_i \cdot e_i$ with $f_i \in i_* \mathcal{O}_{Z^\text{an}}$, for any $1 \leq i \leq r$ and $N = (N_1, \ldots, N_k) \in \mathbb{N}_e$ there exists $C_N > 0$ such that

$$|f_i(x)| \leq C_N |x_1|^{N_1} \cdots |x_k|^{N_k}$$

(3.5)

for any $x \in (c(\Delta^p) \setminus \tilde{D}) \cap V$ with small $|x_1|, \ldots, |x_k|$.

In particular, if $c(\Delta^p) \cap \tilde{D} \cap V = \emptyset$ we do not impose any condition on $s \in i_* \mathcal{L}$.

Note that this definition does not depend on the local coordinate $(x_1, \ldots, x_n)$ nor the local trivialization $(i_* \mathcal{E})^\text{an} \simeq \oplus_{i=1}^n \mathcal{O}_{Z^\text{an}}(*D^\text{an})e_i$. We denote by $C^p_{\tilde{Z}, \tilde{D}}(i_* \mathcal{L})$ the subsheaf of
\( C_{Z,D}^{-\bullet}(\tau_\bullet) \) consisting of rapid decay chains. According to Hien [17] and [18], \( C_{Z,D}^{rd,\bullet}(\tau_\bullet) \) is a fine sheaf. Then we obtain a complex of fine sheaves on \( \tilde{Z} \):

\[
\cdots \to C_{Z,D}^{-(-p+1)}(\tau_\bullet) \to C_{Z,D}^{rd,-p}(\tau_\bullet) \to C_{Z,D}^{-(-p-1)}(\tau_\bullet) \to \cdots
\]  

(3.6)

**Definition 3.2.** (Hien [17] and [18]) For \( p \in \mathbb{Z} \) we set

\[
H_p^{rd}(U; \mathcal{E}) := H^{-p}\Gamma(Z; C_{Z,D}^{rd,\bullet}(\tau_\bullet))
\]

(3.7)

and call it the \( p \)-th rapid decay homology group associated to the integrable connection \( \mathcal{E} \).

In [18] Hien proved that \( H_p^{rd}(U; \mathcal{E}) \) is isomorphic to the dual of the \( p \)-th algebraic de Rham cohomology of the dual connection \( \mathcal{E}^* \) of \( \mathcal{E} \). In this paper, we use only some special integrable connections \((\mathcal{E}, \nabla)\) as the following example.

**Example 3.3.** Let \( U \simeq \mathbb{C}_x \) and \( \mathcal{E} = \mathcal{O}_U \exp(-h(x))x^{-c} \), where \( h(x) = \sum_{i \in \mathbb{C}} a_i x^i \) (\( a_i \in \mathbb{C} \)) is a Laurent polynomial and \( c \in \mathbb{C} \). As usual we endow \( \mathcal{E} = \mathcal{O}_U \exp(-h(x))x^{-c} \) with the connection \( \nabla : \mathcal{E} \to \Omega_U^1 \otimes \mathcal{O}_U \mathcal{E} \) defined by

\[
\nabla \{ f \exp(-h(x))x^{-c} \} = df \otimes \exp(-h(x))x^{-c} - (dh + \frac{c}{x} dx) \otimes f \exp(-h(x))x^{-c}
\]

(3.8)

for \( f \in \mathcal{O}_U \). Then we have \( \mathcal{L} = H^{-1}\text{dR}_U(\mathcal{E}) \simeq \mathbb{C}_{U^\text{an}} \exp(h(x))x^c \subset \mathcal{O}_{U^\text{an}} \). In this case, to define the rapid decay homology groups \( H_p^{rd}(U; \mathcal{E}) \) we consider (relative) twisted chains on which the function \( \exp(h(x))x^c \) decays rapidly at infinity.

If \( \mathcal{E} \simeq \mathcal{O}_U(\frac{1}{g}) \) and we have an isomorphism \( \mathcal{L} = H^{-n}\text{dR}_U(\mathcal{E}) \simeq \mathbb{C}_{U^\text{an}} g \subset \mathcal{O}_{U^\text{an}} \) for a possibly multi-valued holomorphic function \( g : U^\text{an} \to \mathbb{C} \) as the example above, we call \( H_p^{rd}(U; \mathcal{E}) \) the \( p \)-th rapid decay homology group associated to the function \( g \). In the special case where \( g(x) = \exp(h(x))g_0(x) \) for a meromorphic function \( h \) on \( Z^\text{an} \) with poles in \( D^\text{an} \) and a (possibly multi-valued) function \( g_0 \) on \( U^\text{an} \) such that at each point of \( Z^\text{an} \) there exists a local coordinate \( x = (x_1, \ldots, x_n) \) satisfying \( g_0(x) = x_1^{e_1} \cdots x_n^{e_n} \) (\( e_i \in \mathbb{C} \)) and \( D = \{ x_1 \cdots x_k = 0 \} \), we shall give a purely topological interpretation of \( H_p^{rd}(U; \mathcal{E}) \). Since \( Z \) is a good compactification of \( U \) for \( \mathcal{E} \), the meromorphic function \( h \) has no point of indeterminacy on the whole \( Z^\text{an} \) (see [19, Section 2.1]). By \( i : U^\text{an} \hookrightarrow \tilde{Z} \) we consider \( U^\text{an} \) as an open subset of \( \tilde{Z} \) and set

\[
P = \tilde{D} \cap \{ x \in U^\text{an} \mid \text{Re}(h(x)) \geq 0 \}.
\]

(3.9)

Let \( D = D_1 \cup \cdots \cup D_d \) be the irreducible decomposition of \( D \). For \( 1 \leq i \leq d \) let \( b_i \in \mathbb{Z} \) be the order of the meromorphic function \( h \) along \( D_i \). If \( b_i \geq 0 \) we say that the irreducible component \( D_i \) is irrelevant. Namely along a relevant component \( D_i \) the function \( h \) has a pole of order \( -b_i > 0 \). Denote by \( D' \) the union of the irrelevant components of \( D \). Then we set \( Q = D \setminus \{ P \cup \pi^{-1}(D')^\text{an} \} \). Note that \( Q \) is an open subset of \( \tilde{D} \) (i.e. the set of the rapid decay directions of the function \( g \) in \( \tilde{D} \)).

**Proposition 3.4.** In the situation as above (i.e. \( \mathcal{E}^* = \mathcal{O}_U \exp(h(x))g_0(x) \)), we have an isomorphism

\[
H_p^{rd}(U; \mathcal{E}) \simeq H_p(U^\text{an} \cup Q, Q; \iota_*(\mathcal{C}_{U^\text{an}}g_0))
\]

(3.10)

for any \( p \in \mathbb{Z} \), where the right hand side is the \( p \)-th relative twisted homology group of the pair \((U^\text{an} \cup Q, Q)\) with coefficients in the rank-one local system \( \iota_*(\mathcal{C}_{U^\text{an}}g_0) \) on \( \tilde{Z} \) (see [3]).
Proof. Since the function \( \exp(h(x)) \) is single-valued, we have an isomorphism \( \mathcal{L} \simeq \mathbb{C}_{U^an}g_0 \).
First let us consider the case \( n = 1 \). Locally we may assume that \( U = \mathbb{C}^* \), \( Z = \mathbb{C}_x = \mathbb{C}^* \cup \{0\} \), \( D = \{x = 0\} = \{0\} \subset Z \) and \( h(x) = x^{-m} (m > 0) \). Let \( \pi : \tilde{Z} \to Z^an \) be the real oriented blow-up of \( Z^an \) along \( D^an \). In this case we have \( \tilde{D} = \pi^{-1}(D^an) \simeq S^1 \) and \( \tilde{Z} = \{(r, e^{\sqrt{-1}\theta}) \mid r \geq 0 \} \cong [0, \infty) \times S^1 \). For \( 1 \leq i \leq m \) and sufficiently small \( \varepsilon > 0 \) we set
\[
Q^\varepsilon_i = \{e^{\sqrt{-1}\theta} \in \tilde{Z} \simeq S^1 \mid \frac{(2i - \frac{3}{2})\pi}{m} - \varepsilon < \theta < \frac{(2i - \frac{1}{2})\pi}{m} + \varepsilon\},
\]
and \( Q^\varepsilon = \bigcup_{i=1}^m Q^\varepsilon_i \subset \tilde{D} \). Note that \( Q^\varepsilon \subset \tilde{D} \) contains all the rapid decay directions of \( g(x) = \exp(h(x))g_0(x) \) in \( \tilde{D} \). Now let us consider the two topological subspaces \( U^an \cup Q^\varepsilon \) and \( \tilde{D} \) of \( \tilde{Z} \). We patch them on their intersection \( Q^\varepsilon \) and construct a new topological space as follows. By identifying the points of \( Q^\varepsilon \subset U^an \cup Q^\varepsilon \) and those of \( Q^\varepsilon \subset \tilde{D} \) naturally, we obtain a quotient space \( \tilde{Z}^\varepsilon \) of the disjoint union \( (U^an \cup Q^\varepsilon) \cup \tilde{D} \). Recall that \( \tilde{Z}^\varepsilon \) is endowed with the strongest topology for which the quotient map \( (U^an \cup Q^\varepsilon) \cup \tilde{D} \to \tilde{Z}^\varepsilon \) is continuous. Note also that \( \tilde{D} \) is naturally identified with a close subspace of \( \tilde{Z}^\varepsilon \). We denote the local system on \( \tilde{Z}^\varepsilon \) naturally constructed from \( \iota_\varepsilon \mathcal{L} \) by the same letter \( \iota_\varepsilon \mathcal{L} \). For \( p \in \mathbb{Z} \) let \( S_p(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L}) \) be the \( \mathbb{C} \)-vector space of the twisted (piecewise smooth) relative \( p \)-chains on the pair \( (\tilde{Z}^\varepsilon, \tilde{D}) \) with coefficients in \( \iota_\varepsilon \mathcal{L} \). Then by the definition of rapid decay chains, for any \( p \in \mathbb{Z} \) we obtain a natural morphism
\[
\Gamma(\tilde{Z}; C^\text{rd},-p(\iota_\varepsilon \mathcal{L})) \to S_p(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L}).
\]
We can easily show that the chain map
\[
\Gamma(\tilde{Z}; C^\text{rd},\bullet(\iota_\varepsilon \mathcal{L})) \to S_{-\bullet}(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L})
\]
obtained in this way is a homotopy equivalence. Indeed, its homotopy inverse \( S_{-\bullet}(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L}) \to \Gamma(\tilde{Z}; C^\text{rd},\bullet(\iota_\varepsilon \mathcal{L})) \) can be constructed by smooth deformations of chains in \( S_{-\bullet}(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L}) \) in the angular direction \( \theta = \arg x \). We can construct them by a smooth vector field on \( \tilde{Z} \). Hence we obtain an isomorphism
\[
H^{-p}\Gamma(\tilde{Z}; C^\text{rd},\bullet(\iota_\varepsilon \mathcal{L})) \sim \to H_p(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L})
\]
for any \( p \in \mathbb{Z} \). Moreover by excision and homotopy, we have an isomorphism
\[
H_p(\tilde{Z}^\varepsilon, \tilde{D}; \iota_\varepsilon \mathcal{L}) \sim \to H_p(U^an \cup Q^\varepsilon, Q^\varepsilon; \iota_\varepsilon \mathcal{L})
\]
for any \( p \in \mathbb{Z} \). Combining (3.14) with (3.15), we obtain an isomorphism
\[
H^{-p}\Gamma(\tilde{Z}; C^\text{rd},\bullet(\iota_\varepsilon \mathcal{L})) \sim \to H_p(U^an \cup Q^\varepsilon, Q^\varepsilon; \iota_\varepsilon \mathcal{L})
\]
for any \( p \in \mathbb{Z} \).

Finally let us consider the case \( n \geq 2 \). First we assume that for some \( 1 \leq k \leq n \) we have \( Z = \mathbb{C}^*_n \), \( D = \{x_1 \cdots x_k = 0\} \subset Z \), \( U = Z \setminus D \) and \( h(x) = x_1^{-m_1} \cdots x_k^{-m_k} (m_i > 0) \). Let \( \pi : \tilde{Z} \to Z^an \) be the real oriented blow-up of \( Z^an \) along \( D^an \). In this case we have \( \tilde{Z} = \{\left(\{(r_i, e^{\sqrt{-1}\theta(r_i)})\}_{i=1}^k, x_{k+1}, \ldots, x_n\right) \mid r_i \geq 0\} \simeq ([0, \infty) \times S^1)^k \times \mathbb{C}^{n-k} \). For sufficiently small \( \varepsilon > 0 \) we define an open subset \( Q^\varepsilon \subset \tilde{D} \) by \( \left(\{(r_i, e^{\sqrt{-1}\theta(r_i)})\}_{i=1}^k, x_{k+1}, \ldots, x_n\right) \in Q^\varepsilon \iff \ldots \)
\[
\text{Re } e^{\sqrt{-1}(m_1 \theta_1 + \cdots + m_k \theta_k)} < \varepsilon |\text{Im } e^{\sqrt{-1}(m_1 \theta_1 + \cdots + m_k \theta_k)}| \text{ for } ((r_i, e^{\sqrt{-1} \theta_i}))^{k}_{i=1}, x_{k+1}, \ldots, x_n) \in \tilde{D}.
\]
Then \(Q^e\) contains all the rapid decay directions of \(g(x) = \exp(h(x))g_0(x)\) in \(\tilde{D}\). As in the case \(n = 1\), by smooth deformations of chains and excision etc. we obtain an isomorphism
\[
H^{-p}T(\tilde{Z}; \mathcal{O}_{\tilde{Z}}^{an, \bullet}(\iota_* \mathcal{L})) \simto H_p(U^{an} \cup Q^e, Q^e; \iota_* \mathcal{L}) \quad (3.17)
\]
for any \(p \in \mathbb{Z}\). The general case can be proved similarly by patching local smooth deformations of chains (smooth vector fields) as above by a partition of unity. This completes the proof.

The following lemma will be used in Section 4.

**Lemma 3.5.** In the situation of Proposition 3.4, for a point \(q \in D^{an}\) let \(k \geq 0\) be the number of the relevant irreducible components of \(D^{an}\) passing through \(q\). Assume that \(k \geq 2\). Then for a small open neighborhood \(V\) of \(q\) in \(Z^{an}\) we have
\[
\sum_{p \in \mathbb{Z}} (-1)^p \dim H_p((V \cap U^{an}) \cup (\pi^{-1}(V) \cap Q), (\pi^{-1}(V) \cap Q); \iota_* (\mathcal{C}_{U^{an}, g_0})) = 0. \quad (3.18)
\]

**Proof.** The problem being local, we may assume that for some \(k \leq l \leq n\) we have \(V = Z = \mathbb{C}_x^n, D = \{x_1 \cdots x_l = 0\} \subset Z, U = Z \setminus D\) and \(q = 0 \in D^{an}\). For simplicity here we consider the case where \(k = l\) and \(h(x) = x_1^{m_1} \cdots x_l^{m_k}\) (\(m_i > 0\)). Then we have the product decomposition \(\tilde{Z} = \{((r_i, e^{\sqrt{-1} \theta_i}))^{k}_{i=1}, x_{k+1}, \ldots, x_n) \mid r_i \geq 0\} \simeq ([0, \infty) \times S^1)^k \times \mathbb{C}^{n-k}\) of \(\tilde{Z}\) and by the projection \(p_1 : \tilde{Z} \rightarrow S^1\) defined by \(((r_i, e^{\sqrt{-1} \theta_i}))^{k}_{i=1}, x_{k+1}, \ldots, x_n) \mapsto e^{\sqrt{-1} \theta_1}\) the two manifolds \(U^{an}, Q \subset \tilde{Z}\) are fiber bundles over \(S^1\). Let \(S^1 = \cup_{i=1}^{r} I_i\) be an open covering of \(S^1\) such that the restrictions \(p_1^{-1}(I_i) \cap U^{an} \rightarrow I_i\) and \(p_1^{-1}(I_i) \cap Q \rightarrow I_i\) of the above two fiber bundles to \(I_i \subset S^1\) are isomorphic to the trivial ones over \(I_i\) for any \(1 \leq i \leq r\). Then by the Mayer-Vietoris exact sequences for the relative twisted homology groups
\[
H_p((p_1^{-1}(I_i) \cap U^{an}) \cup (p_1^{-1}(I_i) \cap Q), (p_1^{-1}(I_i) \cap Q); \iota_* (\mathcal{C}_{U^{an}, g_0})), \quad (3.19)
\]
the Euler characteristic of the circle \(S^1\) is zero.

In the sequel, we consider the more special case where \(U = \mathbb{C}_x^n\) and \((\mathcal{E}, \nabla)\) is an integrable connection on \(U\) such that \(\mathcal{E}^* = \mathcal{O}_U \exp(h(x))x^c\) and \(\mathcal{L} = H^{-1}DR_U(\mathcal{E}) \simeq \mathbb{C}^{U^{an}} \exp(h(x))x^c\) for a Laurent polynomial \(h(x) = \sum_{i \in \mathbb{Z}} a_i x^i\) \((a_i \in \mathbb{C})\) and \(c \in \mathbb{C}\). Then we can take the projective line \(\mathbb{P}\) to be the good compactification \(\tilde{Z} = U_x^n = \mathbb{C}_x^n\) for \((\mathcal{E}, \nabla)\).

In this case, we have \(D = Z \setminus U = D_1 \cup D_2\), where we set \(D_1 = \{0\}\) and \(D_2 = \{\infty\}\). For the real oriented blow-up \(\pi : \tilde{Z} \rightarrow Z^{an}\) of \(Z^{an}\) the subset \(\tilde{D} = \pi^{-1}(D^{an})\) of \(\tilde{Z}\) is a union of two circles \(\tilde{D}_i = \pi^{-1}(D_i^{an}) \simeq S^1 (i = 1, 2)\). Moreover the open subset \(Q \subset \tilde{D}\) is a union of open intervals in \(\tilde{D}_1 \cup \tilde{D}_2 \simeq S^1 \cup S^1\). Let \(NP(h) \subset \mathbb{R}\) be the Newton polytope of \(h\) i.e. the convex hull of the set \(\{i \in \mathbb{Z} \mid a_i \neq 0\}\) in \(\mathbb{R}\). Finally denote by \(\Delta \subset \mathbb{R}\) the convex hull of \(NP(h) \cup \{0\}\) in \(\mathbb{R}\). Then by Proposition 3.4 and Mayer-Vietoris exact sequences for relative twisted homology groups we can easily prove the following result.

**Lemma 3.6.** In the situation as above \((i.e. \ U = \mathbb{C}_x^n \text{ and } \mathcal{E}^* = \mathcal{O}_U \exp(h(x))x^c\), we have
(i) The dimension of the rapid decay homology group \(H^d_p(U; \mathcal{E})\) is \(\text{Vol}_2(\Delta)\) if \(p = 1\) and zero otherwise.
(ii) Assume that \( \Delta = [-m,0] \) (resp. \( \Delta = [0,m] \)) for some \( m > 0 \). Then \( Q \subset \tilde{D} \) is a union of \( m \) open intervals \( Q_1, Q_2, \ldots, Q_m \) in \( \tilde{D}_1 \simeq S^1 \) (resp. in \( \tilde{D}_2 \simeq S^1 \)) and the first rapid decay homology group \( H_{rd}^1(U;E) \) has a basis formed by the \( m \) elements

\[
[\gamma_i] \in H_{rd}^1(U;E) \quad (i = 1, 2, \ldots, m),
\]

where \( \gamma_i \) is a 1-dimensional twisted chain with values in \( \iota_* \mathcal{L} \) as in Figure 1 below starting from a point in \( Q_i \) and going directly to that in \( Q_{i+1} \) (here we set \( Q_{m+1} = Q_1 \)).

(iii) Assume that \( \Delta = [-m_1,m_2] \) for some \( m_1, m_2 > 0 \). Then \( Q \subset \tilde{D} \) is a union of open intervals \( Q_1, Q_2, \ldots, Q_{m_1} \) in \( \tilde{D}_1 \simeq S^1 \) and the ones \( Q'_1, Q'_2, \ldots, Q'_{m_2} \) in \( \tilde{D}_2 \simeq S^1 \). If moreover \( c \notin \mathbb{Z} \), then the first rapid decay homology group \( H_{rd}^1(U;E) \) has a basis formed by the \( m_1 + m_2 \) elements

\[
[\gamma_i] \in H_{rd}^1(U;E) \quad (i = 1, 2, \ldots, m_1)
\]

and

\[
[\gamma'_i] \in H_{rd}^1(U;E) \quad (i = 1, 2, \ldots, m_2),
\]

where \( \gamma_i \) (resp. \( \gamma'_i \)) is a 1-dimensional twisted chain with values in \( \iota_* \mathcal{L} \) starting from a point in \( Q_i \) (resp. \( Q'_i \)) and going directly to that in \( Q_{i+1} \) (resp. \( Q'_{i+1} \)).

\[\begin{align*}
\tilde{Z} & \quad Q_3 \\
\tilde{D}_1 & \quad Q_2 \\
\tilde{D}_2 & \quad Q_1 \\
\tilde{D}_1 & \quad Q_m \\
\end{align*}\]

Figure 1.

4 A geometric construction of integral representations

In this section we give a geometric construction of Adolphson’s confluent \( A \)-hypergeometric \( D \)-module \( \mathcal{M}_{A,c} \) and apply it to obtain the integral representations of \( A \)-hypergeometric functions. Let \( Y = \mathbb{C}^N_\zeta \) be the dual vector space of \( X = \mathbb{C}^A = \mathbb{C}^N_\zeta \), where \( \zeta \) is the dual coordinate of \( z \). As in [15], to \( A \subset \mathbb{Z}^n \) we associate a morphism

\[
j : T = (\mathbb{C}^*)^n \longrightarrow Y = \mathbb{C}^N_\zeta
\]

defined by \( x \mapsto (x^{a(1)}, x^{a(2)}, \ldots, x^{a(N)}) \). Since we assume here that \( A \) generates \( \mathbb{Z}^n \), \( j \) is an embedding. Let \( I \subset \mathbb{C}[\zeta_1, \ldots, \zeta_N] \) be the defining ideal of the closure \( \overline{j(T)} \) of \( j(T) \subset Y \) in \( Y \). Moreover denote by \( D(Y) \) the Weyl algebra over \( Y \). Then we have a ring isomorphism

\[
\wedge : D(X) \xrightarrow{\sim} D(Y)
\]
defined by
\[
\frac{\partial}{\partial z_j} \wedge = \zeta_j, \quad (z_j)^\wedge = -\frac{\partial}{\partial \zeta_j} \quad (j = 1, 2, \ldots, N).
\] (4.3)

We call \( \wedge \) the Fourier transform (see Malgrange [28] for the details). Via this \( \wedge \), the Adolphson’s system \( M_{A,c} \) is transformed to the one
\[
(Z_{i,c})^\wedge v(\zeta) = 0 \quad (1 \leq i \leq n), \quad f(\zeta)v(\zeta) = 0 \quad (f \in I)
\] (4.4)
on \( Y = \mathbb{C}^N_\zeta \). Note that this system has no non-zero holomorphic solution in general. Let
\[
N_{A,c} = M_{A,c}^\wedge = D(Y)/ \left( \sum_{1 \leq i \leq n} D(Y)(Z_{i,c})^\wedge + \sum_{f \in I} D(Y)f \right)
\] (4.5)
be the corresponding left \( D(Y) \)-module and \( N_{A,c} \) the coherent \( D_Y \)-module associated to it. Now on the algebraic torus \( T = (\mathbb{C}^*)^n_x \) we define a holonomic \( D_T \)-module \( R_c \) by
\[
R_c = D_T/ \sum_{1 \leq i \leq n} D_T \left\{ x_i \frac{\partial}{\partial x_i} + (1 - c_i) \right\} \simeq \mathcal{O}_T x_1^{c_1-1} \cdots x_n^{c_n-1}.
\] (4.6)

This is an integrable connection on \( T \) and we have
\[
\text{DR}_T(R_c) \simeq (\mathbb{C}_T^{*}\cdot x_1^{-c_1+1} \cdots x_n^{-c_n+1})[n].
\] (4.7)

Let \( v = [1] \in N_{A,c} \) and \( w_0 = [1] \in R_c \) be the canonical generators. Recall that the transfer bimodule \( D_{Y \rightarrow T} \) has the canonical section \( 1_{Y \rightarrow T} \). We define a section \( 1_{Y \leftarrow T} \) of \( D_{Y \leftarrow T} = \Omega_T \otimes_{\mathcal{O}_T} D_{T \rightarrow Y} \otimes_{j^{-1}\mathcal{O}_Y} j^{-1}\Omega_Y^\wedge^{-1} \) by
\[
1_{Y \leftarrow T} = (dx_1 \wedge \cdots \wedge dx_n) \otimes 1_{T \rightarrow Y} \otimes j^{-1}(d\zeta_1 \wedge \cdots \wedge d\zeta_N)^\wedge. \] (4.8)

Note that this definition of \( 1_{Y \leftarrow T} \) depends on the coordinates of \( Y \) and \( T \). Then we obtain a section \( w \) of the regular holonomic \( D_Y \)-module
\[
S_{A,c} := \int_j R_c = j_*(D_{Y \leftarrow T} \otimes_{D_T} R_c)
\] (4.9)
defined by \( w = j_*(1_{Y \leftarrow T} \otimes w_0) \). We can easily check that this section \( w \in S_{A,c} \) satisfies the system (4.4). Hence as in [15, page 268-269], we obtain a morphism
\[
\Psi : N_{A,c} \longrightarrow S_{A,c} = \int_j R_c
\] (4.10)
of left \( D_Y \)-modules which sends the canonical generator \( v = [1] \in N_{A,c} \) to \( w \in S_{A,c} \).

**Definition 4.1.** (Gelfand-Kapranov-Zelevinsky [15, page 262]) For a face \( \Gamma \) of \( \Delta \) containing the origin \( 0 \in \mathbb{R}^n \) we denote by \( \text{Lin}(\Gamma) \subset \mathbb{C}^n \) the \( \mathbb{C} \)-linear span of \( \Gamma \). We say that the parameter vector \( c \in \mathbb{C}^n \) is nonresonant (with respect to \( A \)) if for any face \( \Gamma \) of \( \Delta \) of codimension 1 such that \( 0 \in \Gamma \) we have \( c \notin \{ \mathbb{Z}^n + \text{Lin}(\Gamma) \} \).
Recall that if \( c \in \mathbb{C}^n \) is nonresonant then it is semi-nonresonant in the sense of [1, page 284]. The following result was proved by Saito [39] and Schulze-Walther [42], [43] by using commutative algebras (see also Beukers [4] for another approach to this problem). Here we give a geometric proof to it.

**Lemma 4.2.** Assume that the parameter vector \( c \in \mathbb{C}^n \) is nonresonant. Then the regular holonomic \( D_Y \)-module \( S_{A,c} \) is irreducible.

**Proof.** Note that \( DR_T(R_c) \cong (\mathbb{C}_T^a x_1^{c_1+1} \cdots x_n^{c_n+1})[n] \) is an irreducible perverse sheaf on \( T^a \). Then also its minimal extension by the locally closed embedding \( j \) is irreducible (see [22, Corollary 8.2.10]). As in [15, Theorem 3.5 and Propositions 3.2 and 4.4] it suffices to show that the canonical morphism

\[
j_!(\mathbb{C}_T^a x_1^{c_1-1} \cdots x_n^{c_n-1}) \longrightarrow R j_!(\mathbb{C}_T^a x_1^{c_1-1} \cdots x_n^{c_n-1})
\]

is a quasi-isomorphism. For this, we have only to prove the vanishing \( R j_!(\mathbb{C}_T^a x_1^{c_1-1} \cdots x_n^{c_n-1}) q \cong 0 \) for any \( q \in j(T)^0 \setminus j(T) \). Note that by the nonresonance of \( c \in \mathbb{C}^n \) for any \( p \in \mathbb{Z} \) and the local system \( L := \mathbb{C}_T^a x_1^{c_1-1} \cdots x_n^{c_n-1} \) on \( T^a \), we have \( H^p(T^a; L) = 0 \). Let \( S(A) \subset \mathbb{Z}^n \) (resp. \( K(A) \subset \mathbb{R}^n \)) be the semigroup (resp. the convex cone) generated by \( A \). Then by (the proof of) [16, Chapter 5, Theorem 2.3] we have \( j(T) \cong \text{Spec}(\mathbb{C}[S(A)]) \). Let us define an action of \( T \) on \( Y = \mathbb{C}_\zeta^N \) by

\[
(\zeta_1, \ldots, \zeta_N) \mapsto (x^{a(1)} \zeta_1, \ldots, x^{a(N)} \zeta_N)
\]

for \( x \in T \). Then by [16, Chapter 5, Theorem 2.5] there exists a natural bijection between the faces of \( K(A) \) and the \( T \)-orbits in \( j(T) \). In particular, if \( K(A) = \mathbb{R}^n \) we have \( j(T) = j(T) \) and there is nothing to prove. First consider the case where \( 0 \in \mathbb{R}^n \) is an apex of \( K(A) \) and \( q = 0 \in Y = \mathbb{C}_\zeta^N \). If \( 0 \in A \) i.e. \( 0 = a(j) \) for some \( 1 \leq j \leq N \) we have \( j(T) \subset \{ \zeta_j = 1 \} \cong \mathbb{C}^N \). Hence we may assume that \( 0 \notin A \) from the start. In this case, \( \{0\} \subset j(T) \) is the unique 0-dimensional \( T \)-orbit in \( j(T) \) which corresponds to \( \{0\} \subset K(A) \). From now on, we will prove that \( R j_!(L)_0 \cong 0 \). By our assumption there exists a linear function \( l : \mathbb{R}^n \longrightarrow \mathbb{R} \) such that \( l(\mathbb{Z}^n) \subset \mathbb{Z} \) and \( K(A) \setminus \{0\} \subset \{ l > 0 \} \). We define a real-valued function \( \varphi : Y = \mathbb{C}_\zeta^N \longrightarrow \mathbb{R} \) by

\[
\varphi(\zeta) = |\zeta_1|^{\frac{c_1}{|\zeta| \cdot \zeta_1}} + \cdots + |\zeta_N|^{\frac{c_N}{|\zeta| \cdot \zeta_N}},
\]

where we take \( C \in \mathbb{Z}_{>0} \) large enough so that \( \varphi \) and its level sets \( \varphi^{-1}(b) \) (\( b > 0 \)) are smooth. Let \( (l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n \) be the coefficients of the linear function \( l \). Define an action of the multiplicative group \( \mathbb{R}_{>0} \) on \( T \) by

\[
r \cdot (x_1, \ldots, x_n) = (r^{l_1} x_1, \ldots, r^{l_n} x_n)
\]

for \( r \in \mathbb{R}_{>0} \). Then we have

\[
j(r \cdot x) = (r^{l(a(1))} x^{a(1)}, \ldots, r^{l(a(N))} x^{a(N)})
\]

and hence

\[
\varphi(j(r \cdot x)) = r^C \varphi(j(x)).
\]
Therefore by the action of \( \mathbb{R}_{>0} \) on \( Y = \mathbb{C}^N \) defined by

\[
r \cdot (\zeta_1, \ldots, \zeta_N) = (r^{\ell(a(1))}\zeta_1, \ldots, r^{\ell(a(N))}\zeta_N),
\]

(4.17)
a level set \( \varphi^{-1}(t) \) \((t > 0)\) of \( \varphi \) is sent to the one \( \varphi^{-1}(r^C t) \). Moreover this action preserves the \( T \)-orbits in \( j(T) \). Let \( O \subset j(T) \) be such a \( T \)-orbit. Then all the level sets \( \varphi^{-1}(t) \) \((t > 0)\) of \( \varphi \) are transversal to \( O \), or all are not. But the latter case cannot occur by the Sard theorem. Then we obtain an isomorphism

\[
H^pRj_\ast(\mathcal{L})_0 \simeq H^p(\mathbb{C}^N; Rj_\ast(\mathcal{L})) \simeq H^p(T^\text{an}; \mathcal{L}) \simeq 0
\]

(4.18)
for any \( p \in \mathbb{Z} \). Next consider the remaining case where \( q \in O \) for a \( T \)-orbit \( O \) in \( j(T) \) such that \( \dim O \geq 1 \). Then in a neighborhood of \( q \), the variety \( j(T) \) is a product \( W \times O \) for an affine toric variety \( W \subset \mathbb{C}^{N'} \) and \( j(T) = (T_1 \sqcup \cdots \sqcup T_k) \times O \) for some tori \( T_i \simeq (\mathbb{C}^*)^{n - \dim O} \). See [16, Chapter 5, Theorem 3.1] and the proof of [29, Theorem 4.9] for the details. Moreover for the semigroup \( S(A_0) \subset \mathbb{Z}^{n - \dim O} \) generated by a finite subset \( A_0 \subset \mathbb{Z}^{n - \dim O} \) we have \( T_i \simeq \text{Spec}(\mathbb{C}[S(A_0)]) \subset W \) \((i = 1, 2, \ldots, k)\). These varieties \( T_i \) are the irreducible components of \( W \). For the explicit construction of \( T_i \) see the proof of [29, Theorem 4.9]. By this construction \( 0 \in \mathbb{R}^{n - \dim O} \) is an apex of the convex cone \( K(A_0) \subset \mathbb{R}^{n - \dim O} \) generated by \( A_0 \). Let \( p_2: W \times O \to O \) and \( q_2: T_i \times O \to O \) be the second projections. Then it follows from the nonresonance of \( c \in \mathbb{C}^n \) the restriction of \( \mathcal{L} \) to \( q_2^{-1}p_2(q) \simeq T_i \) is a non-constant local system. So we can apply our previous arguments and prove \( Rj_\ast(\mathcal{L})_q \simeq 0 \) in this case, too. This completes the proof.

By Lemma 4.2, if \( c \in \mathbb{C}^n \) is nonresonant the non-trivial morphism \( \Psi \) should be surjective. According to Schulze-Walther [42, Corollary 3.7] the morphism \( \Psi \) is also an isomorphism in this case. Let \( v : D(Y) \xrightarrow{\sim} D(X) \) be the inverse of the Fourier transform \( \wedge \). Then we have an isomorphism \( N^\vee_{A,c} \simeq M_{A,c} \) of left \( D(X) \)-modules. The corresponding coherent \( D_X \)-module \( N^\vee_{A,c} \simeq M_{A,c} \) can be constructed also in the following way. Let \( \sigma = \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C} \) be the canonical pairing defined by \( \langle z, \zeta \rangle = \sum_{j=1}^N z_j \zeta_j \) and \( p_1: X \times Y \to X \) (resp. \( p_2: X \times Y \to Y \)) the first (resp. second) projection. Then we have the following theorem due to Katz-Laumon [23].

**Theorem 4.3.** (Katz-Laumon [23]) For any \( c \in \mathbb{C}^n \) we have an isomorphism

\[
N^\vee_{A,c} \simeq \int_{p_1} \left\{ (p_2^\ast N_{A,c}) \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y} e^\sigma \right\}
\]

(4.19)
where \( \mathcal{O}_{X \times Y} e^\sigma \) is the integrable connection associated to \( e^\sigma : X \times Y \to \mathbb{C} \).

In the same way, for any \( c \in \mathbb{C}^n \) we have

\[
S^\vee_{A,c} \simeq \int_{p_1} \left\{ (p_2^\ast S_{A,c}) \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y} e^\sigma \right\}
\]

(4.20)
If moreover \( c \in \mathbb{C}^n \) is nonresonant, then by Lemma 4.2 we obtain surjective morphisms \( N_{A,c} \to S_{A,c}(Y) \) and \( M_{A,c}(X) \simeq N^\vee_{A,c} \to S^\vee_{A,c}(X) \). For nonresonant \( c \in \mathbb{C}^n \), we thus obtain a surjective morphism

\[
M_{A,c} \simeq N^\vee_{A,c} \to S^\vee_{A,c}
\]

(4.21)
of left $\mathcal{D}_X$-modules. Let $e^\tau : X \times T \longrightarrow \mathbb{C}$ be the function defined by $e^\tau(z, x) = \exp(\sum_{j=1}^N z_j x^{a(j)})$ and $q_1 : X \times T \longrightarrow X$ (resp. $q_2 : X \times T \longrightarrow T$) the first (resp. second) projection. Then by the base change theorem [22, Theorem 1.7.3 and Corollary 1.7.5], we have the isomorphism

$$S_{A,c}^\vee \simeq \int_{q_1} \{(q_2^* \mathcal{R}_c) \otimes \mathcal{O}_{X \times T} e^{\tau}\}.$$  \hfill (4.22)

Namely $S_{A,c}^\vee$ is the direct image of the integrable connection

$$\mathcal{K} = (q_2^* \mathcal{R}_c) \otimes \mathcal{O}_{X \times T} e^{\tau}$$

on $X \times T$ by $q_1$. Define a function $g : X \times T \longrightarrow \mathbb{C}$ by

$$g(z, x) = \exp(\sum_{j=1}^N z_j x^{a(j)})x_1^{c_1-1} \cdots x_n^{c_n-1}.$$  \hfill (4.24)

Then by the results of Hien-Roucairol [19] the holomorphic solution complex

$$\text{Sol}_X(S_{A,c}^\vee) = R \text{Hom}_{\mathcal{D}_{Xan}}((S_{A,c}^\vee)^{\text{an}}, \mathcal{O}_{Xan})$$

of $S_{A,c}^\vee$ is expressed by the rapid decay homology groups associated the function $g$. Indeed, for $z \in \Omega$ let $\mathcal{K}_z$ (resp. $g_z : T \longrightarrow \mathbb{C}$) be the restriction of the connection $\mathcal{K}$ (resp. the function $g$) to $U_z := q_1^{-1}(z) \simeq T \subset \Omega \times T$. Namely we set

$$g_z(x) = \exp(\sum_{j=1}^N z_j x^{a(j)})x_1^{c_1-1} \cdots x_n^{c_n-1}.$$  \hfill (4.26)

for $z \in U_z \simeq T$. Then $\mathcal{K}_z \simeq \mathcal{O}_{U_z} g_z$ and for the dual connection $\mathcal{K}_z^* \simeq \mathcal{O}_{U_z}(\frac{1}{g_z})$ of $\mathcal{K}_z$ we have

$$H^{-n} \text{DR}_T(\mathcal{K}_z^*) \simeq \mathbb{C}_{U_z} g_z.$$  \hfill (4.27)

Moreover for any $p \in \mathbb{Z}$, by Proposition 3.4 (see also the proof of Theorem 4.5 below) the rapid decay homology groups

$$H^p_{\text{rd}}(U_z; \mathcal{K}_z^*) \quad (z \in \Omega^{an})$$

associated to the integrable connections $\mathcal{K}_z^*$ (or to the functions $g_z : T \longrightarrow \mathbb{C}$) are isomorphic to each other and define a local system $\mathcal{H}^{\text{rd}}_p$ on $\Omega^{an}$. See [19] for the details. The following result is essentially due to Hien-Roucairol [19].

**Theorem 4.4.** (Hien-Roucairol [19]) For any $c \in \mathbb{C}^n$ and $p \in \mathbb{Z}$ we have an isomorphism

$$\mathcal{H}^{\text{rd}}_{n+p} \simeq H^p \text{Sol}_X(\int_{q_1} \mathcal{K}) \simeq H^p \text{Sol}_X(S_{A,c}^\vee)$$

(4.29)

of local systems on $\Omega^{an}$. 

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Recall that in [1, Section 3] Adolphson proved that $\mathcal{M}_{A,c}$ is an integrable connection on $\Omega$. From now on, we assume that $c \in \mathbb{C}^n$ is nonresonant. Then we have the surjective morphism $\mathcal{M}_{A,c} \twoheadrightarrow S^\vee_{A,c}$ and $S^\vee_{A,c}$ is also an integrable connection on $\Omega$. This in particular implies that for any $p \neq 0$ we have $H^p\text{Sol}_X(S^\vee_{A,c}) \cong 0$. Hence we get

$$H^r_{n+p}(U_z; \mathcal{K}_z^*) \cong 0 \quad (p \neq 0, \ z \in \Omega^{an}). \quad (4.30)$$

It follows also from the surjection $\mathcal{M}_{A,c} \twoheadrightarrow S^\vee_{A,c}$ that we have an injection

$$\Phi : \mathcal{H}^r_n \cong \text{Hom}_{D^{an}_X}( (S^\vee_{A,c})^{an}, \mathcal{O}_{X^{an}}) \hookrightarrow \text{Hom}_{D^{an}_X}( (\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}}). \quad (4.31)$$

By using the generator

$$u = [1] \in \mathcal{M}_{A,c} = D_X/ \left( \sum_{1 \leq i \leq n} D_XZ_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathbb{Z}^n} D_X\square_\mu \right) \quad (4.32)$$

denote by $\mathcal{H}^r_{an}(\mathcal{M}_{A,c})$ we regard $\text{Hom}_{D^{an}_X}( (\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}})$ as a subsheaf of $\mathcal{O}_{X^{an}}$. Then we have the following result.

**Theorem 4.5.** Assume that the parameter vector $c \in \mathbb{C}^n$ is nonresonant. Then the morphism $\Phi$ induces an isomorphism

$$\mathcal{H}^r_n \cong \text{Hom}_{D^{an}_X}( (\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}}) \quad (4.33)$$

of local systems on $\Omega^{an}$. Moreover this isomorphism is given by the integral

$$\gamma \mapsto \left\{ \Omega^{an} \ni z \mapsto \int_{\gamma^\ast} \exp(\sum_{j=1}^N z_j x^a(j) x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n) \right\}, \quad (4.34)$$

where for a continuous family $\gamma$ of rapid decay $n$-cycles in $\Omega^{an} \times T^{an}$ and $z \in \Omega^{an}$ we denote by $\gamma^\ast$ its restriction $\gamma \cap U_z$ to $U_z = q_1^{-1}(z) \cong T$.

Note that this integral representation of the confluent $A$-hypergeometric functions $\text{Hom}_{D^{an}_X}( (\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}})$ coincides with the one in Adolphson [1, Equation (2.6)].

**Proof.** Recall that the sheaf $\text{Hom}_{D^{an}_X}( (\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}})$ is a local system on $\Omega^{an}$. Moreover by [1, Corollary 5.20] its rank is $\text{Vol}_Z(\Delta)$. So it suffices to show that for any $z \in \Omega^{an}$ the dimension of the $n$-th rapid decay homology group $H^r_n(U_z; \mathcal{K}_z^*)$ is also $\text{Vol}_Z(\Delta)$. Let

$$\text{Eu}^r(U_z; \mathcal{K}_z^*) := \sum_{p \in \mathbb{Z}} (-1)^p \dim H^p_r(U_z; \mathcal{K}_z^*) \quad (4.35)$$

be the rapid decay Euler characteristic. Then by (4.30) we have only to prove the equality

$$\text{Eu}^r(U_z; \mathcal{K}_z^*) = (-1)^n \text{Vol}_Z(\Delta). \quad (4.36)$$

Let $\Sigma_0$ be the dual fan of $\Delta = \text{conv}(A \cup \{0\})$ in $\mathbb{R}^n$ and $\Sigma$ its smooth subdivision. Denote by $Z_\Sigma$ the smooth toric variety associated to the fan $\Sigma$. Then $Z_\Sigma$ is a smooth compactification of $U_z \cong T$ such that $Z_\Sigma \setminus U_z$ is a normal crossing divisor. However on $Z_\Sigma$
there still remain some points where the zero and the pole of the meromorphic extension of $h(z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ to it meet. We call them the points of indeterminacy of $h_z$. By using the non-degeneracy of the Laurent polynomial $h_z(x)$, as in [31, Section 3] we then construct a complex blow-up $Z := \tilde{Z}_\Sigma$ of $Z_\Sigma$ such that the meromorphic extension of $h_z$ to it has no point of indeterminacy. For the reader’s convenience, we briefly recall the construction of $Z$. Recall that $T$ acts on $Z_\Sigma$ and the $T$-orbits are parametrized by the cones in $\Sigma$. For a cone $\sigma \in \Sigma$ we denote by $T_\sigma \simeq (\mathbb{C}^*)^{n-\dim \sigma}$ the corresponding $T$-orbit. Let $\rho_1, \ldots, \rho_m \in \Sigma$ be the rays i.e. the one-dimensional cones in $\Sigma$. By using the primitive vectors $\kappa_i \in \rho_i \cap (\mathbb{Z}^n \setminus \{0\})$ on $\rho_i$ we set

$$m_i = -\min_{a \in \Delta} \langle \kappa_i, a \rangle \geq 0. \quad (4.37)$$

We renumber $\rho_1, \ldots, \rho_m$ so that $m_i > 0$ if and only if $1 \leq i \leq l$ for some $1 \leq l \leq m$. Then for any $1 \leq i \leq l$ the meromorphic extension of $h_z$ to $Z_\Sigma$ has a pole of order $m_i > 0$ along the toric divisor $D_i = \overline{T_{\rho_i}} \subset Z_\Sigma$. By the non-degeneracy of $h_z$ the hypersurface $\frac{1}{h_z}(0) \subset Z_\Sigma$ intersects $D_I = \cap_{i \in I} D_i$ transversally for any subset $I \subset \{1, 2, \ldots, m\}$ such that $I \cap \{1, 2, \ldots, l\} \neq \emptyset$ (see Definition 2.3). The meromorphic extension of $h_z$ has points of indeterminacy in $\bigcup_{i=1}^{l}(\frac{1}{h_z}(0) \cap D_i)$.

First we construct a tower of $m_1$ codimension-two blow-ups over $\frac{1}{h_z}(0) \cap D_1$ (see [31, Section 3] and [32, Section 3 and Lemma 4.9] for the details). Then the indeterminacy of $h_z$ over $D_1 \setminus (\cup_{j \neq 1} D_j)$ is eliminated. By repeating this construction also over (the proper transforms of) $D_2, D_3, \ldots, D_l$ we finally obtain the desired proper morphism $Z = \tilde{Z}_\Sigma \longrightarrow Z_\Sigma$ of $Z_\Sigma$ as the figure below.

![Figure 2.](image-url)
Now we can use this smooth compactification $Z$ of $U_z \simeq T$ and the normal crossing divisor $D := Z \setminus U_z$ in it to define the rapid decay homology groups associated to the function $g_z(x) = \exp(h_z(x)x_1^{c_1-1} \cdots x_n^{c_n-1}$ (see [19, Section 2.1]). In the figure of $Z$ above, the dotted curves stand for the irrelevant components (see the last half of Section 3) of $D$. Note that for any $1 \leq i \leq l$ the inverse image of $\overline{h_z^{-1}(0)} \cap D_i$ by the morphism $Z = \overline{Z}_\Sigma \to \Sigma$ is a union of $\mathbb{P}^1$-bundles on it and only the last one among them is irrelevant. Let $\pi : \tilde{Z} \to Z^\text{an}$ be the real oriented blow-up of $Z^\text{an}$ along $D^\text{an}$ and set $\tilde{D} = \pi^{-1}(D^\text{an})$. Then, as in Section 3 we define the rapid decay homology groups $H^\text{rd}_p(U_z; K^*_\Sigma)$ by using $\pi : \tilde{Z} \to Z^\text{an}$, $\tilde{D}$ and $g_z$ etc. By defining the set $Q \subset \tilde{D}$ of the rapid decay directions of $g_z(x)$ etc. as in Proposition 3.4, for the local system $\mathcal{L} = C_{T^\text{an},x_1^{c_1-1} \cdots x_n^{c_n-1}}$ on $(U_z)^\text{an} \simeq T^\text{an}$ and the inclusion map $\iota : (U_z)^\text{an} \simeq T^\text{an} \to \tilde{Z}$ we obtain isomorphisms

$$H^\text{rd}_p(U_z; K^*_\Sigma) \simeq H_p(T^\text{an} \cup Q, Q; \iota_*(\mathcal{L})) \quad (p \in \mathbb{Z}).$$

For $1 \leq i \leq l$ we define a face $\Gamma_i \prec \Delta$ of $\Delta$ by

$$\Gamma_i = \{ b \in \Delta \mid \langle \kappa_i, b \rangle = \min_{a \in \Delta} \langle \kappa_i, a \rangle \}.$$  

(4.39)

We call it the supporting face of $\rho_i$ in $\Delta$. Denote by $v_i \geq 0$ the normalized (or simplicial) $(n - 1)$-dimensional volume $\text{Vol}_\mathbb{Z}(\Gamma_i) \in \mathbb{Z}_+$ of $\Gamma_i$. Then the $\Gamma_i$-part $h^\Gamma_i$ of $h_z$ is naturally identified with the defining (Laurent) polynomial of the hypersurface $T_{\rho_i} \cap \overline{h_z^{-1}(0)}$ in $T_{\rho_i} \simeq (\mathbb{C}^*)^{n-1}$. Moreover by the Bernstein-Khovanskii-Kouchnirenko theorem (see [24]) its Euler characteristic is equal to $(-1)^n v_i$. Note that we have $\sum_{i=1}^l (v_i \times m_i) = \text{Vol}_\mathbb{Z}(\Delta)$. Indeed, for the convex hulls $\widehat{\Gamma}_i$ of $\Gamma_i \cup \{0\}$ in $\mathbb{R}^n$ we have $\bigcup_{i=1}^l \widehat{\Gamma}_i = \Delta$ and $\sum_{i=1}^l \text{Vol}_\mathbb{Z}(\widehat{\Gamma}_i) = \text{Vol}_\mathbb{Z}(\Delta)$. Since $\text{Vol}_\mathbb{Z}(\widehat{\Gamma}_i)$ (resp. $v_i = \text{Vol}_\mathbb{Z}(\widehat{\Gamma}_i)$) is $n!$ times (resp. $(n-1)!$ times) the Euclidean volume of $\widehat{\Gamma}_i$ (resp $\Gamma_i$) and $m_i > 0$ is the lattice height of $\widehat{\Gamma}_i$ from its base $\Gamma_i \prec \widehat{\Gamma}_i$, we have also $\text{Vol}_\mathbb{Z}(\widehat{\Gamma}_i) = \text{Vol}_\mathbb{Z}(\Gamma_i) \times m_i = v_i \times m_i$ for any $1 \leq i \leq l$.

Let $Z = \bigcup_{\alpha} Z_\alpha$ be the canonical stratification of $Z = \overline{Z}_\Sigma$ associated to the normal crossing divisor $D = Z \setminus T$ and $E \subset Z$ the union of the exceptional divisors of the blow-up $Z = \overline{Z}_\Sigma \to \Sigma$. Then for any $1 \leq i \leq l$ there exists a unique stratum $Z_{\alpha_i}$ such that
For each stratum \( Z_a \) in the stratification we take its sufficiently small tubular neighborhood \( V_a \) in \( Z \). For \( 1 \leq i \leq l \) we denote the alternating sum

\[
\sum_{p \in \mathcal{Z}} (-1)^p H_p((V_a, \cap T^\text{an}) \cup (\pi^{-1}(V_a) \cap Q), (\pi^{-1}(V_a) \cap Q); \iota_*(\mathcal{L}))
\]

(4.40)
simply by \( \text{Eu}^\text{rd}_i \). Then by applying Lemmas 3.5 and 3.6 to the Mayer-Vietoris exact sequences for the relative twisted homology groups

\[
H_p((V_a \cap T^\text{an}) \cup (\pi^{-1}(V_a) \cap Q), (\pi^{-1}(V_a) \cap Q); \iota_*(\mathcal{L}))
\]

(4.41)
and the geometric situation in Figure 3 above, we can easily show that

\[
\text{Eu}^\text{rd}(U; \mathcal{K}_x^+) = \sum_{p \in \mathcal{Z}} (-1)^p \dim H_p(T^\text{an} \cup Q; \iota_*(\mathcal{L})) = \sum_{i=1}^l \text{Eu}^\text{rd}_i.
\]

(4.42)

Moreover by the proof of Lemma 3.5 and and Lemma 3.6, for any \( 1 \leq i \leq l \) we have \( \text{Eu}^\text{rd}_i = (-1)^n v_i \times m_i \). Then the equality (4.36) follows from \( \sum_{i=1}^l (v_i \times m_i) = \text{Vol}_\Sigma(\Delta) \). This completes the proof of the isomorphism (4.33).

Let us prove the remaining assertion. Denote the distinguished section \( (q_2^*w_0) \otimes e^\tau \) of the integrable connection \( \mathcal{K} = (q_2^*\mathcal{R}_c) \otimes \mathcal{O}_{X \times T} \mathcal{O}_{X \times T}e^\tau \) by \( t \). Let \( \Omega_{X \times T/X}^* \otimes \mathcal{O}_{X \times T} \mathcal{K} \) be the relative algebraic de Rham complex of \( \mathcal{K} \) associated to the morphism \( q_1 : X \times T \rightarrow X \). Then we have an isomorphism

\[
S_{A,c}^{i'} \simeq \int_{q_1} \mathcal{K} \simeq H^n \{ (q_1)_*(\Omega_{X \times T/X}^* \otimes \mathcal{O}_{X \times T} \mathcal{K}) \}.
\]

(4.43)
For a relative \( n \)-form \( \omega \in (q_1)_*(\Omega_{X \times T/X}^* \otimes \mathcal{O}_{X \times T} \mathcal{K}) \) denote by \( \text{cl}(\omega \otimes t) \) the section of \( S_{A,c}^{i'} \) which corresponds to the cohomology class \( [(q_1)_*(\omega \otimes t)] \in H^n \{ (q_1)_*(\Omega_{X \times T/X}^* \otimes \mathcal{O}_{X \times T} \mathcal{K}) \} \) by the above isomorphism. According to the result of [19], by the isomorphism

\[
\mathcal{H}_n^\text{rd} \simeq \text{Hom}_{\mathcal{D}_\mathcal{X}_n}(S_{A,c}^{i'}, \mathcal{O}_{X \times T}^\text{an})
\]

(4.44)
of local systems on \( \Omega_{X \times T}^\text{an} \), a family of rapid decay cycles \( \gamma \in \mathcal{H}_n^\text{rd} \) is sent to the section

\[
[(S_{A,c}^{i'})^\text{an} \ni f \otimes \text{cl}(\omega \otimes t) \mapsto \left\{ \Omega_{X \times T}^\text{an} \ni \gamma \mapsto f(z) \int_{\gamma^z} \exp \left( \sum_{j=1}^N z_j e^{a(j)} x_1^{c_j-1} \cdots x_n^{c_n-1} \omega \right) \right\}]
\]

(4.45)
(\( f \in \mathcal{O}_{X \times T} \)) of \( \text{Hom}_{\mathcal{D}_\mathcal{X}_n}(S_{A,c}^{i'}, \mathcal{O}_{X \times T}^\text{an}) \). Then the remaining assertion follows from the lemma below. This completes the proof.

**Remark 4.6.** When \( 0 \in \text{Int}(\Delta) \) the irrelevant components of \( D \) in the proof above are the last \( \mathbb{P}^1 \)-bundles on \( \bar{h}_z^{-1}(0) \cap D_i \) (\( 1 \leq i \leq l = m \)). By the construction of the morphism \( Z = \bar{Z}_\Sigma \rightarrow Z_\Sigma \) we can easily see that for any \( t \in \mathbb{C} \) the hypersurface \( \bar{h}_z^{-1}(t) \subset Z \) intersects them transversally.

**Lemma 4.7.** By the morphism

\[
\mathcal{M}_{A,c} \rightarrow S_{A,c}^{i'} \simeq H^n \{ (q_1)_*(\Omega_{X \times T/X}^* \otimes \mathcal{O}_{X \times T} \mathcal{K}) \}
\]

(4.46)
the canonical section \( u = [1] \in \mathcal{M}_{A,c} \) is sent to the cohomology class \( \text{cl}((dx_1 \wedge \cdots \wedge dx_n) \otimes t) \).
Proof. First note that the morphism $\Psi^*(X) : M_{A,c}(X) \simeq M_{A,c} \simeq N^1_{A,c} \longrightarrow S^\vee_{A,c}(Y) \simeq S_{A,c}(Y)$ sends the canonical generator $u = [1] \in M_{A,c}(X)$ to $w = j_*(1_{Y \times T} \otimes w_0) \in S_{A,c}(Y)$. On the other hand, by (4.20) we have an isomorphism

$$S^\vee_{A,c} \simeq H^N \left( (p_1)_* \left\{ \Omega^\bullet_{X/Y!/X} \otimes_{O_{X/Y}} \left( p_2^* S_{A,c} \right) \otimes_{O_{X/Y}} O_{X/Y} e^o \right\} \right).$$

(4.47)

Then by Malgrange’s simple proof [28, page 135] of Theorem 4.3, via this isomorphism the section $w \in S^\vee_{A,c}(X) \simeq S_{A,c}(Y)$ corresponds to the cohomology class

$$[(p_1)_* \{(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes (p_2^* w) \otimes e^o\}].$$

(4.48)

Let $\tilde{j} : X \times T \hookrightarrow X \times Y$ be the embedding induced by $j$. By the isomorphism

$$S^\vee_{A,c} \simeq H^N \left( (p_1)_* \left\{ \Omega^\bullet_{X/Y!/X} \otimes_{O_{X/Y}} \tilde{j}_*(D_{X/Y!} \times X \times T \otimes_X T \mathcal{K}) \right\} \right),$$

(4.49)

the above cohomology class corresponds to the one

$$\rho := [(p_1)_* \{(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes \tilde{j}_*(1_{X \times Y} \times X \times T \otimes t)\}],$$

(4.50)

where the section $1_{X \times Y} \times X \times T \in D_{X \times Y} \times X \times T$ is defined similarly to $1_{Y \times T} \in D_{Y \times T}$. Then it suffices to show that via the isomorphism

$$S^\vee_{A,c} \simeq \int_{p_1} \int_j \mathcal{K} \simeq \int_{q_1} \mathcal{K}$$

(4.51)

the cohomology class $\rho$ is sent to $\operatorname{cl}((dx_1 \wedge \cdots \wedge dx_n) \otimes t) = [(q_1)_*(\{d\zeta_1 \wedge \cdots \wedge d\zeta_N\}) \otimes \mathcal{K}])$ in

$$\int_{q_1} \mathcal{K} \simeq H^n \{ (q_1)_* \left( \Omega^\bullet_{X/Y!/X} \otimes_{O_{X/Y}} \mathcal{K} \right) \}.$$

(4.52)

Since $X$ and $X \times Y$ are affine, we have only to prove that via the isomorphism

$$H^n \Gamma(X \times Y; \Omega^\bullet_{X/Y!/X} \otimes_{O_{X/Y}} \int_j \mathcal{K}) \simeq H^n \Gamma(X \times T; \Omega^\bullet_{X/Y!/X} \otimes_{O_{X/Y}} \mathcal{K})$$

(4.53)

the cohomology class

$$[(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes \tilde{j}_*(1_{X \times Y} \times X \times T \otimes t)]$$

is sent to $[(dx_1 \wedge \cdots \wedge dx_n) \otimes t]$. Indeed, we have isomorphisms

$$H^0 \Gamma(X \times Y; \Omega^N_{X/Y!/X} \otimes_{O_{X/Y}} \int_j \mathcal{K}) \simeq H^0 \Gamma(X \times Y; p_2^{-1} \Omega^N_{X/Y!} \otimes_{O_{X/Y}} q_2^{-1} \mathcal{D}_{Y \times T} \otimes_{q_2^{-1} \mathcal{D}_{Y \times T}} \mathcal{K}) \simeq H^0 \Gamma(X \times Y; \tilde{j}_*(j^{-1} p_2^{-1} \Omega^N_{X/Y!} \otimes_{j^{-1} p_2^{-1} O_{X/Y}} q_2^{-1} \mathcal{D}_{Y \times T} \otimes_{q_2^{-1} \mathcal{D}_{Y \times T}} \mathcal{K})).$$

(4.55)

(4.56)

(4.57)

(4.58)

by which the element $[(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes \tilde{j}_*(1_{X \times Y} \times X \times T \otimes t)]$ is sent to the one $[q_2^{-1} \{j^{-1} (d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes 1_{Y \times T} \otimes t\}]$. Let $P^\bullet \to \mathcal{K}$ be a free resolution of the left $D_{X \times T}$-module $\mathcal{K}$. Since $X \times T$ is affine, we obtain a surjective homomorphism

$$\Gamma(X \times T; P^0) \longrightarrow \Gamma(X \times T; \mathcal{K})$$

(4.59)
and can take a lift \( \hat{t} \in \Gamma(X \times T; \mathcal{D}) \) of \( t \in \Gamma(X \times T; K) \). Moreover by the flatness of the right \( \mathcal{D}_T \)-module \( \mathcal{D}_{Y \leftarrow T} \) and the well-known formula

\[
j^{-1}\Omega_Y^{N+} \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \leftarrow T} \cong j^{-1}\Omega_Y^N \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \leftarrow T} \cong \Omega^N_T
\]

(4.60)

there exists an isomorphism

\[
H^0 \Gamma(X \times T; q_2^{-1}(j^{-1}\Omega_Y^{N+} \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \leftarrow T}) \otimes q_2^{-1} \mathcal{D}_T \mathcal{K}) \\
\cong H^0 \Gamma(X \times T; q_2^{-1}(j^{-1}\Omega_Y^{N+} \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \leftarrow T}) \otimes q_2^{-1} \mathcal{D}_T \mathcal{P}^*) \\
\cong H^0 \Gamma(X \times T; q_2^{-1}\Omega^N_T \otimes q_2^{-1} \mathcal{P}^*)
\]

(4.61)

(4.62)

(4.63)

by which \([q_2^{-1}(d\zeta_1 \land \cdots \land d\zeta_N) \otimes 1_{Y \leftarrow T}] \otimes \hat{t}] \) is sent to \([q_2^{-1}(dx_1 \land \cdots \land dx_n) \otimes \hat{t}] \). Similarly, by the isomorphism

\[
H^0 \Gamma(X \times T; q_2^{-1}\Omega^N_T \otimes q_2^{-1} \mathcal{P}^*) \cong H^0 \Gamma(X \times T; \Omega_X^{N+} \otimes \mathcal{O}_{X \times T} \mathcal{K})
\]

(4.64)

the element \([q_2^{-1}(dx_1 \land \cdots \land dx_n) \otimes \hat{t}] \) is sent to \([(dx_1 \land \cdots \land dx_n) \otimes \hat{t}] \). This completes the proof.

As a corollary of Theorem 4.5, we recover the following Saito and Schulze-Walther’s construction of Adolphson’s confluent \( A \)-hypergeometric \( \mathcal{D} \)-module \( \mathcal{M}_{A,c} \) on \( \Omega \subset X = \mathbb{C}^A \).

**Corollary 4.8.** (Saito [39] and Schulze-Walther [42, 43]) Assume that the parameter vector \( c \in \mathbb{C}^n \) is nonresonant. Then we have an isomorphism \( \mathcal{M}_{A,c} \cong \mathcal{S}_{A,c}' \) of integrable connections on \( \Omega \). In particular, \( \mathcal{M}_{A,c} \) is an irreducible connection there.

This result was first obtained in Saito [39] and Schulze-Walther [42, 43] by using totally different methods. In fact, Saito [39] proved moreover that we have an isomorphism \( \mathcal{M}_{A,c} \cong \mathcal{S}_{A,c}' \) on the whole \( X \).

**Remark 4.9.** Since \( \mathcal{S}_{A,c} \) is regular holonomic by a theorem of Hotta [21], it is also regular at infinity in the sense of Daia [7]. Then by using the Fourier-Sato transforms (see [28]) we can apply the main theorem of Daia [7] to get another sheaf-theoretical (or functorial) construction of the sheaf \( \mathcal{H}^n_{\text{rd}} \cong \text{Hom}_{\mathcal{D}_{X^\infty}}((\mathcal{S}_{A,c}' \cap \mathcal{O}_{X^\infty})^\text{an}, \mathcal{O}_{X^\infty}) \). This construction is valid even when the parameter \( c \in \mathbb{C}^n \) is resonant. However if \( c \in \mathbb{C}^n \) is resonant, we cannot prove that the morphism (4.31) is an isomorphism. Namely for such \( c \in \mathbb{C}^n \), the sheaf \( \mathcal{H}^n_{\text{rd}} \cong \text{Hom}_{\mathcal{D}_{X^\infty}}((\mathcal{S}_{A,c}' \cap \mathcal{O}_{X^\infty})^\text{an}, \mathcal{O}_{X^\infty}) \) may be different from the one \( \text{Hom}_{\mathcal{D}_{X^\infty}}((\mathcal{M}_{A,c} \cap \mathcal{O}_{X^\infty})^\text{an}, \mathcal{O}_{X^\infty}) \) of confluent \( A \)-hypergeometric functions.

**Example 4.10.** Assume that \( n = 1 \) and \( T = \mathbb{C}^* \).

(i) If \( A = \{1, -1\} \subset \mathbb{Z} \) our integral representation of the \( A \)-hypergeometric functions \( u(z_1, z_2) \) on \( \mathbb{C}_x^2 \) is

\[
u(z_1, z_2) = \int_{\gamma_z} \exp(z_1 x + \frac{z_2}{x}) x^{c-1} \, dx.
\]

(4.65)

Restricting the function \( u(z_1, z_2) \) to \( \mathbb{C}_t \) by the inclusion map \( \mathbb{C}_t \rightarrow \mathbb{C}_x^2, t \rightarrow (\frac{t}{2}, -\frac{t}{2}) \) we obtain the classical Bessel function

\[
v(t) = \frac{1}{2\pi i} \int_{\gamma(\frac{1}{2}, -\frac{1}{2})} \exp\left(\frac{tx}{2}\right) x^{-\nu - 1} \, dx
\]

(4.66)
for the parameter $\nu = -c$. Here $\gamma(\frac{1}{2}, -\frac{1}{2}) \subset \mathbb{C}$ is the path which comes from infinity along the line $\arg x = -\pi$, turns around the origin and goes back to infinity along $\arg x = \pi$.

(ii) If $A = \{3, 1\} \subset \mathbb{Z}$ our integral representation of the $A$-hypergeometric functions $u(z_1, z_2)$ on $\mathbb{C}_z^2$ is

$$u(z_1, z_2) = \int_{\gamma} \exp(z_1 x^3 + z_2 x) x^{c-1} dx.$$  \hfill (4.67)

Restricting the function $u(z_1, z_2)$ to $\mathbb{C}_z$ by the inclusion map $\mathbb{C}_z \hookrightarrow \mathbb{C}_z^2$, $t \mapsto (\frac{1}{3}, -t)$ we obtain the classical Airy function

$$v(t) = \frac{1}{2\pi i} \int_{\gamma(\frac{1}{3}, -t)} \exp(x^3 - tx) dx$$  \hfill (4.68)

for $c = 1$. Here $\gamma(\frac{1}{3}, -t) \subset \mathbb{C}$ is the path which comes from infinity along the line $\arg x = -\frac{\pi}{3}$ and goes back to infinity along $\arg x = \frac{\pi}{3}$.

5 Asymptotic expansions at infinity of confluent $A$-hypergeometric functions

In this section, assuming the condition $0 \in \text{Int}(\Delta)$ we construct natural bases of the rapid decay homology groups $(H_{\text{rd}}^n)_z \simeq H_{\text{rd}}^n(U_z; \mathcal{K}_z^n)$ and apply them to prove a formula for the asymptotic expansions at infinity of Adolphson’s confluent $A$-hypergeometric functions.

5.1 Preliminary results

For the construction of the bases of the rapid decay homology groups, we first prove some preliminary results.

**Definition 5.1.** We define a subset $\Omega_0$ of $\Omega \subset X = \mathbb{C}_z^N$ by: $z \in \Omega_0 \iff z \in \Omega$ and the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ has only non-degenerate (Morse) critical points in $T = (\mathbb{C}^*)^n$.

It is clear that $\Omega_0 \subset X = \mathbb{C}_z^N$ is stable by the multiplication of $\mathbb{C}^*$ (i.e. homothety) on $X = \mathbb{C}_z^N$. Let $b(1), b(2), \ldots, b(n) \in A$ be elements of $A$ such that $\{b(1), b(2), \ldots, b(n)\}$ is a basis of the vector space $\mathbb{R}^n$. By our assumption that $A$ generates $\mathbb{Z}^n$, we can take such elements of $A$.

**Proposition 5.2.** Let $h(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ be a Laurent polynomial with support in $A \subset \mathbb{Z}^n$ on $T = (\mathbb{C}^*)^n$. Assume that $h$ is non-degenerate i.e. $z = (z_1, z_2, \ldots, z_N) \in \Omega$. Then for generic $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ the perturbation

$$\tilde{h}(x) = h(x) - \sum_{i=1}^{n} \alpha_i x^{b(i)}$$  \hfill (5.1)

of $h$ is non-degenerate and has only non-degenerate (Morse) critical points in $T$. 

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Proof. It is clear that \( \tilde{h} \) is non-degenerate for generic \( \alpha \in \mathbb{C}^n \) (see for example [32, Lemma 5.2]). Let \( l_1, l_2, \ldots, l_n \in (\mathbb{R}^n)^* \) be the dual basis of \( b(1), b(2), \ldots, b(n) \) and set

\[
g_i(x) = \sum_{j=1}^{N} l_i(a(j)) z_j x^{a(j) - b(i)} \quad (i = 1, 2, \ldots, n). \tag{5.2}
\]

Note that for \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) we have

\[
a_j = \sum_{i=1}^{n} l_i(a(b(i)) \quad (j = 1, 2, \ldots, n). \tag{5.3}
\]

Then we can easily prove the equality

\[
(x_1 \frac{\partial \tilde{h}}{\partial x_1}, \ldots, x_n \frac{\partial \tilde{h}}{\partial x_n}) = (x_1^{b(1)}(g_1 - \alpha_1), \ldots, x_1^{b(n)}(g_n - \alpha_n)) \cdot B,
\]

where \( B \in GL_n(\mathbb{C}) \) is an invertible matrix defined by \( B = (b_{ij})_{i,j=1}^{n} = (b(i))_{i,j=1}^{n} \). Hence we obtain

\[
\{x \in T \mid \frac{\partial \tilde{h}}{\partial x_1}(x) = \cdots = \frac{\partial \tilde{h}}{\partial x_n}(x) = 0\} = \{x \in T \mid g_i(x) = \alpha_i \ (1 \leq i \leq n)\}. \tag{5.5}
\]

Moreover degenerate critical points of \( \tilde{h} \) in \( T \) correspond to critical points \( x \in T \) of the map \( (g_1, g_2, \ldots, g_n) : T \rightarrow \mathbb{C}^n \) such that \( g_i(x) = \alpha_i \ (1 \leq i \leq n) \). By the Bertini-Sard theorem, generic \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n \) are not such critical values. This implies that for generic \( \alpha \in \mathbb{C}^n \) the Laurent polynomial \( \tilde{h} \) has no degenerate critical point. This completes the proof. \( \square \)

**Corollary 5.3.** The subset \( \Omega_0 \) of \( \Omega \) is open dense in \( X = \mathbb{C}^n \) and stable by the multiplication of \( \mathbb{C}^* \) (i.e. homothety) on \( X = \mathbb{C}^n \).

**Proposition 5.4.** Assume that \( 0 \in \text{Int}(\Delta) \). Then for any \( z \in \Omega_0 \) the Laurent polynomial \( h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)} \) has exactly \( \text{Vol}_2(\Delta) \) non-degenerate (Morse) critical points in \( T \).

**Proof.** Let us fix \( z \in \Omega_0 \) and set \( h(x) = h_z(x) \). By an invertible matrix \( C \in GL_n(\mathbb{C}) \) we define new Laurent polynomials \( h_1, h_2, \ldots, h_n \) on \( T \) by

\[
(h_1, \ldots, h_n) = (x_1 \frac{\partial h}{\partial x_1}, \ldots, x_n \frac{\partial h}{\partial x_n}) \cdot C. \tag{5.6}
\]

By our assumption \( 0 \in \text{Int}(\Delta) \), taking sufficiently generic \( C \) we may assume that all the Newton polytopes of \( h_1, h_2, \ldots, h_n \) are equal to \( \Delta \). Then for any face \( \Gamma \prec \Delta \) of \( \Delta \) the set

\[
\{x \in T \mid h_1^\Gamma(x) = \cdots = h_n^\Gamma(x) = 0\} \tag{5.7}
\]

coincides with that of the critical points of \( h^\Gamma \) in \( T \). In this correspondence for the special case \( \Gamma = \Delta \), multiple roots of the equation \( h_1(x) = \cdots = h_n(x) = 0 \) in \( T \) correspond to degenerate critical points of \( h : T \rightarrow \mathbb{C} \). But by our assumption \( z \in \Omega_0 \) there is no such point in \( T \). Moreover by the non-degeneracy of \( h \) (\( \iff \) \( z \in \Omega \)), for any face \( \Gamma \prec \Delta \) of \( \Delta \) such that \( 0 \notin \Gamma \) (i.e. \( \Gamma \neq \Delta \) when \( 0 \in \text{Int}(\Delta) \)) we have

\[
\{x \in T \mid h_1^\Gamma(x) = \cdots = h_n^\Gamma(x) = 0\} = \emptyset. \tag{5.8}
\]

This means that the (0-dimensional) subvariety \( \{x \in T \mid h_1(x) = \cdots = h_n(x) = 0\} \) of \( T \) is a non-degenerate complete intersection (for the definition, see [30, Definition 2.7] and [36]). Then by Bernstein’s theorem its cardinality is equal to \( \text{Vol}_2(\Delta) \). \( \square \)
5.2 A basis of the rapid decay homology group

From now on, assuming the condition \( 0 \in \text{Int}(\Delta) \), for any \( z \in \Omega_0 \) we construct a natural basis of the rapid decay homology group \( (H^\text{rd}_n)_{\zeta} \cong H^\text{rd}_n(U_z;\mathcal{K}^\ast_\zeta) \) by using the (relative) twisted Morse theory for the function \( \text{Re}(h_z) : T^\text{an} \rightarrow \mathbb{R} \). For the twisted Morse theory and its applications to period integrals, we refer to Aomoto-Kita [3], Pajitnov [37] and Pham [38]. Our construction of the basis is similar to the ones of Dubrovin [8] and Tanabe-Ueda [45] in the untwisted case. Note that by our assumption \( 0 \in \text{Int}(\Delta) \) any parameter vector \( c \in \mathbb{C}^n \) is nonresonant. This implies that Theorem 4.5 holds for any \( c \in \mathbb{C}^n \). For \( z \in \Omega_0 \) let \( \alpha(i) \in T \) (\( 1 \leq i \leq \text{Vol}_2(\Delta) \)) be the non-degenerate (Morse) critical points of the Laurent polynomial \( h_z(x) = \sum_{j=1}^N z_j x^{a(j)} \) in Proposition 5.4. By the Cauchy-Riemann equation, they are also non-degenerate (Morse) critical points of the real-valued function \( \text{Re}(h_z) : T^\text{an} \rightarrow \mathbb{R} \). We can observe this fact more explicitly by taking a holomorphic Morse coordinate around each \( \alpha(i) \in T \) as follows. For a fixed \( 1 \leq i \leq \text{Vol}_2(\Delta) \) let \( y = (y_1, \ldots, y_n) \) where \( y_j = \xi_j + \sqrt{-1} \eta_j \) (\( 1 \leq j \leq n \)) be a holomorphic Morse coordinate for \( h_z \) around its critical point \( \alpha(i) \in T \) such that \( h_z(x) = h_z(\alpha(i)) + y_1^2 + y_2^2 + \cdots + y_n^2 \) in a neighborhood of \( \alpha(i) \in T \). Since we have

\[
\text{Re}(h_z)(x) = \text{Re}(h_z)(\alpha(i)) + \left( \xi_1^2 + \cdots + \xi_n^2 \right) - \left( \eta_1^2 + \cdots + \eta_n^2 \right),
\]

we regard the smooth submanifold \( \{ \xi_1 = \cdots = \xi_n = 0 \} \) in it as the stable manifold of the gradient flow of the Morse function \( \text{Re}(h_z) : T^\text{an} \rightarrow \mathbb{R} \) in a neighborhood of \( \alpha(i) \in T^\text{an} \) and denote it by \( S_i \). By shrinking \( S_i \) if necessary, we may assume that \( S_i \) is homeomorphic to the \( n \)-dimensional disk. For \( 1 \leq i \leq \text{Vol}_2(\Delta) \) let \( R_i \subset \mathbb{C} \) be the ray in \( \mathbb{C} \) defined by

\[
R_i = \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \leq \text{Re}(h_z)(\alpha(i)), \quad \text{Im} \lambda = \text{Im}(h_z)(\alpha(i)) \}. \tag{5.10}
\]

Namely \( R_i \) emanates from the critical value \( h_z(\alpha(i)) \in \mathbb{C} \) of \( h_z \) and goes to the left in the complex plane \( \mathbb{C} \) so that we have \( \text{Re} \lambda \rightarrow -\infty \) along it. By shrinking the stable manifold \( S_i \) if necessary, we may assume also that the image of \( \overline{S_i} \subset T^\text{an} \) by the map \( h_z : T^\text{an} \rightarrow \mathbb{C} \) is the closed interval

\[
R_i^\varepsilon = \{ \lambda \in R_i \mid \text{Re}(h_z)(\alpha(i)) - \varepsilon \leq \text{Re} \lambda \leq \text{Re}(h_z)(\alpha(i)) \} \tag{5.11}
\]

in \( R_i \) for some \( \varepsilon > 0 \) and \( h_z(\partial S_i) \) is just the one point \( \{ h_z(\alpha(i)) - \varepsilon \} \) in \( R_i \). We drag \( \partial S_i \simeq S^{n-1} \) over the complement of \( R_i^\varepsilon \) in \( R_i \) to construct a tube \( M_i \simeq (-\infty,0] \times S^{n-1} \) in \( T^\text{an} \). Finally we set \( \gamma_i := S_i \cup M_i \subset T^\text{an} \). From now we shall use the notations in the proof of Theorem 4.5. Then by Proposition 3.4, for \( U_z = T \) and the local system

\[
\mathcal{L} = C_{T^\text{an}}x_1^{c_1-1} \cdots x_n^{c_n-1}. \tag{5.12}
\]

there exists an isomorphism

\[
H^\text{rd}_n(U_z;K^\ast_\zeta) \cong H_n(T^\text{an} \cup Q, Q; \iota_* \mathcal{L}). \tag{5.13}
\]

Since \( \gamma_i \subset T^\text{an} \) is a singular \( n \)-chain in \( T^\text{an} \) whose boundary in the real oriented blow-up \( \hat{Z} \) is contained in \( Q \subset \hat{D} \), we obtain an element \([\gamma_i]\) of the relative twisted homology group \( H_n(T^\text{an} \cup Q, Q; \iota_* \mathcal{L}) \). Namely \([\gamma_i]\) \in \( H_n(T^\text{an} \cup Q, Q; \iota_* \mathcal{L}) \) thus obtained is a rapid decay \( n \)-cycle in \( T^\text{an} \) for the function

\[
g_z(x) = \exp(h_z(x))x_1^{c_1-1} \cdots x_n^{c_n-1}. \tag{5.14}
\]
satisfying the conditions

\[(i) : \quad S_i \subset \gamma_i, \quad (5.15)\]
\[(ii) : \quad \gamma_i \setminus \bar{\gamma}_i \subset \{ x \in T^\an \mid \text{Re}(h_z)(x) < \text{Re}(h_z)(\alpha(i)) - \varepsilon \} \quad \text{for some } \varepsilon > 0. \quad (5.16)\]

**Theorem 5.5.** In the situation as above (i.e. \( 0 \in \text{Int}(\Delta) \) and \( z \in \Omega_0 \)), the elements \([\gamma_1, \gamma_2, \ldots, [\gamma_{\text{Vol}(\Delta)}]] \in (H^d_{n})_z \simeq H^d_{n}(U_z; \mathcal{K}^*_\mathcal{L}) \simeq H_n(T^\an \cup Q; Q; t_\ast \mathcal{L}) \) form a basis of the rapid decay homology group \( H^d_{n}(U_z; \mathcal{K}^*_\mathcal{L}) \simeq H_n(T^\an \cup Q; Q; t_\ast \mathcal{L}) \).

**Proof.** First note that by (4.30) we have

\[ \dim H_n(T^\an \cup Q; Q; t_\ast \mathcal{L}) = \text{Vol}_2(\Delta) = \sharp \{ \alpha(i) \}. \quad (5.17) \]

For \( t \in \mathbb{R} \) we define an open subset \( T^\an_t \subset T^\an \) of \( T^\an \) by

\[ T^\an_t = \{ x \in T^\an \mid \text{Re}(h_z)(x) < t \}. \quad (5.18) \]

Then by Remark 4.6 for any \( t \in \mathbb{R} \) the closure of \( \partial T^\an_t \subset T^\an \) in \( Z \) intersects each irrelevant divisor \( D_i \subset Z \) transversally. This implies that for any \( p \in \mathbb{Z} \) and \( t \ll 0 \) we have

\[ H_p(T^\an_t \cup Q; Q; t_\ast \mathcal{L}) \simeq 0. \quad (5.19) \]

Moreover for any \( p \in \mathbb{Z} \) and \( t \gg 0 \) we have an isomorphism

\[ H_p(T^\an_t \cup Q; Q; t_\ast \mathcal{L}) \simeq H_p(T^\an \cup Q; Q; t_\ast \mathcal{L}). \quad (5.20) \]

Now let \(-\infty < t_1 < t_2 < \cdots < t_r < +\infty \) be the critical values of \( \text{Re}(h_z) : T^\an \rightarrow \mathbb{R} \). Then by Remark 4.6 for any \( p \in \mathbb{Z} \) and \( s_1, s_2 \in \mathbb{R} \) such that \( s_1 < s_2, [s_1, s_2] \cap \{ t_1, t_2, \ldots, t_r \} = \emptyset \) we have a natural isomorphism

\[ H_p(T^\an_{s_1} \cup Q; Q; t_\ast \mathcal{L}) \simeq H_p(T^\an_{s_2} \cup Q; Q; t_\ast \mathcal{L}). \quad (5.21) \]

For \( 1 \leq j \leq r \) let \( \alpha(i_1), \alpha(i_2), \ldots, \alpha(i_{t_j}) \in T^\an \) be the critical points of \( \text{Re}(h_z) : T^\an \rightarrow \mathbb{R} \) such that \( \text{Re}(h_z)(\alpha(i_j)) = t_j \). Then, for sufficiently small \( 0 < \varepsilon \ll 1 \) we obtain a short exact sequence

\[ 0 \rightarrow H_n(T^\an_{t_j - \varepsilon} \cup Q; Q; t_\ast \mathcal{L}) \rightarrow H_n(T^\an_{t_j - \varepsilon} \cup (\cup_{q=1}^{n_j} S_{t_q}) \cup Q; Q; t_\ast \mathcal{L}) \rightarrow \oplus_{q=1}^{n_j} H_n(\overline{S_{t_q}}; \partial S_{t_q}; t_\ast \mathcal{L}) \rightarrow 0 \quad (5.22) \]

by induction on \( j \) with the help of (5.19) and the fact \( H_p(\overline{S_{t_q}}; \partial S_{t_q}; t_\ast \mathcal{L}) \simeq 0 \) \( (p \neq n) \). Moreover there exist \( [S_{t_q}] \in H_n(\overline{S_{t_q}}; \partial S_{t_q}; t_\ast \mathcal{L}) \simeq \mathbb{C} \) which can be lifted to the elements \([\gamma_{t_q}] \) of \( H_n(T^\an_{t_j - \varepsilon} \cup (\cup_{q=1}^{n_j} S_{t_q}) \cup Q; Q; t_\ast \mathcal{L}) \subset H_n(T^\an \cup Q; Q; t_\ast \mathcal{L}) \). This implies that \([\gamma_1], [\gamma_2], \ldots, [\gamma_{\text{Vol}(\Delta)}] \in H_n(T^\an \cup Q; Q; t_\ast \mathcal{L}) \) form a basis of \( H_n(T^\an \cup Q; Q; t_\ast \mathcal{L}) \). \( \square \)

Note that for a connected open neighborhood \( V \) of the point \( z \) in \( \Omega_0^\an \) the basis \([\gamma_1], \ldots, [\gamma_{\text{Vol}(\Delta)}] \in (H^d_{n})_z \) constructed in Theorem 5.5 can be naturally extended to a family of the bases \([\gamma^w], \ldots, [\gamma_{\text{Vol}(\Delta)}^w] \in (H^d_{n})_w \) \( (w \in V) \) i.e. a basis of the local system \( H^d_{n} \) on \( V \). We can extend it so that \( V \subset (\Omega_0^\an) \) is stable by the multiplication of the
group $\mathbb{R}_{>0}$ on $X = \mathbb{C}^N$ and the rapid decay $n$-cycles $\gamma_1^w, \ldots, \gamma_{\mathbb{Vol}(\Delta)}^w$ ($w \in V$) satisfy the conditions

$$(i): \quad S_i^w \subset \gamma_i^w, \quad (5.24)$$

$$(ii): \quad \gamma_i^w \setminus S_i^w \subset \{ x \in T^an \mid \text{Re}(h_w)(x) < \text{Re}(h_w)(\alpha(i)^w) - \varepsilon \} \quad \text{for some } \varepsilon > 0, (5.25)$$

where $S_i^w \subset T^an$ is the stable manifold of the gradient flow of $\text{Re}(h_w)$ passing through its $i$-th non-degenerate critical point $\alpha(i)^w \in T^an$. For $1 \leq i \leq \mathbb{Vol}(\Delta)$ we define a confluent $A$-hypergeometric function $u_i$ on $V \subset (\Omega_0)^an$ by

$$u_i(w) = \int_{\gamma_i^w} \exp \left( \sum_{j=1}^{N} w_j a^{(j)} x_j^{a^{-1}_1} \cdots x_j^{c_n^{-1}} dx_1 \wedge \cdots \wedge dx_n \right) \quad (5.26)$$

for $w \in V$.

### 5.3 Asymptotic expansions at infinity

Now by applying the higher-dimensional saddle point (steepest descent) method to holomorphic Morse coordinates around the critical points of $\text{Re}(h_w) : T^an \to \mathbb{R}$ ($w \in V$) in $T^an$ we obtain the following result. For $\delta > 0$ let $\Lambda \subset \mathbb{C}$ be the open sector in $\mathbb{C}$ defined by $\Lambda = \{ \lambda \in \mathbb{C} \mid -\delta < \arg \lambda < \delta \}$. By taking a sufficiently small $\delta > 0$ such that $\lambda \cdot z \in V$ for any $\lambda \in \Lambda$ we set $\Lambda := \Lambda_\delta$.

**Theorem 5.6.** In the situation as above (i.e. $0 \in \text{Int}(\Delta)$), if $\delta > 0$ is sufficiently small, for any $1 \leq i \leq \mathbb{Vol}(\Delta)$ and $\lambda \in \Lambda$ we have an asymptotic expansion:

$$u_i(\lambda \cdot z) = \int_{\gamma_i^\lambda} \exp(\lambda \sum_{j=1}^{N} z_j a^{(j)} x_j^{a^{-1}_1} \cdots x_j^{c_n^{-1}} dx_1 \wedge \cdots \wedge dx_n) \quad (5.27)$$

$$\sim \left( \frac{1}{\sqrt{H_i(z)}} \cdot \frac{\beta_1(z)}{\lambda^{\frac{1}{2}}} + \frac{\beta_2(z)}{\lambda^{\frac{3}{2}}} + \cdots \right) \quad (5.28)$$

as $|\lambda| \to +\infty$ in the sector $\Lambda$, where $\beta_i(z) \in \mathbb{C}$ are functions of $z$ and

$$H_i(z) = \det \left( \frac{\partial^2 h_z}{\partial x_j \partial x_k} \right)_{x=\alpha(i)} \quad (5.30)$$

is the Hessian of $h_z$ at $x = \alpha(i) \in T^an$.

**Proof.** First, it is clear that for any $\lambda \in \Lambda$ the critical points of the function $\text{Re}(h_{\lambda,z}) = \text{Re}(\lambda \cdot h_z) : T^an \to \mathbb{R}$ are $\alpha(i)$ ($1 \leq i \leq \mathbb{Vol}(\Delta)$). Fix $1 \leq i \leq \mathbb{Vol}(\Delta)$. Let $y = (y_1, \ldots, y_n)$, $y_j = \xi_j + \sqrt{-1} \eta_j$ ($1 \leq j \leq n$) be the holomorphic Morse coordinate for the function $h_z$ around its $i$-th critical point $\alpha(i) \in T^an$ such that $h_z(x) = h_z(\alpha(i)) + y_1^2 + \cdots + y_n^2$. For sufficiently small $\varepsilon > 0$ we define an open neighborhood $W_\varepsilon$ of $\alpha(i) \in T^an$ by $W_\varepsilon = \{ y = (y_1, \ldots, y_n) \mid |y_j| < \varepsilon (1 \leq j \leq n) \} \simeq B(0;\varepsilon) \times \cdots \times B(0;\varepsilon) \subset T^an$ and set

$$S_i^\varepsilon = \{ \xi_1 = \cdots = \xi_n = 0 \} = \{ \eta \in \mathbb{R}^n \mid |\eta_j| < \varepsilon (1 \leq j \leq n) \} \subset W_\varepsilon \quad (5.31)$$
in it. Then $S^n_I$ is the stable manifold of the gradient flow of the function $\text{Re}(h_z)$ passing through its non-degenerate critical point $\alpha(i) \in T^\alpha_n$. For $\lambda = |\lambda|e^{\sqrt{-1}\theta} \in \Lambda \subset \mathbb{C}$ ($-\delta < \theta = \text{arg}\lambda < \delta$) we set $(y'_1, \ldots, y'_n) = (e^{\sqrt{-1}\theta}y_1, \ldots, e^{\sqrt{-1}\theta}y_n)$. Then the Laurent polynomial $h_{\lambda z} = \lambda \cdot h_z$ can be rewritten as

$$h_{\lambda z}(x) = \lambda \cdot h_z(\alpha(i)) + |\lambda|(y'_1)^2 + \cdots + |\lambda|(y'_n)^2. \quad (5.32)$$

By setting $y'_j = \xi'_j + \sqrt{-1}\eta'_j \ (1 \leq j \leq n)$ we see also that the subset

$$S^\lambda_i = \{\xi'_1 = \cdots = \xi'_n = 0\} = \{\eta'_1 \in \mathbb{R}^n \mid |\eta'_j| < \varepsilon \ (1 \leq j \leq n)\} \subset W_\varepsilon \quad (5.33)$$

of $W_\varepsilon$ is the stable manifold of the gradient flow of $\text{Re}(h_{\lambda z})$ through $\alpha(i) \in T^\alpha_n$. By our construction of the rapid decay $n$-cycle $\gamma^\lambda_i S^\lambda_i z$ we may assume that $S^\lambda_i \subset S^\lambda_i z$ and

$$\text{Re}(h_{\lambda z})(x) - \text{Re}(h_{\lambda z})(\alpha(i)) < -\varepsilon^2 |\lambda|, \quad \text{Im}(h_{\lambda z})(x) = \text{Im}(h_{\lambda z})(\alpha(i)) \quad (5.34)$$

for any $x \in S^\lambda_i \setminus S^\lambda_i z$. We may assume also that for any $\lambda, \lambda' \in \Lambda$ such that $\text{arg}\lambda = \text{arg}\lambda'$ we have $\gamma^\lambda_i = \gamma^{\lambda'}_i$. This implies that (if $\delta > 0$ is sufficiently small) there exists a positive real numbers $C > 0$ such that

$$\int_{\gamma^\lambda_i S^\lambda_i z} |\exp(h_z(x) - h_z(\alpha(i)))x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} dx_1 \wedge \cdots \wedge dx_n| < C \quad (5.35)$$

for any $\lambda \in \Lambda$. Then we have

$$\left| \int_{\gamma^\lambda_i S^\lambda_i z} \exp(\lambda \cdot h_z(x))x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} dx_1 \wedge \cdots \wedge dx_n \right| = |\exp(\lambda \cdot h_z(\alpha(i)))| \quad (5.36)$$

$$\times \left| \int_{\gamma^\lambda_i S^\lambda_i z} \exp(\lambda \cdot \{h_z(x) - h_z(\alpha(i))\})x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} dx_1 \wedge \cdots \wedge dx_n \right| \quad (5.37)$$

$$\leq C |\exp(\lambda \cdot h_z(\alpha(i)))| \times \sup_{x \in \gamma^\lambda_i S^\lambda_i z} \left| \exp\left(\frac{\lambda - 1}{\lambda}(h_{\lambda z}(x) - h_{\lambda z}(\alpha(i)))\right)\right| \quad (5.38)$$

$$\leq C |\exp(\lambda \cdot h_z(\alpha(i)))| \times \exp\left(-\frac{\varepsilon^2}{2}|\lambda|\right) \quad (5.39)$$

for any $\lambda \in \Lambda$ satisfying $|\lambda| \gg 0$. Hence, to prove the theorem, it suffices to calculate the asymptotic expansion of the integral

$$\tilde{u}_i(\lambda, z) = \int_{S^\lambda_i z} \exp(\lambda \sum_{j=1}^N z_j x^{a(j)}_j x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} dx_1 \wedge \cdots \wedge dx_n) \quad (5.40)$$

as $|\lambda| \to +\infty$ in the sector $\Lambda$. For the Morse coordinate $y = (y_1, \ldots, y_n)$ of $h_z$ we can easily show

$$\det \left( \frac{\partial y_j}{\partial x_k} \right)_{x=\alpha(i)} = \sqrt{\frac{H_i(z)}{2^n}}. \quad (5.41)$$

Also by using the coordinate $y = (y_1, \ldots, y_n)$ set

$$f(y_1, \ldots, y_n) := x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} \times \det \left( \frac{\partial x_j}{\partial y_k} \right) \quad (5.42)$$

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and let 
\[ f(y_1, \ldots, y_n) = \sum_{a \in \mathbb{Z}_+^n} f_a y_a \quad (f_a \in \mathbb{C}) \]  

be its Taylor expansion at \( y = 0 \) i.e. \( x = \alpha(i) \). Then by (5.41) we obtain

\[ f_0 = f(0, \ldots, 0) = \alpha(i)^{c_1-1} \cdots \alpha(i)^{c_n-1} \times \sqrt{\frac{2n}{H_i(z)}}. \]  

(5.44)

Now the restriction of the \( n \)-form
\[ \exp(\lambda \cdot h_z(x))x_1^{c_1-1} \cdots x_n^{c_n-1}dx_1 \land \cdots \land dx_n \]  

to the stable manifold \( S_i^{\lambda,z} = \{ \eta \in \mathbb{R}^n \mid |\eta'_j| < \varepsilon \ (1 \leq j \leq n) \} \subset \mathbb{R}^n \) has the following form:

\[ (\sqrt{-1})^n e^{-\frac{\sqrt{n} t_0}{2}} \exp \{ \lambda \cdot h_z(\alpha(i)) - |\lambda|(\eta'_1)^2 - \cdots - |\lambda|(\eta'_n)^2 \} \]  

× \[ \sum_{a \in \mathbb{Z}_+^n} f_a \cdot e^{-\frac{\sqrt{n} |a|}{2}} \cdot \{ \sqrt{-1} \eta'_1 \}^a \right] d\eta'_1 \land \cdots \land d\eta'_n. \]  

(5.47)

For any \( a = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n \) we can show that the integral of the \( n \)-form

\[ \omega_a := (\sqrt{-1})^n e^{-\frac{\sqrt{n} t_0}{2}} \exp \{ \lambda \cdot h_z(\alpha(i)) - |\lambda|(\eta'_1)^2 - \cdots - |\lambda|(\eta'_n)^2 \} \]  

× \[ f_a \cdot e^{-\frac{\sqrt{n} |a|}{2}} \cdot \{ \sqrt{-1} \eta'_1 \}^a d\eta'_1 \land \cdots \land d\eta'_n \]  

over the whole \( \mathbb{R}_0^n \) is equal to

\[ (\sqrt{-1})^n \lambda^{-\frac{n}{2}} \exp(\lambda \cdot h_z(\alpha(i))) \times f_a \cdot \lambda^{-|a|} \]  

× \[ \int_{\mathbb{R}^n} \exp(-t_1^2 - \cdots - t_n^2) \{ \sqrt{-1} t \}^a dt_1 \land \cdots \land dt_n \]  

(5.50)

by setting \((t_1, \ldots, t_n) = (\sqrt{|\lambda|} \eta_1', \ldots, \sqrt{|\lambda|} \eta_n')\). Note that the integral

\[ \int_{\mathbb{R}^n} \exp(-t_1^2 - \cdots - t_n^2) \{ \sqrt{-1} t \}^a dt_1 \land \cdots \land dt_n \]  

(5.52)

is zero if \( a_i \in \mathbb{Z}_+ \) is odd for some \( 1 \leq i \leq n \). As the previous part of this proof, we can show also that there exists \( M > 0 \) such that

\[ \left| \int_{\mathbb{R}^n \setminus S_i^{\lambda,z}} \omega_a \right| \leq M \exp(\lambda \cdot h_z(\alpha(i))) \times \exp(-\frac{\varepsilon^2}{2} |\lambda|) \]  

(5.53)

for any \( \lambda \in \Lambda \) satisfying \(|\lambda| \gg 0\). Then the result follows immediately from (5.44). This completes the proof. \( \square \)

**Remark 5.7.** If \( z \in \Omega_0 \) and the critical point \( \alpha(i) \) of \( \text{Re}(h_z) : T^\mathbb{R} \longrightarrow \mathbb{R} \) in \( T^\mathbb{R} \) is given, by using the holomorphic Morse coordinate in the proof above we can calculate also the coefficients \( \beta_1(z), \beta_2(z), \ldots \in \mathbb{C} \) explicitly.

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For the point $z \in \Omega_0$ let

$$L_z = \{ \lambda \cdot z \in X = \mathbb{C}^N \mid \lambda \in \mathbb{C} \} \simeq \mathbb{C}_\lambda$$

be the complex line in $X = \mathbb{C}^N$ passing through $z \in \Omega_0$. Then by Theorems 5.5 and 5.6 we can observe Stokes’ phenomena for the restrictions $u_i|_{L_z}$ of the functions $u_i$ ($1 \leq i \leq \text{Vol}_2(\Delta)$) to the line $L_z \simeq \mathbb{C}_\lambda$. Indeed, by Theorem 5.6 the dominance ordering of the functions $u_i|_{L_z}$ at infinity (i.e. where $|\lambda| \gg 0$) changes as $\text{arg}(\lambda)$ increases. More precisely, the asymptotic expansions at infinity of the restrictions of the $A$-hypergeometric functions to $L_z \simeq \mathbb{C}_\lambda$ may jump at the Stokes lines:

$$\{ \lambda \in \mathbb{C} \mid \text{Re} [\lambda \cdot \{ h_z(\alpha(i)) - h_z(\alpha(j))] = 0 \} \quad (i \neq j).$$

(5.55)

It would be an interesting problem to determine the Stokes multipliers in this case.

**Remark 5.8.** When $\Delta' = \text{conv}(A)$ does not contain the origin $0 \in \mathbb{R}^n$, the last half of the proof of Proposition 5.4 does not work. Namely for $z \in \Omega_0$ the number of the non-degenerate (Morse) critical points of $h_z(x)$ may be smaller than $\text{Vol}_2(\Delta') < \text{Vol}_2(\Delta)$. Nevertheless, as in Theorems 5.5 and 5.6 we can construct a part of a basis of $H_n^{rd}(U_z; \mathbb{C}_z^*) \simeq H_n(T^{an} \cup Q, Q; \iota_\ast \mathcal{L})$ by the corresponding rapid decay $n$-cycles and obtain the asymptotic expansions at infinity of the confluent $A$-hypergeometric functions associated to them. Note that the number of the critical points of $h_z(x)$ in such a case is given by [10, Lemma 2.10].

## 6 The two-dimensional case

In this section, we shall construct a natural basis of the rapid decay homology group $H_n(T^{an} \cup Q, Q; \iota_\ast \mathcal{L})$ in the two-dimensional case i.e. $n = 2$.

### 6.1 Some results on relative twisted homology groups

First, we prepare some elementary results on relative twisted homology groups. Set $Z = \mathbb{C}^2_{x_1, x_2}$ and let $h_0$ be the meromorphic function on $Z^{an}$ defined by

$$h_0(x_1, x_2) = \frac{1}{x_1^{m_1} x_2^{m_2}} \quad (m_1, m_2 \in \mathbb{Z}_{>0}).$$

(6.1)

Let $\pi_0 : \widetilde{Z}_0 \longrightarrow Z^{an}$ be the real oriented blow-up of $Z^{an}$ along the normal crossing divisor $D_0^{an} = \{ x_1 = 0 \} \cup \{ x_2 = 0 \}$ and set $\widetilde{D}_0 = \pi_0^{-1}(D_0^{an}) \subset \widetilde{Z}_0$ and $U^{an}_0 = Z^{an} \setminus D_0^{an} \simeq (\mathbb{C}^*)^2$. By the inclusion map $\iota_0 : U^{an}_0 \hookrightarrow \widetilde{Z}_0$ we consider $U^{an}_0$ as an open subset of $\widetilde{Z}_0$ and set

$$P_0 = \widetilde{D}_0 \cap \{ x \in U^{an}_0 \mid \text{Re} h_0(x) \geq 0 \}$$

(6.2)

and $Q_0 = \widetilde{D}_0 \setminus P_0$. Finally let $\mathcal{L}_0$ be the local system of rank one on $U^{an}_0$ defined by

$$\mathcal{L}_0 = \mathbb{C}_{U^{an}_0} x_1^{\beta_1} x_2^{\beta_2} \quad (\beta = (\beta_1, \beta_2) \in \mathbb{C}^2).$$

(6.3)

Then by homotopy and Lemma 3.6 we obtain the following lemma.
Lemma 6.1. (i) For $0 < \varepsilon \ll 1$ set

$$U_0^{an}(\varepsilon) = \{x = (x_1, x_2) \in U_0^{an} \mid \varepsilon < |x_1| < \frac{1}{\varepsilon}\} \subset U_0^{an}. \quad (6.4)$$

Then for any $p \in \mathbb{Z}$ the natural morphism

$$H_p(U_0^{an}(\varepsilon) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \longrightarrow H_p(U_0^{an} \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \quad (6.5)$$

is an isomorphism.

(ii) Assume that $\beta = (\beta_1, \beta_2) \in \mathbb{C}^2$ satisfies the condition $m_2\beta_1 - m_1\beta_2 \notin \mathbb{Z}$. Then we have

$$H_p(U_0^{an} \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \simeq 0 \quad (6.6)$$

for any $p \in \mathbb{Z}$.

Proof. The assertion (i) can be easily shown by homotopy. We will prove (ii). Let $S^1$ be the unit circle $\{x_1 \in \mathbb{C} \mid |x_1| = 1\}$ in $\mathbb{C}_{x_1}$. Then by (i) and homotopy we have an isomorphism

$$H_p((S^1 \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \longrightarrow H_p(U_0^{an} \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \quad (6.7)$$

for any $p \in \mathbb{Z}$. Let us take the base point $e := 1 \in S^1$ of $S^1$. Then by Lemma 3.6 we have

$$H_p((\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \simeq 0 \quad (6.8)$$

for $p \neq 1$ and there exists a natural basis $[\gamma_1], \ldots, [\gamma_{m_2}]$ of $H_1((\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0)$.

Let

$$\Psi_0 : H_1((\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \longrightarrow H_1((\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0) \quad (6.9)$$

be the linear automorphism, i.e. the monodromy of $H_1((\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0)$ induced by the (clockwise) rotation along the circle $S^1$. By the matrix representation of $\Psi_0$ with respect to the basis $[\gamma_1], \ldots, [\gamma_{m_2}]$ we see that the eigenvalues of $\Psi_0$ are contained in the set

$$\{t \in \mathbb{C} \mid t^{m_2} = \exp[2\pi \sqrt{-1}(m_2\beta_1 - m_1\beta_2)]\}. \quad (6.10)$$

In particular, our assumption $m_2\beta_1 - m_1\beta_2 \notin \mathbb{Z}$ implies that id$-\Psi_0$ is an automorphism of $H_1((\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0)$. Now for $0 < \varepsilon << 1$ we define two arcs $G_\pm \subset S^1$ in $S^1$ by

$$G_\pm = \{x_1 \in S^1 \mid \pm \text{Re}x_1 > -\varepsilon|\text{Im}x_1|\} \subset S^1. \quad (6.11)$$

Then $S^1 = G_+ \cup G_-$. By the Mayer-Vietoris exact sequence for relative twisted homology groups associated to the open covering $S^1 \times \mathbb{C}^* = (G_+ \times \mathbb{C}^*) \cup (G_- \times \mathbb{C}^*)$ of $S^1 \times \mathbb{C}^*$ we can calculate $H_p((S^1 \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0)$ ($p \in \mathbb{Z}$) from $H_p((G_\pm \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*, \mathcal{L}_0)$ ($p \in \mathbb{Z}$). Then the assertion (ii) follows from the invertibility of id$-\Psi_0$. □

Next consider the meromorph function $h_1$ on $Z^{an} = \mathbb{C}^2$ defined by

$$h_1(x_1, x_2) = \frac{1}{(x_1 - \lambda_1)^{\alpha_1} \cdots (x_1 - \lambda_k)^{\alpha_k} x_1^{m_1} x_2^{m_2}} \quad (n_j, m_1, m_2 \in \mathbb{Z}_{>0}), \quad (6.12)$$

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where \( \lambda_1, \ldots, \lambda_k \) are distinct non-zero complex numbers. Let \( \pi_1 : \widetilde{Z}_1 \rightarrow Z^{an} \) be the real oriented blow-up of \( Z^{an} \) along \( D_1^{an} = \bigcup_{j=1}^k \{ x_1 = \lambda_j \} \cup \{ x_1 = 0 \} \cup \{ x_2 = 0 \} \) and define \( \widetilde{D}_1 \subset \widetilde{Z}_1, \iota_1 : U_1^{an} = Z^{an} \setminus D_1^{an} \rightarrow Z_1, \) \( P_1 \subset \widetilde{D}_1 \) and \( Q_1 = \widetilde{D}_1 \setminus P_1 \) as above. Moreover let \( L_1 \) be the local system of rank one on \( U_1^{an} \) defined by

\[
L_1 = \mathbb{C}_{U_1^{an}} x_1^{\beta_1} x_2^{\beta_2} \prod_{j=1}^k (x_1 - \lambda_j)^{\beta_j} \quad (\beta = (\beta_1, \beta_2) \in \mathbb{C}^2, \beta' = (\beta'_1, \ldots, \beta'_k) \in \mathbb{C}^k). \quad (6.13)
\]

Then by the proof of Lemma 6.1 (ii) and Mayer-Vietoris exact sequences for relative twisted homology groups we obtain the following proposition.

**Proposition 6.2.** (i) For \( 0 < \varepsilon \ll \) set

\[
U_1^{an}(\varepsilon) = \{ (x_1, x_2) \in U_1^{an} \mid \varepsilon < |x_1| < \frac{1}{\varepsilon}, \quad |x_1 - \lambda_j| > \varepsilon \quad (1 \leq j \leq k) \} \subset U_1^{an}. \quad (6.14)
\]

Then for any \( p \in \mathbb{Z} \) the natural morphism

\[
H_p(U_1^{an}(\varepsilon) \cup Q_1, (\iota_1)_* L_1) \rightarrow H_p(U_1^{an} \cup Q_1, (\iota_1)_* L_1) \quad (6.15)
\]

is an isomorphism.

(ii) Assume that \( k \geq 1 \) and \( \beta = (\beta_1, \beta_2) \in \mathbb{C}^2, \quad \beta' = (\beta'_1, \ldots, \beta'_k) \in \mathbb{C}^k \) satisfy the conditions \( m_2 \beta_1 - m_1 \beta_2 \notin \mathbb{Z} \) and \( m_2 \beta'_j - n_j \beta_2 \notin \mathbb{Z} \) for any \( 1 \leq j \leq k \). Then we have

\[
\dim H_p(U_1^{an} \cup Q_1, (\iota_1)_* L_1) = \begin{cases} k \times m_2 & (p = 2), \\ 0 & (p \neq 2) \end{cases} \quad (6.16)
\]

and can explicitly construct a basis of the vector space \( H_2(U_1^{an} \cup Q_1, (\iota_1)_* L_1) \) over \( \mathbb{C} \).

In the special but important case where \( k \geq 2 \) and \( n_1 = n_2 = \cdots = n_k > 0 \), we can construct the basis of \( H_2(U_1^{an} \cup Q_1, (\iota_1)_* L_1) \) in Proposition 6.2 (ii) very elegantly as follows. By homotopy we may assume that \( \lambda_j = \exp(\frac{2\pi j}{k} \sqrt{-1}) \) (\( 1 \leq j \leq k \)) from the start. For \( 1 \leq j \leq k \) let \( G_j \subset S^1 = \{ x_1 \in \mathbb{C} \mid |x_1| = 1 \} \) be the arc in the unit circle \( S^1 \) between the two points \( \lambda_j, \lambda_{j+1} \in S^1 \), where we set \( \lambda_{k+1} = \lambda_1 \). For sufficiently small \( \varepsilon > 0 \) let \( F_j \) be the boundary of the set

\[
B(\lambda_j; \varepsilon) \cup G_j \cup B(\lambda_{j+1}; \varepsilon) \subset \mathbb{C}^1_{x_1} \quad (6.17)
\]

and denote the central point of the arc \( G_j \) by \( e_j \in G_j \).
We regard $e_j \in F_j$ as the base point of the one-dimensional complex $F_j$. By Lemma 3.6 we have

$$H_p((\{e_j\} \times \mathbb{C}^*) \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1) \simeq 0$$  \hspace{1cm} (6.18)

for $p \neq 1$ and there exists a natural basis $[\gamma_{j1}], \ldots, [\gamma_{jm_2}]$ of $H_1((\{e_j\} \times \mathbb{C}^*) \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1) \simeq \mathbb{C}^{m_2}$. Moreover by the proof of Lemma 6.1 (ii) and Mayer-Vietoris exact sequences we obtain

$$H_p((F_j \times \mathbb{C}^*) \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1) \simeq 0$$  \hspace{1cm} (6.19)

for $p \neq 2$. Note that the shape of $F_j$ looks like that of the figure-8. We start from the base point $e_j \in F_j$, go along the one-dimensional complex $F_j$ in the way of the usual drawing of the figure-8 and come back to the same place $e_j \in F_j$. Along this path on $F_j$ we drag the twisted 1-cycles $\gamma_{j1}, \ldots, \gamma_{jm_2}$ over the point $e_j \in F_j$ keeping their end points in the rapid decay direction $Q_1$ of $\exp(h_1)$. Then by our assumption $n_j = n_{j+1}$ we obtain the twisted 2-cycles $[\delta_{j1}], \ldots, [\delta_{jm_2}]$ in $H_2((F_j \times \mathbb{C}^*) \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1)$. It is easy to see that they form a basis of $H_2((F_j \times \mathbb{C}^*) \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1) \simeq \mathbb{C}^{m_2}$. On the other hand, by Proposition 6.2 (i) and homotopy there exists an isomorphism

$$H_p(\{(\bigcup_{j=1}^k F_j) \times \mathbb{C}^*\} \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1) \sim H_p(U_1^{an} \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1)$$  \hspace{1cm} (6.20)

for any $p \in \mathbb{Z}$. Moreover it follows from our assumption $m_2\beta'_j - n_j \beta \notin \mathbb{Z}$ that we have

$$H_p((F_j \cap F_{j-1}) \times \mathbb{C}^*) \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1) \simeq 0$$  \hspace{1cm} (6.21)

for any $1 \leq j \leq k$ and $p \in \mathbb{Z}$. Hence by the Mayer-Vietoris exact sequences associated to the covering $(\bigcup_{j=1}^k F_j) \times \mathbb{C}^* = \bigcup_{j=1}^k (F_j \times \mathbb{C}^*)$ of $(\bigcup_{j=1}^k F_j) \times \mathbb{C}^*$ we obtain the following result.

**Proposition 6.3.** Assume that $k \geq 2$, $n_1 = n_2 = \cdots = n_k > 0$ and $\beta = (\beta_1, \beta_2) \in \mathbb{C}^2$, $\beta' = (\beta'_1, \ldots, \beta'_k) \in \mathbb{C}^k$ satisfy the condition $m_2\beta'_j - n_j \beta \notin \mathbb{Z}$ for $1 \leq j \leq k$. Then the elements $[\delta_{j1}], \ldots, [\delta_{jm_2}] \in H_2(U_1^{an} \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1)$ ($1 \leq j \leq k$) constructed above are linearly independent over $\mathbb{C}$ and form a basis of $H_2(U_1^{an} \cup Q_1, Q_1; (t_1)_*\mathcal{L}_1)$.

### 6.2 A construction of the basis in the two-dimensional case

Now let us consider the situation in Sections 4 and 5 in the two-dimensional case. For $z \in \Omega$ we define $Q \subset \tilde{D} \subset \tilde{Z}$ in the real oriented blow-up $\pi: \tilde{Z} \to Z^{an}$ of $Z^{an} = (\tilde{Z}_2)^{an}$ as in the proof of Theorem 4.5. For the local system $\mathcal{L} = \mathcal{C}_{T^{an}x_1^{c_1-1}x_2^{c_2-1}}$ on $T^{an}$ we shall construct a basis of the rapid decay homology group $H_2^{rd}(T^{an}) := H_2(T^{an} \cup Q, Q; t_*\mathcal{L})$. By abuse of notations, for an open subset $W$ of $T^{an}$ and $p \in \mathbb{Z}$ we set

$$H_p^{rd}(W) := H_p(W \cup Q, Q; t_*\mathcal{L})$$  \hspace{1cm} (6.22)

for short. Recall that $\Sigma$ is a smooth subdivision of the dual fan of $\Delta = \text{conv}(A \cup \{0\}) \subset \mathbb{R}^2$ and $\rho_1, \ldots, \rho_l \in \Sigma$ are the rays i.e. the one-dimensional cones in $\Sigma$ which correspond to the relevant divisors $D_1, \ldots, D_l$ in $Z = \tilde{Z}_\Sigma$. We renumber $\rho_1, \ldots, \rho_l$ in the clockwise order so that we have $D_i \cap D_{i+1} \neq \emptyset$ for any $1 \leq i \leq l - 1$. By the primitive vector
\( \kappa_i \in \rho_i \cap (\mathbb{Z}^2 \setminus \{0\}) \) on \( \rho_i \) the order \( m_i > 0 \) of the pole of \( h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)} \) along \( D_i \) is explicitly given by

\[
m_i = -\min_{a \in \Delta} \langle \kappa_i, a \rangle.
\]  

(6.23)

For \( 1 \leq i \leq l \) we set

\[
\beta_i = \langle \kappa_i, (c_1 - 1, c_2 - 1) \rangle \in \mathbb{C}.
\]  

(6.24)

Then at each point of \( D_i \setminus (\bigcup_{j \neq i} D_j) \) there exists a local coordinate system \((y_1, y_2)\) of \( Z_{\Sigma} \) such that \( D_i = \{ y_1 = 0 \} \) and the function \( x_1^{c_1-1}x_2^{c_2-1} \) has the form \( y_i^{\beta_i} \). Namely the function \( x_1^{c_1-1}x_2^{c_2-1} \) has the order \( \beta_i \in \mathbb{C} \) along \( D_i \). By the non-degeneracy of \( h_z \) the complex curve \( \overline{h_z^{-1}(0)} \subset Z_{\Sigma} \) intersects each relevant divisor \( D_i \) transversally. Set \( v_i = \sharp \{ D_i \cap \overline{h_z^{-1}(0)} \} \geq 0 \) and \( \{ q_{i1}, \ldots, q_{iv_i} \} = D_i \cap \overline{h_z^{-1}(0)} \). By our construction of the complex blow-up \( Z = \overline{Z_{\Sigma}} \rightarrow Z_{\Sigma} \) of \( Z_{\Sigma} \), the fiber of the point \( q_{ij} \) is a union \( E_{ij} = E_{ij1} \cup \cdots \cup E_{ijm_i} \) of exceptional divisors \( E_{ij} \) is the exceptional divisor constructed by the \( k \)-th blow-up over \( q_{ij} \). See Figure 5 below. Then the order of the pole of \( h_z(p) \) along \( E_{ij} \) is \( m_i - k \). Moreover the order of the function \( x_1^{c_1-1}x_2^{c_2-1} \) along \( E_{ij} \) is \( \beta_i \) for any \( 1 \leq k \leq m_i \). Let \( T_i \simeq \mathbb{C}^* \subset D_i \) be the one-dimensional \( T \)-orbit in \( Z_{\Sigma} \) which corresponds to \( \rho_i \) and denote by the same letter \( T_i \) its strict transform in the blow-up \( Z = \overline{Z_{\Sigma}} \). Assume that \( \beta_i \notin \mathbb{Z} \) \( (1 \leq i \leq l) \) and \( m_{i+1} \beta_i - m_i \beta_i \neq \mathbb{Z} \) \( (1 \leq i \leq l - 1) \). Then by Propositions 6.2 (ii) and 6.3 there exists a sufficiently small tubular neighborhood \( W_i \) of \( T_i^{an} \) in \( Z^{an} \) such that for its open subset \( W_i^o = W_i \cap T^{an} \subset T^{an} \) we have

\[
\dim H^d_p(W_i^o) = \begin{cases} v_i \times m_i & (p = 2), \\ 0 & (p \neq 2). \end{cases}
\]  

(6.25)

Furthermore we can explicitly construct a basis \( \delta_{ij} \in H^d_p(W_i^o) \) \( (1 \leq j \leq v_i, 1 \leq k \leq m_i) \) of the vector space \( H^d_p(W_i^o) = H_2(W_i^o \cup Q; \mathbb{Q}; \iota_*, \mathcal{L}) \) over \( \mathbb{C} \).

\[ Z = \overline{Z_{\Sigma}} \]

Figure 5.

**Theorem 6.4.** Assume that \( c = (c_1, c_2) \in \mathbb{C}^2 \) is nonresonant, \( \beta_i \notin \mathbb{Z} \) \( (1 \leq i \leq l) \) and \( m_{i+1} \beta_i - m_i \beta_{i+1} \neq \mathbb{Z} \) \( (1 \leq i \leq l - 1) \). Then the natural morphisms

\[
\Theta_i : H^d_2(W_i^o) \rightarrow H^d_2(T^{an}) \quad (1 \leq i \leq l)
\]  

(6.26)

are injective and induce an isomorphism

\[
\Theta : \oplus_{i=1}^l H^d_2(W_i^o) \xrightarrow{\sim} H^d_2(T^{an}).
\]  

(6.27)
In particular, the cycles \( \gamma_{ijk} := \Theta_i(\delta_{ijk}) \in H^2_{\text{rd}}(T^\text{an}) \) \( (1 \leq i \leq l, 1 \leq j \leq v, 1 \leq k \leq m_i) \) form a basis of the vector space \( H^2_{\text{rd}}(T^\text{an}) = H_2(T^\text{an} \cup Q, Q; \iota_* \mathcal{L}) \) over \( \mathbb{C} \).

Proof. By the repeated use of Lemma 6.1 (i) and homotopy, we can find a small neighborhood \( \hat{W}_i \) of \( T^\text{an} \cup \cup_{j=1}^v (E_{ij}^\text{an} \setminus E_{ijm_j}^\text{an}) \) in \( Z^\text{an} \) containing \( W_i \) such that for its open subset \( \hat{W}_i^\circ = \hat{W}_i \cap T^\text{an} \subset T^\text{an} \) the natural morphism

\[
H^p_{\text{rd}}(W_i^\circ) \longrightarrow H^p_{\text{rd}}(\hat{W}_i^\circ)
\]

is an isomorphism for any \( p \in \mathbb{Z} \). By our assumption \( (c_1, c_2) \notin \mathbb{Z}^2 \) we have the vanishing of the usual twisted homology group \( H_p(T^\text{an}; \mathcal{L}) \) for any \( p \in \mathbb{Z} \). Similarly we obtain the vanishings of \( H_p(\hat{W}_i^\circ; \mathcal{L}) \) etc. Then the assertion can be proved by patching these results with the help of Lemma 6.1 (i) and the Mayer-Vietoris exact sequences for the relative twisted homology groups in the proof of Theorem 4.5.

Let \( \Gamma_i \prec \Delta \) be the supporting face of \( \rho_i \) in \( \Delta \). Then the lattice length of \( \Gamma_i \) is equal to \( v_i \geq 0 \) and we have the equality \( \sum_{i=1}^l (v_i \times m_i) = \text{Vol}_Z(\Delta) \) as expected from the result of Theorem 6.4.

7 Higher-dimensional cases

In this section, we shall extend the construction of the basis of the rapid decay homology group \( H_n(T^\text{an} \cup Q, Q; \iota_* \mathcal{L}) \) for \( n = 2 \) in Section 6 to higher-dimensional cases.

7.1 Some results on twisted Morse theory

Let \( T_0 = (\mathbb{C}^*)^k \) be a \( k \)-dimensional algebraic torus and \( h_0(x) \) a Laurent polynomial on it whose Newton polytope \( \Delta_0 = \text{NP}(h_0) \subset \mathbb{R}^k \) is \( k \)-dimensional. We assume that \( h_0 \) is non-degenerate in the sense of Kouchnirenko [26]. Namely we impose the condition in Definition 2.3 for any face \( \Gamma \prec \Delta_0 \) of \( \Delta_0 \).

Proposition 7.1. In the situation as above, for generic \( a = (a_1, \ldots, a_k) \in \mathbb{C}^k \) the (possibly multi-valued) function \( g(a, x) = h_0(x)x^a \) on \( T_0 \) has exactly \( \text{Vol}_Z(\Delta_0) \) critical points in \( T_0 \) and all of them are non-degenerate (i.e. of Morse type) and contained in \( T_0 \setminus \{h_0 = 0\} = \{x \in T_0 \mid g(a, x) \neq 0\} \).

Proof. We follow the argument in [9, page 10]. For \( 1 \leq i \leq k \) set \( \partial_i = \partial_{x_i} \). Then for \( x \in T_0 \) we have

\[
\partial_i g(a, x) = 0 \iff x_i \partial_i h_0(x) - a_i h_0(x) = 0.
\]

Since the hypersurface \( \{h_0 = 0\} \subset T_0 \) is smooth, by (7.1) all the critical points of the function \( g(a, \ast) \) are contained in \( T_0 \setminus \{h_0 = 0\} \). Set \( f_i(a, x) = x_i \partial_i h_0(x) - a_i h_0(x) \). Then by Bernstein’s theorem we have

\[
\#\{x \in T_0 \mid f_i(a, x) = 0 \mid 1 \leq i \leq k\} = \text{Vol}_Z(\Delta_0)
\]

if the Newton polytopes \( \text{NP}(f_i(a, \ast)) \) of the Laurent polynomials \( f_i(a, \ast) \) \( (1 \leq i \leq k) \) are equal to \( \Delta_0 \) and the subvariety \( K = \{x \in T_0 \mid f_i(a, x) = 0 \mid 1 \leq i \leq k\} \) of \( T_0 \) is a non-degenerate complete intersection (see [36]). From now on, we will show that these two
conditions are satisfied for generic \( a \in \mathbb{C}^k \). First of all, it is clear that \( NP(f_i(a, *)) = \Delta_0 \) (\( 1 \leq i \leq k \)) for generic \( a \in \mathbb{C}^k \). Note that \( x \in T_0 \) is in \( K \) if and only if \( x \in T_0 \setminus \{ h_0 = 0 \} \) and

\[
\frac{x_i \partial_i h_0(x)}{h_0(x)} = a_i \quad (1 \leq i \leq k),
\]

that is, \( x \in T_0 \setminus \{ h_0 = 0 \} \) is sent to the point \( a \in \mathbb{C}^k \) by the map \( T_0 \setminus \{ h_0 = 0 \} \rightarrow \mathbb{C}^k \) defined by

\[
x \mapsto \left( \frac{x_1 \partial_1 h_0(x)}{h_0(x)}, \ldots, \frac{x_k \partial_k h_0(x)}{h_0(x)} \right).
\]

By the Bertini-Sard theorem generic \( a \in \mathbb{C}^k \) are regular values of this map. If \( a \in \mathbb{C}^k \) is such a point, we can easily show that \( \det \{ \partial_j f_i(a, x) \}_{j,i=1}^k \neq 0 \) for any \( x \in K \subset T_0 \setminus \{ h_0 = 0 \} \). Now let \( \Gamma \not\subseteq \Delta_0 \) be a proper face of \( \Delta_0 \). Then for generic \( a \in \mathbb{C}^k \) we have

\[
\{ x \in T_0 \mid f_i(a, x) = 0 \quad (1 \leq i \leq k) \} = \emptyset.
\]

Indeed, let us assume the converse. Then the first projection from the variety

\[
\{ (a, x) \in \mathbb{C}^k \times T_0 \mid f_i(a, x) = 0 \quad (1 \leq i \leq k) \} \subset \mathbb{C}^k \times T_0
\]

to \( \mathbb{C}^k \) is dominant. Moreover by \( \Gamma \not\subseteq \Delta_0 \) this variety is quasi-homogeneous with respect to the second variables \( x = (x_1, \ldots, x_k) \). In particular, its dimension is greater than \( k \). Then by considering the second projection from it to \( T_0 \), we find that there exist \( x \in T_0 \) and \( a \neq a' \in \mathbb{C}^k \) such that \( f_i(a, x)x^0 = f_i(a', x)x^0 = 0 \) for any \( 1 \leq i \leq k \). Namely we have

\[
x_i \partial_i h_0^x(x) - a_i h_0^x(x) = 0 \quad (1 \leq i \leq k)
\]

and

\[
x_i \partial_i h_0^{x'}(x) - a'_i h_0^{x'}(x) = 0 \quad (1 \leq i \leq k).
\]

Comparing (7.7) with (7.8) for \( 1 \leq i \leq k \) such that \( a_i \neq a'_i \), we obtain \( h_0^x(x) = \partial_1 h_0^x(x) = \cdots = \partial_k h_0^x(x) = 0 \) for the point \( x \in T_0 \). This contradicts our assumption that the Laurent polynomial \( h_0 \) is non-degenerate. We thus proved that the subvariety \( K = \{ x \in T_0 \mid f_i(a, x) = 0 \quad (1 \leq i \leq k) \} \) of \( T_0 \) is a non-degenerate complete intersection and its cardinality is \( \text{Vol}_\mathbb{Z}(\Delta_0) \) for generic \( a \in \mathbb{C}^k \). Let us fix such a point \( a \in \mathbb{C}^k \). Recall that \( K = \{ x \in T_0 \mid f_i(a, x) = 0 \quad (1 \leq i \leq k) \} \) is the set of the critical points of the function \( g(a, *) \) in \( T_0 \setminus \{ h_0 = 0 \} \). At such a critical point \( x \in T_0 \setminus \{ h_0 = 0 \} \) we have

\[
\frac{\partial^2 g}{\partial x_j \partial x_i}(a, x) = \partial_j \left\{ f_i(a, x) \frac{x^{-a}}{x_i} \right\} = \partial_j f_i(a, x) \cdot \frac{x^{-a}}{x_i}.
\]

Since \( \det \{ \partial_j f_i(a, x) \}_{j,i=1}^k \neq 0 \), we obtain

\[
\det \left\{ \frac{\partial^2 g}{\partial x_j \partial x_i}(a, x) \right\}_{j,i=1}^k \neq 0.
\]

Namely all the critical points of the function \( g(a, *) \) are non-degenerate. \( \square \)
Let $h_0$ be as above and $\mathcal{L}_0$ a local system of rank one on $T_0^{an}$ defined by
\begin{equation}
\mathcal{L}_0 = \mathbb{C} T_0^{an} \frac{x_1^{\beta_1} \cdots x_k^{\beta_k}}{(\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k)}.
\end{equation}
From now on, we will calculate the twisted homology groups $H_p(T_0^{an} \setminus \{h_0 = 0\}^{an}; \mathcal{L}_0)$ ($p \in \mathbb{Z}$) by using our twisted Morse theory. Taking a sufficiently generic $a \in \text{Int}(\Delta_0) \subset \mathbb{R}^k$ we set $h_1(x) = h_0(x) x^{-a}$. Then by Proposition 7.1 the real-valued function $f := |h_1|^{-2}: T_0^{an} \setminus \{h_0 = 0\}^{an} \to \mathbb{R}$ has only Vol$_\mathbb{Z}(\Delta_0)$ non-degenerate (Morse) critical points in $M := T_0^{an} \setminus \{h_0 = 0\}^{an}$. Moreover we can easily verify that the index of such a critical point is always $k$. For $t \in \mathbb{R}_{>0}$ we define an open subset $M_t \subset M$ of $M$ by
\begin{equation}
M_t = \{x \in M \mid f(x) = |h_1|^{-2}(x) < t\}.
\end{equation}
Then we have the following result.

**Proposition 7.2.** For generic $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k$ and $0 < \varepsilon \ll 1$ we have
\begin{equation}
H_p(M_\varepsilon; \mathcal{L}_0) \simeq 0
\end{equation}
for any $p \in \mathbb{Z}$.

**Proof.** Let $\Sigma'$ be a smooth subdivision of the dual fan of $\Delta_0$ and $Z_{\Sigma'}$ the (smooth) toric variety associated to it. Then the divisor at infinity $D' = Z_{\Sigma'} \setminus T_0$ is normal crossing. By the non-degeneracy of $h_0$ the divisor $\overline{h_0^{-1}(0)} \cup D'$ is also normal crossing in a neighborhood of $D'$. Moreover by the condition $a \in \text{Int}(\Delta_0)$ the neighborhood $M_\varepsilon \cup (D' \setminus \overline{h_0^{-1}(0)})$ of $D' \setminus \overline{h_0^{-1}(0)}$ retracts to $D' \setminus \overline{h_0^{-1}(0)}$ as $\varepsilon \to +0$. Since for generic $\beta \in \mathbb{C}^k$ the local system $\mathcal{L}_0$ has a non-trivial monodromy around each irreducible component of $D'$, the assertion follows.

By this proposition we can apply the argument in the proof of Theorem 5.5 to the Morse function $f = |h_1|^{-2}: M = T_0^{an} \setminus \{h_0 = 0\}^{an} \to \mathbb{R}$ and obtain the following theorem.

**Theorem 7.3.** For generic $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k$ we have
\begin{equation}
\dim H_p(T_0^{an} \setminus \{h_0 = 0\}^{an}; \mathcal{L}_0) = \begin{cases} \text{Vol}_\mathbb{Z}(\Delta_0) & (p = k), \\ 0 & (p \neq k) \end{cases}
\end{equation}
and there exists a basis of $H_k(T_0^{an} \setminus \{h_0 = 0\}^{an}; \mathcal{L}_0)$ indexed by the Vol$_\mathbb{Z}(\Delta_0)$ non-degenerate (Morse) critical points of the (possibly multi-valued) function $h_1(x) = h_0(x) x^{-a}$ in $T_0 \setminus \{h_0 = 0\}$.

Note that Theorem 7.3 partially solves the famous problem in the paper Gelfand-Kapranov-Zelevinsky [15] of constructing a basis of the twisted homology group in their integral representation of A-hypergeometric functions. Indeed Theorem 7.3 holds even if we replace the local system $\mathcal{L}_0$ with the one
\begin{equation}
\mathcal{L}_1 = \mathbb{C} T_0^{an} \setminus \{h_0 = 0\}^{an} h_0(x)^{\alpha} x_1^{\beta_1} \cdots x_k^{\beta_k} \quad (\alpha \in \mathbb{C}, \quad \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k)
\end{equation}
on $T_0^{an} \setminus \{h_0 = 0\}^{an}$. 34
7.2 A construction of the basis in the higher-dimensional case

Now we consider the situation in Sections 4 and 5. We inherit the notations there. We fix a point \( z \in \Omega \) and define \( Q \subset \tilde{D} \subset \tilde{Z} \) etc. in the real oriented blow-up \( \pi : \tilde{Z} \to Z^\text{an} \) of \( Z^\text{an} = (\tilde{Z}_\Sigma)^\text{an} \) as in the proof of Theorem 4.5. For the local system \( \mathcal{L} = C_{\mathcal{T}^*} x_1^{c_1-1} \cdots x_n^{c_n-1} \) on \( T^\text{an} \) we shall construct a basis of the rapid decay homology group \( H_p^\text{rd}(T^\text{an}) := H_p(T^\text{an} \cup Q, Q; t_\ast \mathcal{L}) \). As in Section 6, for an open subset \( W \) of \( T^\text{an} \) and \( p \in \mathbb{Z} \) we set

\[
H_p^\text{rd}(W) := H_p(W \cup Q, Q; t_\ast \mathcal{L})
\]  

(7.16)

for short. Recall that \( \rho_1, \ldots, \rho_l \) are the rays in the smooth fan \( \Sigma \) which correspond to the relevant divisors \( D_1, \ldots, D_l \) in \( Z = Z_\Sigma \). By the primitive vector \( \kappa_i \in \rho_i \cap (\mathbb{Z}^n \setminus \{0\}) \) on \( \rho_i \) the order \( m_i > 0 \) of the pole of \( h_z(x) = \sum_j \zeta_j x^{a(j)} \) along \( D_i \) is explicitly described by

\[
m_i = -\min_{a \in \Delta} \langle \kappa_i, a \rangle.
\]  

(7.17)

By the non-degeneracy of \( h_z \) the complex hypersurface \( \overline{h_z^{-1}(0)} \subset Z_\Sigma \) intersects each relevant divisor \( D_i \) transversally. Let \( T_i \simeq (\mathbb{C}^*)^{n-1} \subset D_i \) be the \( T \)-orbit in \( Z_\Sigma \) which corresponds to \( \rho_i \) and denote by the same letter \( T_i \) its strict transform in the blow-up \( Z = \tilde{Z}_\Sigma \). Recall also that for \( 1 \leq i \leq l \) the Euler characteristic of the hypersurface \( \{ h_z^{\Gamma_i} = 0 \} = T_i \cap \overline{h_z^{-1}(0)} \) in \( T_i \) is equal to \((-1)^n v_i = (-1)^n \text{Vol}_\Sigma(T_i) \). Let \( y = (y_1, \ldots, y_n) \) be the coordinates on an affine chart \( U_i \simeq \mathbb{C}^n \subset Z_\Sigma \) of \( Z_\Sigma \) containing \( T_i \) such that \( T_i = \{ y_n = 0 \} \) and \( \mathcal{L} \simeq C_{\mathcal{T}^*} y_{\beta_1}^{\beta_1} \cdots y_{\beta_{n-1}}^{\beta_{n-1}} \). Define a local system \( \mathcal{L}_i \) on \( T_i^\text{an} \) by \( \mathcal{L}_i = C_{\mathcal{T}^*} y_{\beta_1}^{\beta_1} \cdots y_{\beta_{n-1}}^{\beta_{n-1}} \).

**Proposition 7.4.** If \( (\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1} \) is generic, we have

\[
\dim H_p(T_i^\text{an} \setminus \{ h_z^{\Gamma_i} = 0 \}^\text{an}; \mathcal{L}_i) = \begin{cases} v_i & (p = n - 1), \\ 0 & (p \neq n - 1) \end{cases}
\]  

(7.18)

and can construct a basis of \( H_{n-1}(T_i^\text{an} \setminus \{ h_z^{\Gamma_i} = 0 \}^\text{an}; \mathcal{L}_i) \) by the twisted Morse theory.

**Proof.** If \( \dim \Gamma_i = n - 1 \) (\( \iff \) \( v_i > 0 \)), the assertion follows immediately from Theorem 7.3. If \( \dim \Gamma_i < n - 1 \) (\( \iff \) \( v_i = 0 \)), we have \( T_i^\text{an} \setminus \{ h_z^{\Gamma_i} = 0 \}^\text{an} \simeq \mathbb{C}^* \times W \) for an open subset \( W \) of \( (\mathbb{C}^*)^{n-2} \). Hence for generic \( (\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1} \) there exists an isomorphism \( H_p(T_i^\text{an} \setminus \{ h_z^{\Gamma_i} = 0 \}^\text{an}; \mathcal{L}_i) \simeq 0 \) for any \( p \in \mathbb{Z} \).

Similarly we can prove also the following proposition.

**Proposition 7.5.** For each generic parameter vector \( c \in \mathbb{C}^n \) and \( 1 \leq i \leq l \) there exists a sufficiently small tubular neighborhood \( W_i \) of \( T_i^\text{an} \) in \( Z^\text{an} \) such that for its open subset \( W_i^\circ = W_i \cap T^\text{an} \subset T^\text{an} \) we have

\[
\dim H_p^\text{rd}(W_i^\circ) = \begin{cases} v_i \times m_i & (p = n), \\ 0 & (p \neq n). \end{cases}
\]  

(7.19)

and can construct a basis \( \delta_{ijk} \in H_p^\text{rd}(W_i^\circ) \) \((1 \leq j \leq v_i, 1 \leq k \leq m_i)\) of \( H_p^\text{rd}(W_i^\circ) = H_n(W_i^\circ \cup Q, Q; t_\ast \mathcal{L}) \) by the twisted Morse theory.
Proof. Let \( f_i : T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \to \mathbb{R} \) be the function on \( T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \) defined by
\[
f_i(x) = |h_z^i(x) x^{-a}|^{-2}
\]
for a sufficiently generic \( a \in \text{Int}(\Gamma_i) \subset \mathbb{R}^{n-1} \). Then by Proposition 7.1 the function \( f_i \) has only \( v_i \) non-degenerate (Morse) critical points in \( T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \). By the product decomposition \( U_i^{an} \simeq C_{y_i}^{an} \times \cdots \times C_{y_n}^{an} \) we consider \( f_i \) also as a function on the open subset
\[
U_i^o = (T_i^{an} \setminus \{ h_z^i = 0 \}^{an}) \times C_{y_n} \subset U_i^{an}
\]
of \( U_i^{an} \). For \( t \in \mathbb{R}_{>0} \) we set
\[
W_{i,t}^o = \{ y \in U_i^o \cap W_i^o \mid f_i(y) < t \}.
\]
Then it follows from Lemma 6.1 (i) that by shrinking \( W_i \) and taking large enough \( t_0 \gg 1 \) we obtain isomorphisms
\[
H_p^{rd}(W_{i,t_0}^o) \cong H_p^{rd}(W_i^o) \quad (p \in \mathbb{Z}).
\]
In the same way as the proof of Proposition 7.2, for generic \( c \in C^n \) and sufficiently small \( 0 < \varepsilon \ll 1 \) we can show that \( H_p^{rd}(W_i^o) \simeq 0 \) \((p \in \mathbb{Z})\). Let \( 0 < t_1 < t_2 < \cdots < t_r < +\infty \) be the critical values of \( f_i : T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \to \mathbb{R} \). We may assume that \( t_r < t_0 \). For \( 1 \leq j \leq r \) let \( \alpha(1), \alpha(2), \ldots, \alpha(n_j) \in T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \) be the critical points of \( f_i \) such that \( f_i(\alpha(q)) = t_j \). Let \( S_q \subset T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \) be the stable manifold of the gradient flow of \( f_i \) passing through its critical point \( \alpha(q) \in T_i^{an} \setminus \{ h_z^i = 0 \}^{an} \) and \( B_i^* \subset C_{y_n} \), the punctured disk \( \{ y_n \in C \mid 0 < |y_n| < \varepsilon \} \) for sufficiently small \( 0 < \varepsilon \ll 1 \). We assume here that \( S_q \) are homeomorphic to the \((n-1)\)-dimensional disk. Then by Lemma 3.6 and the Künneth formula we have
\[
H_p((S_q \times B_i^*) \cup Q, (\partial S_q \times B_i^*) \cup Q; \iota_* L) \simeq 0 \quad (p \neq n)
\]
and the dimension of the vector space \( H_n((S_q \times B_i^*) \cup Q, (\partial S_q \times B_i^*) \cup Q; \iota_* L) \) is \( m_i \). This implies that for any \( 1 \leq j \leq r \) and generic \( c \in C^n \) there exists a short exact sequence
\[
0 \to H_n^{rd}(W_{i,t_j-\varepsilon}) \to H_n^{rd}(W_{i,t_j+\varepsilon}) \to \bigoplus_{q=1}^{n_j} H_n((S_q \times B_i^*) \cup Q, (\partial S_q \times B_i^*) \cup Q; \iota_* L) \to 0.
\]
Hence by (7.22) we get
\[
\dim H_p^{rd}(W_i^o) = \begin{cases} v_i \times m_i & (p = n), \\ 0 & (p \neq n). \end{cases}
\]
Moreover by Lemma 3.6 we can construct a natural basis of the vector space \( H_n((S_q \times B_i^*) \cup Q, (\partial S_q \times B_i^*) \cup Q; \iota_* L) \simeq C^{m_i} \). Lifting these bases to \( H_n^{rd}(W_{i,t_0}^o) \simeq H_n^{rd}(W_i^o) \) with the help of the above short exact sequences we obtain the one \( \delta_{ijk} \in H_n^{rd}(W_i^o) \) \((1 \leq j \leq v_i, 1 \leq k \leq m_i)\) of \( H_n^{rd}(W_i^o) \).

Since for generic \( c \in C^n \) we have the vanishings of the usual twisted homology groups \( H_p(T_i^{an}; L) \) and \( H_p(W_i^o; L) \) etc., by Mayer-Vietoris exact sequences for relative twisted homology groups and Proposition 7.5 we obtain the following theorem.
Theorem 7.6. For generic nonresonant parameter vectors $c \in \mathbb{C}^n$ the natural morphisms
\[ \Theta_i : H^\mathrm{rd}_n(W_i^\circ) \to H^\mathrm{rd}_n(T^\mathrm{an}) \quad (1 \leq i \leq l) \] (7.27)
are injective and induce an isomorphism
\[ \Theta : \bigoplus_{i=1}^l H^\mathrm{rd}_n(W_i^\circ) \sim H^\mathrm{rd}_n(T^\mathrm{an}). \] (7.28)

In particular, the cycles $\gamma_{ijk} := \Theta_i(\delta_{ijk}) \in H^\mathrm{rd}_n(T^\mathrm{an})$ ($1 \leq i \leq l$, $1 \leq j \leq v_i$, $1 \leq k \leq m_i$) form a basis of the vector space $H^\mathrm{rd}_n(T^\mathrm{an}) = H_n(T^\mathrm{an} \cup Q, Q; \iota_\ast \mathcal{L})$ over $\mathbb{C}$.

References


